Helicity Amplitudes for QCD with Massive Quarks

arXiv:1802.06730 [hep-ph]

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Invitation

Parke-Taylor formula:

Parke, Taylor (1986)

$$A(1^-, 2^+, 3^-, 4^+, \dots, n^+) = \frac{i\langle 1\,3\rangle^4}{\langle 1\,2\rangle\langle 2\,3\rangle \dots \langle n\,1\rangle}$$

Simplification w.r.t Feynman rules due to

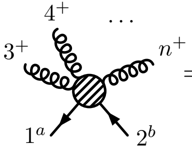
- Gauge invariance
- Massless spinor-helicity variables

This talk:

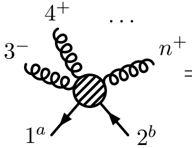
- possible for massive quarks!

Main results

AO (2018)



$$= \frac{i\, m\langle 1^a 2^b\rangle [3| \prod_{j=3}^{n-2} \{ \not{p}_{13\ldots j}\, \not{p}_{j+1} + (s_{13\ldots j} - m^2)\} |n]}{(s_{13} - m^2)(s_{134} - m^2)\ldots(s_{13\ldots(n-1)} - m^2)\,\langle 34\rangle\langle 45\rangle \ldots \langle n-1|n\rangle}$$



$$= -\frac{i\langle 3|1|2|3\rangle\,\left(\langle 1^a 3\rangle [2^b|1+2|3\rangle + \langle 2^b 3\rangle [1^a|1+2|3\rangle\right)}{s_{12}\langle 3\,4\rangle \ldots \langle n-1|n\rangle \langle 3|1|1+2|n\rangle}$$

$$+ \sum_{k=4}^{n-1} \frac{i\, m\langle 3|\not{p}_1\, \not{p}_{3\ldots k}|3\rangle\,\left(\langle 1^a 2^b\rangle\langle 3|\not{p}_1\, \not{p}_{3\ldots k}|3\rangle + \langle 1^a 3\rangle\langle 2^b 3\rangle s_{3\ldots k}\right)}{s_{3\ldots k}\,(s_{13\ldots k} - m^2)\ldots(s_{13\ldots(n-1)} - m^2)\,\langle 3\,4\rangle \ldots \langle k-1|k\rangle\langle 3|\not{p}_1\, \not{p}_{3\ldots k}|k\rangle}$$

$$\times \frac{\langle 3|\not{p}_{3\ldots k} \prod_{j=k}^{n-2}\{\not{p}_{13\ldots j}\, \not{p}_{j+1} + (s_{13\ldots j} - m^2)\}|n]}{\langle 3|\not{p}_1\, \not{p}_{3\ldots k}|k+1\rangle\langle k+1|k+2\rangle \ldots \langle n-1|n\rangle}$$

Outline

1. Massive spinor helicity
2. 4-pt Compton amplitude
3. n-pt amplitudes

4. Summary & outlook

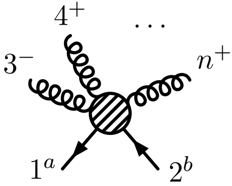
Massive spinor helicity

Why spinor helicity?

Consider color-ordered QCD amplitude $A(\underline{1}^a, 3^-, 4^+, \ldots, n^+, \overline{2}^b)^*$

Feynman rules give function of

- momenta p_i^μ
- polarization vectors $\varepsilon_\pm^\mu(p_i)$
- external spinors $\bar{u}^a(p_1)$, $v^b(p_2)$



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[*] Disclaimer: all momenta outgoing

Why spinor helicity?

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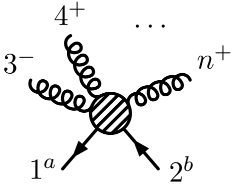

But all vector, spinor indices must be contracted

Remaining indices	$\Leftrightarrow$	physical quantum numbers:
▸ helicities $\pm$	$\Leftrightarrow$	spins $\{\pm 1/2\}_p$, $\{\pm \mathbf{1}\}_{\mathbf{p}}$, etc.
▸ SU(2) labels a, b	$\Leftrightarrow$	spins $\{\pm \mathbf{1/2}\}_{\mathbf{q}}$, $\{\pm 1, 0\}_q$, etc.

Crucial on-shell notion — LITTLE GROUP

*Disclaimer: all momenta outgoing

Little groups

- Quantum fields $\Leftarrow$ reps of $\mathrm{SO}(1,3)$
- Quantum states $\Leftarrow$ reps of LITTLE GROUP
  - massless states $\Leftarrow$ $\mathrm{SO}(2)$
  - massive states $\Leftarrow$ $\mathrm{SO}(3)$

Little groups

- Quantum fields $\Leftarrow$ reps of $SO(1,3)$ $\subset$ $SL(2,\mathbb{C})$
- Quantum states $\Leftarrow$ reps of LITTLE GROUP's dbl cover
  - massless states $\Leftarrow$ $SO(2)$ $\subset$ $\mathbf{U(1)}$
  - massive states $\Leftarrow$ $SO(3)$ $\subset$ $\mathbf{SU(2)}$

