Functional reduction of Feynman integrals

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July 24th, CALC-2018, Dubna

Contents

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Method for deriving FE

Three methods for deriving functional equations (FE) were proposed

- FE from recurrence relations
- FE from algebraic relations for propagators
- FE from algebraic relations for deformed propagators

The simplest is the method based on algebraic relations for propagators.

To some extent such FE for Feynman integrals resemble Abel's addition theorem for algebraic integrals:
Algebraic relation for an integrand leads to some relationships between integrals considered as functions of some variables

Algebraic relation for propagators

The following algebraic relation between the products of n propagators was discovered:

$$\boxed{\prod_{r=1}^{n} \frac{1}{P_r} = \frac{1}{P_0} \sum_{r=1}^{n} x_r \prod_{\substack{j=1 \\ j \neq r}}^{n} \frac{1}{P_j},}$$

where

$$P_j = (k_1 - p_j)^2 - m_j^2 + i\eta.$$

and x_j, m_0, p_0 satisfy the following system of equations

$$\sum_{r=1}^{n} x_r = 1, \qquad p_0 = \sum_{j=1}^{n} x_j p_j.$$

$$m_0^2 - \sum_{k=1}^{n} x_k m_k^2 + \sum_{j=1}^{n} \sum_{l=1}^{j-1} x_j \, x_l s_{lj} = 0.$$

Considering k_1 as integration momentum and p_j as external ones, after integration w.r.t. k_1 we will get functional equation.

The last equation leads to a quadratic equation w.r.t. one of the variables x_j, i.e. square roots of Gram determinants can appear.

Solutions of this system of equations will depend on $n-2$ arbitrary parameters x_i and one arbitrary mass m_0.

Functional reduction of integrals

Is it possible to use this arbitrariness and to express complicated integral in terms of simpler ones?

In O.V.T., Phys.Lett. B670 (2008) 67-72 exploiting FE the one-loop vertex integral depending on six variables

$$I_3^{(d)}(m_1^2, m_2^2, m_3^2;\; s_{23}, s_{13}, s_{12}),$$

was represented as a combination of integrals depending only on three variables:

$$I_3^{(d)}(m^2, 0, 0;\; s_{23}, 0, s_{12}),$$
$$I_3^{(d)}(0, 0, 0;\; s_{23}, s_{13}, s_{12}).$$

The goal of my talk : to present a systematic, computer oriented method for obtaining FE, reducing complicated integrals to simpler ones.

Simpler integrals mean integrals with lesser number of different nonzero kinematical invariants or some invariants being equal to each other.

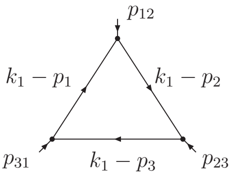
Vertex type integral

Let's consider integral with 3 massless propagators:

$$I_3^{(d)}(s_{23}, s_{13}, s_{12}) = \int \frac{d^d k_1}{i\pi^{d/2}} \frac{1}{P_1 P_2 P_3},$$

where

$$P_i = (k_1 - p_i)^2 + i\eta, \qquad s_{ij} = p_{ij}^2, \qquad p_{ij} = p_i - p_j.$$



Vertex type integral

In order to obtain FE for this integral we will use relationship for 3 propagators

$$\frac{1}{P_1P_2P_3} = \frac{x_1}{P_0P_2P_3} + \frac{x_2}{P_1P_0P_3} + \frac{x_3}{P_1P_2P_0},$$

where

$$P_j = (k_1 - p_j)^2 - m_j^2 + i\eta, \qquad p_0 = x_1p_1 + x_2p_2 + x_3p_3,$$

and parameters x_j, m_0 obey the following conditions:

$$x_1 + x_2 + x_3 = 1,$$
$$x_1x_2s_{12} + x_1x_3s_{13} + x_2x_3s_{23} - x_1m_1^2 - x_2m_2^2 - x_3m_3^2 + m_0^2 = 0.$$

Integration of this relationship with respect to k_1 yields FE:

$$\begin{aligned} I_3^{(d)}&(m_1^2, m_2^2, m_3^2;\; s_{23}, s_{13}, s_{12}) \\ &= x_1\; I_3^{(d)}(m_0^2, m_2^2, m_3^2;\; s_{23}, s_{30}, s_{20}) \\ &+ x_2\; I_3^{(d)}(m_1^2, m_0^2, m_3^2;\; s_{30}, s_{13}, s_{10}) \\ &+ x_3\; I_3^{(d)}(m_1^2, m_2^2, m_0^2;\; s_{20}, s_{10}, s_{12}). \end{aligned}$$