Minor complication: spinorial reps use groups' double covers

U(1) and SU(2) arise naturally in spinor helicity

Spinor map

Basis for spinor helicity

- Minkowski space isomorphism:*

$$
\begin{aligned}
\mathrm{M}^{2\times 2,\mathbb{C}}_{\text{Hermitian}} \quad &\leftrightarrow \quad \mathbb{R}^{1,3} \\
p_{\alpha\dot\beta} = p_\mu \sigma^\mu_{\alpha\dot\beta} &= \begin{pmatrix} p^0 - p^3 & -p^1 + ip^2 \\ -p^1 - ip^2 & p^0 + p^3 \end{pmatrix} \\
\det\{p_{\alpha\dot\beta}\} \;&=\; m^2
\end{aligned}
$$

* $\sigma^0 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, $\sigma^1 = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, $\sigma^2 = \left(\begin{smallmatrix} 0 & -i \\ i & 0 \end{smallmatrix}\right)$, $\sigma^3 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, $\epsilon^{\alpha\beta} = -\epsilon_{\alpha\beta} = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$.

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\det\{p_{\alpha\dot\beta}\} \; &= \; m^2
\end{aligned}
$$

- Lorentz group homomorphism:

$$
\begin{aligned}
\mathrm{SL}(2,\mathbb{C}) \quad &\rightarrow \quad \mathrm{SO}(1,3) \\
p_{\alpha\dot\delta} \rightarrow S_\alpha{}^\beta p_{\beta\dot\gamma} \left(S_\delta{}^\gamma\right)^* \quad &\Rightarrow \quad p^\mu \rightarrow L^\mu{}_\nu p^\nu, \quad L^\mu{}_\nu = \frac{1}{2}\,\mathrm{tr}\left(\bar\sigma^\mu S \sigma_\nu S^\dagger\right)
\end{aligned}
$$

$^*\sigma^0 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, $\sigma^1 = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, $\sigma^2 = \left(\begin{smallmatrix} 0 & -i \\ i & 0 \end{smallmatrix}\right)$, $\sigma^3 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, $\epsilon^{\alpha\beta} = -\epsilon_{\alpha\beta} = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$.

Massless vs massive spinor helicity

Arkani-Hamed, Huang, Huang (2017)

MASSLESS	MASSIVE
$\det\{p_{\alpha\dot\beta}\} = 0$	$\det\{p_{\alpha\dot\beta}\} = m^2$
$p_{\alpha\dot\beta} = \lambda_{p\alpha}\tilde\lambda_{p\dot\beta} \equiv \lvert p\rangle_\alpha [p\rvert_{\dot\beta}$	$p_{\alpha\dot\beta} = \lambda_{p\alpha}^{\ a}\epsilon_{ab}\tilde\lambda_{p\dot\beta}^{\ b} \equiv \lvert p^a\rangle_\alpha [p_a\rvert_{\dot\beta}$ $\det\{\lambda_{p\alpha}^{\ a}\} = \det\{\tilde\lambda_{p\dot\alpha}^{\ a}\} = m$
$p^\mu = \frac{1}{2}\langle p\lvert\sigma^\mu\rvert p]$	$p^\mu = \frac{1}{2}\langle p^a\lvert\sigma^\mu\rvert p_a]$
$p_{\alpha\dot\beta}\tilde\lambda_p^{\dot\beta} = 0$	$p_{\alpha\dot\beta}\tilde\lambda_p^{a\dot\beta} = m\lambda_{p\alpha}^{\ a}$
$\langle p\,q\rangle = -\langle q\,p\rangle \quad\Rightarrow\quad \langle p\,p\rangle = 0$ $[p\,q] = -[q\,p] \quad\Rightarrow\quad [p\,p] = 0$ $\langle p\,q\rangle[q\,p] = 2p\cdot q$	$\langle p^a q^b\rangle = -\langle q^b p^a\rangle$ e.g. $\langle p^a p^b\rangle = -m\epsilon^{ab}$ $[p^a q^b] = -[q^b p^a]$ e.g. $[p^a p^b] = m\epsilon^{ab}$ $\langle p^a q^b\rangle[q_b\,p_a] = 2p\cdot q$

Wavefunctions from helicity spinors

$$\varepsilon^\mu_{p+} = \frac{1}{\sqrt{2}}\frac{\langle q|\sigma^\mu|p]}{\langle q\,p\rangle} \qquad\qquad \varepsilon^\mu_{p-} = \frac{1}{\sqrt{2}}\frac{\langle p|\sigma^\mu|q]}{[p\,q]}$$

$$\Rightarrow \quad \begin{cases} \varepsilon_p^{\pm}\cdot p = \varepsilon_p^{\pm}\cdot q = 0 \\ \varepsilon^\mu_{p+}\varepsilon^\nu_{p-} + \varepsilon^\mu_{p-}\varepsilon^\nu_{p+} = -\eta^{\mu\nu} + \dfrac{p^\mu q^\nu + q^\mu p^\nu}{p\cdot q} \\ \varepsilon_p^{h_1}\cdot\varepsilon_p^{h_2} = -\delta^{h_1(-h_2)} \end{cases}$$

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$$u^a_p = \begin{pmatrix} |p^a\rangle \\ |p^a] \end{pmatrix} \qquad \bar{u}^a_p = \begin{pmatrix} -\langle p^a| \\ [p^a| \end{pmatrix} \quad \Rightarrow \quad \begin{cases} (\not{p} - m)u^a_p = \bar{u}^a_p(\not{p} - m) = 0 \\ \bar{u}^a_p u^b_p = 2m\epsilon^{ab} \\ \bar{u}^a_p\gamma^\mu u^b_p = 2p^\mu\epsilon^{ab} \\ u^a_p\bar{u}_{pa} = u^a_p\epsilon_{ab}\bar{u}^b_p = \not{p} + m \end{cases}$$

$$v^a_p = \begin{pmatrix} -|p^a\rangle \\ |p^a] \end{pmatrix} \qquad \bar{v}^a_p = \begin{pmatrix} \langle p^a| \\ [p^a| \end{pmatrix} \quad \Rightarrow \quad \begin{cases} (\not{p} + m)v^a_p = \bar{v}^a_p(\not{p} + m) = 0 \\ \bar{v}^a_p v^b_p = 2m\epsilon^{ab} \\ \bar{v}^a_p\gamma^\mu v^b_p = -2p^\mu\epsilon^{ab} \\ v^a_p\bar{v}_{pa} = v^a_p\epsilon_{ab}\bar{v}^b_p = -\not{p} + m \end{cases}$$

Little group transformations

Consider Lorentz transformation $p^\mu \to L^\mu{}_\nu p^\nu$

MASSLESS:

$$|p\rangle \to S|p\rangle = e^{i\phi/2}|Lp\rangle \qquad \langle p| \to \langle p|S^{-1} = e^{i\phi/2}\langle Lp|$$

$$|p] \to S^{\dagger\,-1}|p] = e^{-i\phi/2}|Lp] \qquad [p| \to [p|S^\dagger = e^{-i\phi/2}[Lp|$$

$\Rightarrow\ \varepsilon_p^\pm \to L\varepsilon_p^\pm \sim e^{\mp i\phi}\varepsilon_{Lp}^\pm$

$e^{ih\phi} \in \mathrm{U}(1)$ encode $2d$ rotations in frame where $p = (E, 0, 0, E)$

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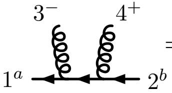
MASSIVE:

$$|p^a\rangle \to S|p^a\rangle = \omega^a{}_b|Lp^b\rangle \qquad |p^a\rangle \to |p^a\rangle S^{-1} = \omega^a{}_b|Lp^a\rangle$$
$$|p^a] \to S^{\dagger\,-1}|p^a] = \omega^a{}_b[Lp^b| \qquad [p^a| \to [p^a|S^\dagger = \omega^a{}_b[Lp^b|$$

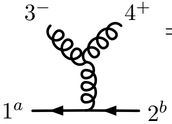
$\omega \in \mathrm{SU}(2)$ encode $3d$ rotations in rest frame where $p = (m, 0, 0, 0)$

4-pt Compton amplitude

Feynman-rules calculation of $A(\underline{1}^a, 3^-, 4^+, \overline{2}^b)$

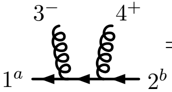


$$= -\frac{i}{2(s_{13}-m^2)}\,(\bar{u}_1^a \not{\varepsilon}_3^- (\not{p}_{13}+m)\not{\varepsilon}_4^+ v_2^b)$$

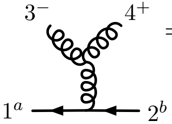


$$= \frac{i}{2s_{34}}\Big\{(\varepsilon_3^-\cdot\varepsilon_4^+)(\bar{u}_1^a(\not{p}_3-\not{p}_4)v_2^b) + 2(p_4\cdot\varepsilon_3^-)(\bar{u}_1^a\not{\varepsilon}_4^+ v_2^b) - 2(p_3\cdot\varepsilon_4^+)(\bar{u}_1^a\not{\varepsilon}_3^- v_2^b)\Big\}$$

Feynman-rules calculation of $A(\underline{1}^a, 3^-, 4^+, \overline{2}^b)$

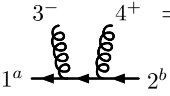


$$= -\frac{i}{2(s_{13}-m^2)}\left(\bar{u}_1^a \not\varepsilon_3^- (\not p_{13}+m) \not\varepsilon_4^+ v_2^b\right)$$

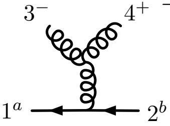


$$= \frac{i}{2s_{34}}\Big\{(\varepsilon_3^-\cdot\varepsilon_4^+)(\bar{u}_1^a(\not p_3-\not p_4)v_2^b) + 2(p_4\cdot\varepsilon_3^-)(\bar{u}_1^a \not\varepsilon_4^+ v_2^b) - 2(p_3\cdot\varepsilon_4^+)(\bar{u}_1^a \not\varepsilon_3^- v_2^b)\Big\}$$