Vertex type integral

If $m_1 = m_2 = m_3 = 0$ then

$$s_{10} = m_0^2 + s_{13} - s_{13}x_1 + (s_{12} - s_{13})x_2,$$
$$s_{20} = m_0^2 + s_{23} + (s_{12} - s_{23})x_1 - x_2 s_{23},$$
$$s_{30} = m_0^2 + s_{13}x_1 + s_{23}x_2.$$

To find FE expressing initial integral in terms of simpler integrals we considered 1330 systems of equations, each consisting of 3 equations composed out of 21 equations

$$s_{10} = 0, \quad s_{10} - s_{20} = 0, \quad s_{10} - s_{30} = 0, \quad s_{20} = 0, \quad s_{20} - s_{30} = 0, \quad s_{30} = 0,$$
$$s_{10} - s_{12} = 0, \quad s_{20} - s_{12} = 0, \quad s_{30} - s_{12} = 0, \quad s_{10} - s_{23} = 0, \quad s_{20} - s_{23} = 0,$$
$$s_{30} - s_{23} = 0, \quad s_{10} - s_{13} = 0, \quad s_{20} - s_{13} = 0, \quad s_{30} - s_{13} = 0, \quad s_{10} - m_0^2 = 0,$$
$$s_{20} - m_0^2 = 0, \quad s_{30} - m_0^2 = 0, \quad s_{10} + m_0^2 = 0, \quad s_{20} + m_0^2 = 0, \quad s_{30} + m_0^2 = 0.$$

In 35 sec of CPU time, 7 solutions without square roots of Gram determinants were discovered.

Vertex type integral

The most compact and simple equation:

$$\boxed{\begin{aligned} & I_3^{(d)}(0,0,0;\ s_{23},s_{13},s_{12}) \\ & \qquad = \phi_{231}\ \xi_3^{(d)}(\mu_{123},s_{23}) + \phi_{132}\ \xi_3^{(d)}(\mu_{123},s_{13}) + \phi_{123}\ \xi_3^{(d)}(\mu_{123},s_{12}), \end{aligned}}$$

where

$$\xi_3^{(d)}(\mu_{123},s_{ij}) = I_3^{(d)}(0,0,\mu_{123};\ -\mu_{123},-\mu_{123},s_{ij}),$$

$$\phi_{ijk} = -\frac{\partial}{\partial m_i^2}\ \frac{\Delta_3(\{M\},\{S\})}{G_2(\{S\})}\bigg|_{m_i=m_j=m_k=0} = \frac{s_{ij}-s_{ik}-s_{jk}}{s_{ik}s_{jk}}\ \mu_{ijk},$$

$$\mu_{ijk} = \frac{\Delta_3(\{M\},\{S\})}{G_2(\{S\})}\bigg|_{m_i=m_j=m_k=0} = \frac{s_{ij}s_{ik}s_{jk}}{s_{ij}^2+s_{ik}^2+s_{jk}^2-2s_{ij}s_{ik}-2s_{ij}s_{jk}-2s_{ik}s_{jk}},$$

$$\{S\} = \{s_{ij},s_{jk},s_{ik}\}, \qquad \{M\} = \{m_i^2,m_j^2,m_k^2\}.$$

Vertex-type integral

where

$$\Delta_3(\{M\},\{S\}) = \begin{vmatrix} Y_{ii} & Y_{ij} & Y_{ik} \\ Y_{ij} & Y_{jj} & Y_{jk} \\ Y_{ik} & Y_{jk} & Y_{kk} \end{vmatrix}, \qquad Y_{nl} = -s_{nl} + m_n^2 + m_l^2,$$

$$G_2(\{S\}) = -2 \begin{vmatrix} S_{ii} & S_{ij} \\ S_{ij} & S_{jj} \end{vmatrix}, \qquad S_{nl} = s_{nk} + s_{lk} - s_{nl},$$