- plug in external wavefunctions:

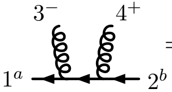


$$= \frac{-i}{(s_{13}-m^2)[3\,q_3]\langle 4\,q_4\rangle}\Big\{\langle 1^a 3\rangle[q_3|p_{13}|q_4\rangle[4\,2^b] + [1^a q_3]\langle 3|p_{13}|4]\langle q_4 2^b\rangle - m\langle 1^a 3\rangle[q_3 4]\langle q_4 2^b\rangle - m[1^a q_3]\langle 3\,q_4\rangle[4\,2^b]\Big\}$$

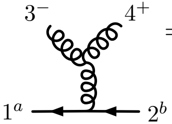


$$= \frac{-i}{s_{34}[3\,q_3]\langle 4\,q_4\rangle}\Big\{-\frac{1}{2}\langle 3\,q_4\rangle[4\,q_3]\big(\langle 1^a|p_3-p_4|2^b] + [1^a|p_3-p_4|2^b\rangle\big) - \langle 3|4|q_3]\big(\langle 1^a q_4\rangle[4\,2^b] + [1^a 4]\langle q_4 2^b\rangle\big) + \langle q_4|3|4]\big(\langle 1^a 3\rangle[q_3 2^b] + [1^a q_3]\langle 3\,2^b\rangle\big)\Big\}$$

Feynman-rules calculation of $A(\underline{1}^a, 3^-, 4^+, \overline{2}^b)$

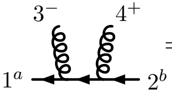


$$= -\frac{i}{2(s_{13}-m^2)}\left(\bar{u}_1^a \not{\varepsilon}_3^- (\not{p}_{13}+m) \not{\varepsilon}_4^+ v_2^b\right)$$

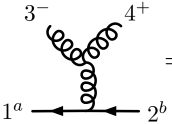


$$= \frac{i}{2s_{34}}\Big\{(\varepsilon_3^-\cdot\varepsilon_4^+)(\bar{u}_1^a(\not{p}_3-\not{p}_4)v_2^b) + 2(p_4\cdot\varepsilon_3^-)(\bar{u}_1^a \not{\varepsilon}_4^+ v_2^b) - 2(p_3\cdot\varepsilon_4^+)(\bar{u}_1^a \not{\varepsilon}_3^- v_2^b)\Big\}$$

- plug in external wavefunctions with $q_3 = p_4$, $q_4 = p_3$:



$$= \frac{i\langle 3|1|4]}{(s_{13}-m^2)s_{34}}\left(\langle 1^a 3\rangle[2^b 4] + [1^a 4]\langle 2^b 3\rangle\right)$$

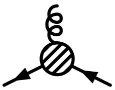


$$= 0$$

- spinor helicity helps no matter method

3-pt amplitudes

Modern methods require on-shell 3-pt input only

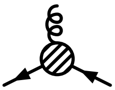


$$\mathcal{A}(1_i^a, 2_{\bar{\jmath}}^b, 3_c^+) = -\frac{iT^c_{i\bar{\jmath}}}{\langle 3\, q\rangle}\big(\langle 1^a q\rangle[2^b 3] + [1^a 3]\langle 2^b q\rangle\big) = -iT^c_{i\bar{\jmath}}\frac{\langle 1^a 2^b\rangle[3|1|q\rangle}{m\langle 3\, q\rangle}$$

$$\mathcal{A}(1_i^a, 2_{\bar{\jmath}}^b, 3_c^-) = \frac{iT^c_{i\bar{\jmath}}}{[3\, q]}\big(\langle 1^a 3\rangle[2^b q] + [1^a q]\langle 2^b 3\rangle\big) = iT^c_{i\bar{\jmath}}\frac{[1^a 2^b]\langle 3|1|q]}{m[3\, q]}$$

3-pt amplitudes

Modern methods require on-shell 3-pt input only



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$$\mathcal{A}(1_i^a,2_{\bar{\jmath}}^b,3_c^-) = \frac{iT^c_{i\bar{\jmath}}}{[3\,q]}\big(\langle 1^a 3\rangle[2^b q] + [1^a q]\langle 2^b 3\rangle\big) = iT^c_{i\bar{\jmath}}\frac{[1^a 2^b]\langle 3|1|q]}{m[3\,q]}$$

NB! Independent of ref. momentum q

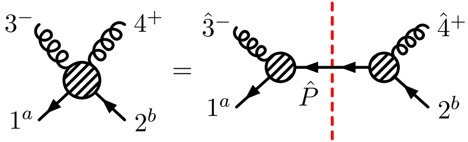
$$p_2^2 - m^2 = \langle 3|1|3] = 0 \qquad \Rightarrow \qquad \exists\, x_3 \in \mathbb{C} : |1|3] = -m x_3|3\rangle$$

$$\Rightarrow \qquad x_3 = \frac{[3|1|q\rangle}{m\langle 3\,q\rangle} \quad \text{indep. of } q$$

BCFW calculation of $A(\underline{1}^a, 3^-, 4^+, \overline{2}^b)$

BCFW shift: $\begin{cases} |3] \to |\hat{3}] = |3] - z|4] \\ |4\rangle \to |\hat{4}\rangle = |4\rangle + z|3\rangle \end{cases}$

Britto, Cachazo, Feng, Witten (2005)



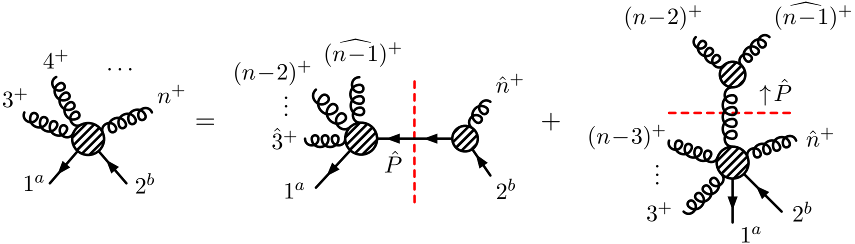
$$= \underset{z=z_{13}}{\text{Res}}\, A(\underline{1}^a, \hat{3}^-, \hat{4}^+, \overline{2}^b) = A(\underline{1}^a, \hat{3}^-, -\hat{P}^c)\frac{i}{s_{13} - m^2} A(\hat{P}_c, \hat{4}^+, \overline{2}^b)$$

$$= \frac{-i}{(s_{13} - m^2)[3\,4]\langle 4\,3\rangle} \big(\langle 1^a 3\rangle [4\hat{P}^c] - [1^a 4]\langle 3\hat{P}^c\rangle\big)\big(\langle \hat{P}_c\, 3\rangle [2^b 4] + [\hat{P}_c\, 4]\langle 2^b 3\rangle\big)$$

$$= \frac{i\langle 3|1|4]}{(s_{13} - m^2)s_{34}} \big(\langle 1^a 3\rangle [2^b 4] + [1^a 4]\langle 2^b 3\rangle\big)$$

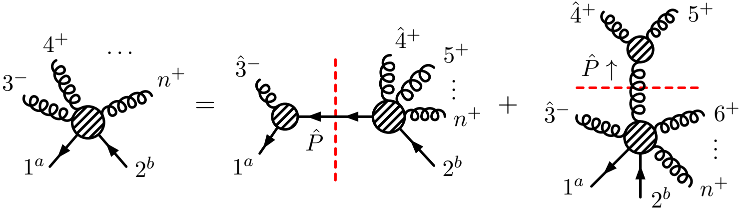
n-pt amplitudes

BCFW recursion for $A(\underline{1}^a, 3^+, \ldots, n^+, \overline{2}^b)$



$$= \ldots = \frac{i\, m \langle 1^a 2^b \rangle [3| \prod_{j=3}^{n-2} \left\{ \not{p}_{13\ldots j}\, \not{p}_{j+1} + (s_{13\ldots j} - m^2) \right\} |n]}{(s_{13} - m^2)(s_{134} - m^2) \ldots (s_{13\ldots(n-1)} - m^2)\, \langle 34 \rangle \langle 45 \rangle \ldots \langle n-1|n \rangle}$$

BCFW recursion for $A(\underline{1}^a, 3^-, 4^+, \ldots, n^+, \overline{2}^b)$



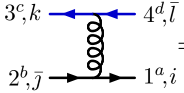
$$= \ldots = -\frac{i\langle 3|1|2|3\rangle\left(\langle 1^a 3\rangle[2^b|1+2|3\rangle + \langle 2^b 3\rangle[1^a|1+2|3\rangle\right)}{s_{12}\langle 3\,4\rangle\ldots\langle n-1|n\rangle\langle 3|1|1+2|n\rangle}$$

$$+\sum_{k=4}^{n-1}\frac{i\,m\langle 3|\not{p}_1\,\not{p}_{3\ldots k}|3\rangle\left(\langle 1^a 2^b\rangle\langle 3|\not{p}_1\,\not{p}_{3\ldots k}|3\rangle + \langle 1^a 3\rangle\langle 2^b 3\rangle s_{3\ldots k}\right)}{s_{3\ldots k}\,(s_{13\ldots k}-m^2)\ldots(s_{13\ldots(n-1)}-m^2)\,\langle 3\,4\rangle\ldots\langle k-1|k\rangle\langle 3|\not{p}_1\,\not{p}_{3\ldots k}|k\rangle}$$

$$\times\frac{\langle 3|\not{p}_{3\ldots k}\prod_{j=k}^{n-2}\{\not{p}_{13\ldots j}\,\not{p}_{j+1} + (s_{13\ldots j}-m^2)\}|n]}{\langle 3|\not{p}_1\,\not{p}_{3\ldots k}|k+1\rangle\langle k+1|k+2\rangle\ldots\langle n-1|n\rangle}$$