Vertex type integral

Integral $\xi_3^{(d)}(m^2, q^2)$ can be evaluated as a solution of simple dimensional recurrence relation:

$$(d-2)\ \xi_3^{(d+2)}(m^2, q^2) = -2\widetilde{m}^2\ \xi_3^{(d)}(m^2, q^2) - \xi_2^{(d)}(q^2),$$

where

$$\xi_2^{(d)}(q^2) = I_2^{(d)}(0, 0, q^2) = -\frac{\pi^{3/2}(-\widetilde{q}^2)^{\frac{d}{2}-2}}{2^{d-3}\Gamma\left(\frac{d-1}{2}\right)\sin\frac{\pi d}{2}},$$

and

$$\widetilde{q}^2 = q^2 + 4i\eta,$$
$$\widetilde{m}^2 = m^2 - i\eta.$$

Vertex type integral

Solution of dimensional recurrence relation is

$$\xi_3^{(d)}(m^2,q^2) = -\frac{1}{2m^2}\xi_2^{(d)}(q^2)\,{}_2F_1\left[\begin{array}{c} 1,\frac{d-2}{2};\\ \frac{d-1}{2}; \end{array} \frac{-\widetilde{q}^{\,2}}{4\widetilde{m}^2}\right] + \frac{(-\widetilde{m}^2)^{d/2}}{\Gamma\left(\frac{d-2}{2}\right)} C_3(q^2,d),$$

where $C_3(q^2,d)$ is a periodic function $C_3(q^2,d)=C_3(q^2,d+2)$. To find it we use differential equation for $\xi_3^{(d)}(m^2,q^2)$:

$$\frac{\partial}{\partial q^2}\xi_3^{(d)}(m^2,q^2) = \frac{-(q^2+2m^2)}{q^2(q^2+4m^2)}\xi_3^{(d)}(m^2,q^2)$$
$$-\frac{(d-3)}{\widetilde{q}^{\,2}(q^2+4m^2)}\xi_2^{(d)}(q^2) + \frac{d-2}{2\widetilde{m}^2 q^2(q^2+4m^2)}\xi_1^{(d)}(m^2).$$

From this equation it follows that

$$q^2\frac{\partial C_3(q^2,d)}{\partial q^2} + \frac{(q^2+2m^2)}{q^2+4m^2}C_3(q^2,d) + \frac{\Gamma\left(\frac{d}{2}\right)}{(-\widetilde{m}^2)^{d/2+1}(q^2+4m^2)}\xi_1^{(d)}(m^2) = 0.$$

where

$$\xi_1(m^2) = -\frac{\pi(\widetilde{m}^2)^{d/2-1}}{\Gamma\left(\frac{d}{2}\right)\sin\frac{\pi d}{2}}.$$

Vertex type integral

Solution of this equation:

$$C_3(q^2,d) = \frac{-\Gamma\left(\frac{d}{2}\right)\xi_1^{(d)}(m^2)}{\sqrt{q^2(q^2+4m^2)}(-\widetilde{m}^2)^{d/2+1}} \ln\left(2m^2+q^2+\sqrt{(q^2+4m^2)q^2}\right) + \frac{K_3}{\sqrt{q^2(q^2+4m^2)}},$$

depends on an arbitrary constant K_3. $\xi_3^{(d)}(m^2,q^2)$ is finite at $q^2 \to 0$, therefore

$$K_3 = \frac{\Gamma\left(\frac{d}{2}\right)\xi_1^{(d)}(m^2)}{(-\widetilde{m}^2)^{d/2+1}} \ln(2m^2)$$

Finally

$$\boxed{\begin{aligned}\xi_3^{(d)}(m^2,q^2) &= -\frac{1}{2m^2}\xi_2^{(d)}(q^2)\ {}_2F_1\left[\begin{matrix}1, \frac{d-2}{2}; \\ \frac{d-1}{2};\end{matrix}\ \frac{-\widetilde{q}^{\,2}}{4\widetilde{m}^2}\right] \\ &+ \frac{(d-2)\xi_1^{(d)}(m^2)}{2m^2\sqrt{q^2(q^2+4m^2)}} \ln\left(1+\frac{q^2+\sqrt{q^2(q^2+4m^2)}}{2m^2}\right).\end{aligned}}$$

Vertex type integral

Some Remarks:

To obtain result for $I_3^{(d)}(0,0,0;\ s_{23},s_{13},s_{12})$ in the whole kinematical domain we must know simpler integral $\xi_3^{(d)}(m^2,q^2)$ for arbitrary q^2 and m^2.

Our result for $I_3^{(d)}(0,0,0;\ s_{23},s_{13},s_{12})$ was compared numerically with the results obtained with the program SecDec version 3.0 by S. Borowka, G. Heinrich, S. Jones, M. Kerner, J. Schlenk, T. Zirke

Perfect agreement was found in Euclidean as well as in Minkowski and mixed regions of kinematical variables.

For some kinematical regions logarithmic terms cancel each other but in some regions they give contribution.