Four-quark amplitudes

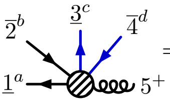
Lazopoulos, AO, Shi (in progress)



$$= -\frac{iT^a_{i\bar{j}}T^a_{k\bar{l}}}{s_{12}}\left(\langle 1^a 4^d\rangle[2^b 3^c] + [1^a 4^d]\langle 2^b 3^c\rangle + \langle 1^a 3^c\rangle[2^b 4^d] + [1^a 3^c]\langle 2^b 4^d\rangle\right)$$

Four-quark amplitudes with 1 gluon

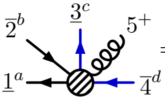
Lazopoulos, AO, Shi (in progress)



$$= i\Bigg\{ \frac{[1^a5]\langle 2^b3^c\rangle[4^d5] + [1^a5]\langle 2^b4^d\rangle[3^c5]}{(s_{15}-m_1^2)s_{34}} + \frac{[1^a5]\langle 2^b3^c\rangle[4^d5] + \langle 1^a3^c\rangle[2^b5][4^d5]}{s_{12}(s_{45}-m_3^2)}$$

$$+ \frac{\langle 1^a4^d\rangle[2^b5][3^c5] + [1^a5]\langle 2^b3^c\rangle[4^d5] + \langle 1^a3^c\rangle[2^b5][4^d5] + [1^a5]\langle 2^b4^d\rangle[3^c5]}{s_{12}s_{34}}$$

$$+ \frac{s_{12}[5|1|4|5] - (s_{15}-m_1^2)[5|3|4|5]}{s_{12}s_{34}(s_{15}-m_1^2)(s_{45}-m_3^2)}\left(\langle 1^a4^d\rangle[2^b3^c] + [1^a4^d]\langle 2^b3^c\rangle + \langle 1^a3^c\rangle[2^b4^d] + [1^a3^c]\langle 2^b4^d\rangle\right)\Bigg\}$$

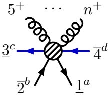


$$= i\Bigg\{ \frac{\langle 1^a4^d\rangle[2^b5][3^c5] + [1^a5]\langle 2^b4^d\rangle[3^c5]}{s_{12}(s_{35}-m_3^2)} - \frac{\langle 1^a3^c\rangle[2^b5][4^d5] + [1^a5]\langle 2^b3^c\rangle[4^d5]}{s_{12}(s_{45}-m_3^2)}$$

$$- \frac{[5|3|4|5]}{s_{12}(s_{35}-m_3^2)(s_{45}-m_3^2)}\left(\langle 1^a4^d\rangle[2^b3^c] + [1^a4^d]\langle 2^b3^c\rangle + \langle 1^a3^c\rangle[2^b4^d] + [1^a3^c]\langle 2^b4^d\rangle\right)\Bigg\}$$

Four-quark amplitudes with plus-helicity gluons

Lazopoulos, AO, Shi (in progress)



$$
= \frac{-im_3[a_n|n]}{D_n\big(\langle n{-}1|P_{3,n-1}|n]S_{4,n} + \langle n{-}1|P_{4,n}|n]S_{3,n-1}\big)}\bigg[\langle 34\rangle\langle 1|P_{4,n}|n][2n]
$$

$$
- \frac{[n|P_{3,n-1}P_{4,n}|n]}{S_{3,n}}\big(\langle 14\rangle[2|1{+}2|3\rangle + \langle 34\rangle\langle 1|P_{4,n}|2] + \langle 34\rangle\langle 1n\rangle[2n]\big)\bigg]
$$

$$
+ \frac{i}{D_6^n s_{12}S_{4,5}\langle 5|3|b_n^5]}\bigg[-\langle 14\rangle[23][b_n^5|p_3P_{4,5}|b_n^5] - m_3\langle 13\rangle[2|b_n^5]\langle 4|3|b_n^5] + \langle 14\rangle[2|b_n^5][3|b_n^5]S_{4,5}
$$

$$
+ \langle 13\rangle[b_n^5|p_3P_{4,5}|2]\bigg([4n]\prod_{j=5}^{n-1}S_{4,j} + m_3\sum_{i=5}^{n-1}\langle 4|p_i|b_n^{i+1}]\prod_{j=5}^{i-1}S_{4,j}\bigg)\bigg]
$$

$$
+ \sum_{i=1}^{n-6}\frac{-im_3[a_{n-i}|b_n^{n-i}]}{D_{n-i}D_{n-i+1}^n\big(S_{3,n-i-1}\langle n{-}i{-}1|P_{4,n-i}|b_n^{n-i}] + S_{4,n-i}\langle n{-}i{-}1|P_{3,n-i-1}|b_n^{n-i}]\big)}
$$

$$
\times\bigg[\frac{\langle 34\rangle\langle 1|P_{4,n-i}|b_n^{n-i}][2|b_n^{n-i}]}{\langle n{-}i|P_{4,n-i+1}|b_n^{n-i+1}]}
$$

$$
- \frac{[b_n^{n-i}|P_{3,n-i-1}P_{4,n-i}|b_n^{n-i}]}{S_{3,n-i}\langle n{-}i|P_{4,n-i+1}|b_n^{n-i+1}] + S_{4,n-i+1}\langle n{-}i|P_{3,n-i}|b_n^{n-i+1}]}
$$

$$
\times\bigg(\langle 14\rangle[2|1{+}2|3\rangle + \langle 34\rangle\langle 1|P_{4,n-i}|2] + \frac{\langle 34\rangle\langle 1|n{-}i\rangle[2|b_n^{n-i}]}{\langle n{-}i|P_{4,n-i+1}|b_n^{n-i+1}]}\bigg)\bigg]
$$

$$
+ (1 \leftrightarrow 2)
$$

$$
P_{3,i} = p_{356\ldots i}, \qquad S_{3,i} = P_{3,i}^2 - m_3^2, \qquad D_n = s_{12}S_{3,5}\prod_{i=6}^{n-1}S_{3,i}\langle i{-}1|i\rangle
$$

$$
P_{4,i} = p_{i(i+1)\ldots n4}, \qquad S_{4,i} = P_{4,i}^2 - m_3^2, \qquad D_k^l = \prod_{j=k}^{l}S_{4,j}\langle j{-}1|j\rangle
$$

$$
[a_n| = [5|\prod_{j=5}^{n-2}(\not P_{3,j}\not p_{j+1} + S_{3,j}), \qquad |b_k^l] = \prod_{i=l+1}^{k}(S_{4,i} + p_{i-1}P_{4,i})|k]
$$

Summary & outlook

- SU(2) covariance $\Leftrightarrow$ arbitrary spin projections
- Elegant form for two-quark amplitudes with
  - all gluons of same helicity (e.g. all plus)
  - one gluon of different helicity (e.g. one minus)

  AO (2018)
- Preliminary results for four-quark amplitudes

  Lazopoulos, AO, Shi (in progress)
- Applicable to any massive particles with spin

STAY TUNED!

Thank you!

Backup slides

Solution to BCJ relations

Bern, Carrasco, Johansson (2008)

Johansson, AO (2015)

BCJ relations:

$$A(\underline{1},\overline{2},\alpha,3,\beta) = \sum_{\sigma\in S(\alpha)\sqcup\beta} A(\underline{1},\overline{2},3,\sigma) \prod_{i=1}^{|\alpha|} \frac{\mathcal{F}(q,\sigma,1|i)}{s_{2,\alpha_1,\ldots,\alpha_i} - m_2^2}$$

Kleiss-Kuijf basis of $(n-2)!$ primitives $\left\{A(\underline{1},\overline{2},\sigma)\right\}$

$\Rightarrow$ BCJ basis of $(n-3)!$ primitives $\left\{A(\underline{1},\overline{2},3,\sigma)\right\}$

Solution to BCJ relations for QCD

Johansson, AO (2015)

General BCJ relations:

$$A(\underline{1},\overline{2},\alpha,\underline{q},\beta)=\sum_{\sigma\in S(\alpha)\sqcup\beta}A(\underline{1},\overline{2},\underline{q},\sigma)\prod_{i=1}^{|\alpha|}\frac{\mathcal{F}(q,\sigma,1|i)}{s_{2,\alpha_1,\ldots,\alpha_i}-m_2^2},$$

where α is purely gluonic

Melia basis of $(n-2)!/k!$ primitives

$$\left\{A(\underline{1},\overline{2},\sigma)\;\middle|\;\sigma\in\mathrm{Dyck}_{k-1}\times\{\text{gluon insertions}\}_{n-2k}\right\}$$

$\Rightarrow$ new BCJ basis of $(n-3)!(2k-2)/k!$ primitives

$$\left\{A(\underline{1},\overline{2},\underline{q},\sigma)\;\middle|\;\{\underline{q},\sigma\}\in\mathrm{Dyck}_{k-1}\times\{\text{gluon insertions in }\sigma\}_{n-2k}\right\}$$

Helicity basis

Arkani-Hamed, Huang, Huang (2017)

Take $p^\mu = (E, P\cos\varphi\sin\theta, P\sin\varphi\sin\theta, P\cos\theta)$

$$|p^a\rangle = \lambda_{p\alpha}^{\ a} = \begin{pmatrix} \sqrt{E-P}\cos\frac{\theta}{2} & -\sqrt{E+P}e^{-i\varphi}\sin\frac{\theta}{2} \\ \sqrt{E-P}e^{i\varphi}\sin\frac{\theta}{2} & \sqrt{E+P}\cos\frac{\theta}{2} \end{pmatrix}$$

$$[p^a| = \tilde{\lambda}_{p\dot{\alpha}}^{\ a} = \begin{pmatrix} -\sqrt{E+P}e^{i\varphi}\sin\frac{\theta}{2} & -\sqrt{E-P}\cos\frac{\theta}{2} \\ \sqrt{E+P}\cos\frac{\theta}{2} & -\sqrt{E-P}e^{-i\varphi}\sin\frac{\theta}{2} \end{pmatrix}$$

Then

$$s^\mu(u_p^a) = \frac{1}{2m}\bar{u}_{pa}\gamma^\mu\gamma^5 u_p^a = (-1)^{a-1}s_p^\mu$$

$$s_p^\mu = \frac{1}{m}(P, E\cos\varphi\sin\theta, E\sin\varphi\sin\theta, E\cos\theta)$$

Comparison with earlier results

Older reference-momentum-dep. spinors:

$$\bar{u}_p^{a=1} = \begin{pmatrix} -\langle p^1| \equiv \frac{m\langle q|}{\langle q\,p^\flat\rangle} \\ [p^1| \equiv [p^\flat| \end{pmatrix} = \bar{u}_p^-(q) \qquad v_p^{a=1} = \begin{pmatrix} -|p^1\rangle \equiv -\frac{m|q\rangle}{\langle p^\flat q\rangle} \\ |p^1] \equiv |p^\flat] \end{pmatrix} = v_p^-(q)$$

$$\bar{u}_p^{a=2} = \begin{pmatrix} -\langle p^2| \equiv -\langle p^\flat| \\ [p^2| \equiv -\frac{m[q|}{[q\,p^\flat]} \end{pmatrix} = -\bar{u}_p^+(q) \qquad v_p^{a=2} = \begin{pmatrix} -|p^2\rangle \equiv -|p^\flat\rangle \\ |p^2] \equiv \frac{m|q]}{[p^\flat q]} \end{pmatrix} = -v_p^+(q)$$

Kleiss, Stirling (1986), Dittmaier (1998), Schwinn, Weinzierl (2005)

$\Rightarrow$ Analytically retrieve older non-SU(2)-covariant formulae

Schwinn, Weinzierl (2007)

$$A(\underline{1}^1, 3^-, 4^+, \dots, n^+, \overline{2}^1) = 0$$

$$A(\underline{1}^1, 3^-, 4^+, \dots, n^+, \overline{2}^2) = \frac{-i\langle 2^\flat 3\rangle}{\langle 1^\flat 3\rangle\langle 34\rangle\dots\langle n{-}1|n\rangle} \sum_{k=4}^{n} \frac{\langle 3|\not{p}_1\,\not{p}_{3\dots k}|3\rangle^2}{s_{3\dots k}\langle 3|\not{p}_1\,\not{p}_{3\dots k}|k\rangle}$$
$$\times \left\{ \delta_{k=n} + \delta_{k\neq n} \frac{m^2\langle k|k{+}1\rangle\langle 3|\not{p}_{3\dots k} \prod_{j=k+1}^{n-1}\{(s_{13\dots j}-m^2) - \not{p}_j\,\not{p}_{13\dots j}\}|n]}{(s_{13\dots k}-m^2)\dots(s_{13\dots(n-1)}-m^2)\langle 3|\not{p}_1\,\not{p}_{3\dots k}|k{+}1\rangle} \right\}$$

$$A(\underline{1}^2, 3^-, 4^+, \dots, n^+, \overline{2}^1) = \frac{i\langle 1^\flat 3\rangle}{\langle 2^\flat 3\rangle\langle 34\rangle\dots\langle n{-}1|n\rangle} \sum_{k=4}^{n} \frac{\langle 3|\not{p}_1\,\not{p}_{3\dots k}|3\rangle^2}{s_{3\dots k}\langle 3|\not{p}_1\,\not{p}_{3\dots k}|k\rangle}$$
$$\times \left\{ \delta_{k=n} + \delta_{k\neq n} \frac{m^2\langle k|k{+}1\rangle\langle 3|\not{p}_{3\dots k} \prod_{j=k+1}^{n-1}\{(s_{13\dots j}-m^2) - \not{p}_j\,\not{p}_{13\dots j}\}|n]}{(s_{13\dots k}-m^2)\dots(s_{13\dots(n-1)}-m^2)\langle 3|\not{p}_1\,\not{p}_{3\dots k}|k{+}1\rangle} \right\}$$

$$A(\underline{1}^2, 3^-, 4^+, \dots, n^+, \overline{2}^2) = \frac{i\langle 1^\flat 2^\flat\rangle}{m\langle 34\rangle\dots\langle n{-}1|n\rangle} \sum_{k=4}^{n} \frac{\langle 3|\not{p}_1\,\not{p}_{3\dots k}|3\rangle^2}{s_{3\dots k}\langle 3|\not{p}_1\,\not{p}_{3\dots k}|k\rangle} \left[1 + \frac{s_{3\dots k}\langle 3\,2^\flat\rangle}{\langle 3|\not{p}_{3\dots k}\,\not{p}_1^\flat|2^\flat\rangle}\right]$$
$$\times \left\{ \delta_{k=n} + \delta_{k\neq n} \frac{m^2\langle k|k{+}1\rangle\langle 3|\not{p}_{3\dots k} \prod_{j=k+1}^{n-1}\{(s_{13\dots j}-m^2) - \not{p}_j\,\not{p}_{13\dots j}\}|n]}{(s_{13\dots k}-m^2)\dots(s_{13\dots(n-1)}-m^2)\langle 3|\not{p}_1\,\not{p}_{3\dots k}|k{+}1\rangle} \right\}$$