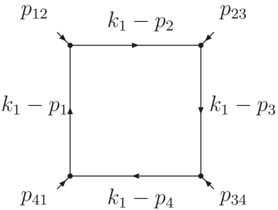
Box type integral

Let's consider integral with 4 propagators:

$$I_4^{(d)}(m_1^2, m_2^2, m_3^2, m_4^2;\ s_{12}, s_{23}, s_{34}, s_{14}, s_{24}, s_{13}) = \int \frac{d^d k_1}{i\pi^{d/2}} \frac{1}{P_1 P_2 P_3 P_4}.$$

where

$$P_i = (k_1 - p_i)^2 - m_i^2 + i\eta, \qquad s_{ij} = p_{ij}^2, \qquad p_{ij} = p_i - p_j.$$



Box type integral

To obtain FE we use relation for product of 4 propagators:

$$\frac{1}{P_1P_2P_3P_4} = \frac{x_1}{P_0P_2P_3P_4} + \frac{x_2}{P_1P_0P_3P_4} + \frac{x_3}{P_1P_2P_0P_4} + \frac{x_4}{P_1P_2P_3P_0},$$

where

$$p_0 = x_1p_1 + x_2p_2 + x_3p_3 + x_4p_4,$$

and parameters x_j, m_0 obey the following conditions:

$$x_1 + x_2 + x_3 + x_4 = 1,$$
$$x_1x_2s_{12} + x_1x_3s_{13} + x_1x_4s_{14} + x_2x_3s_{23} + x_2x_4s_{24} + x_3x_4s_{34}$$
$$-x_1m_1^2 - x_2m_2^2 - x_3m_3^2 - x_4m_4^2 + m_0^2 = 0.$$

Integration of this relationship with respect to k_1 yields FE:

$$\begin{aligned}
I_4^{(d)}(m_1^2, m_2^2, m_3^2, m_4^2;\ s_{12}, s_{23}, s_{34}, s_{14}, s_{24}, s_{13})& \\
= x_1\ I_4^{(d)}(m_0^2, m_2^2, m_3^2, m_4^2;\ s_{20}, s_{23}, s_{34}, s_{40}, s_{24}, s_{30})& \\
+ x_2\ I_4^{(d)}(m_1^2, m_0^2, m_3^2, m_4^2;\ s_{10}, s_{30}, s_{34}, s_{14}, s_{40}, s_{13})& \\
+ x_3\ I_4^{(d)}(m_1^2, m_2^2, m_0^2, m_4^2;\ s_{12}, s_{20}, s_{40}, s_{14}, s_{24}, s_{10})& \\
+ x_4\ I_4^{(d)}(m_1^2, m_2^2, m_3^2, m_0^2;\ s_{12}, s_{23}, s_{30}, s_{10}, s_{20}, s_{13})&.
\end{aligned}$$

Box type integral

When $m_1 = m_2 = m_3 = m_4 = 0$

$$
\begin{aligned}
s_{10} &= s_{14} + m_0^2 - s_{14}x_1 + (s_{12} - s_{14})x_2 + (s_{13} - s_{14})x_3, \\
s_{20} &= s_{24} + m_0^2 + (s_{12} - s_{24})x_1 - x_2 s_{24} + (s_{23} - s_{24})x_3, \\
s_{30} &= s_{34} + m_0^2 + (s_{13} - s_{34})x_1 + (s_{23} - s_{34})x_2 - s_{34}x_3, \\
s_{40} &= m_0^2 + s_{14}x_1 + s_{24}x_2 + s_{34}x_3,
\end{aligned}
$$

and in this case

$$
\begin{aligned}
& I_4^{(d)}(0,0,0,0; s_{12}, s_{23}, s_{34}, s_{14}, s_{24}, s_{13}) \\
= \;\; & x_1 \; I_4^{(d)}(m_0^2, 0, 0, 0; s_{20}, s_{23}, s_{34}, s_{40}, s_{24}, s_{30}) \\
& + x_2 \; I_4^{(d)}(0, m_0^2, 0, 0; s_{10}, s_{30}, s_{34}, s_{14}, s_{40}, s_{13}) \\
& + x_3 \; I_4^{(d)}(0, 0, m_0^2, 0; s_{12}, s_{20}, s_{40}, s_{14}, s_{24}, s_{10}) \\
& + x_4 \; I_4^{(d)}(0, 0, 0, m_0^2; s_{12}, s_{23}, s_{30}, s_{10}, s_{20}, s_{13}).
\end{aligned}
$$

Box-type integral

We combined out of 42 equations

$$s_{i0}=0,\quad s_{i0}-s_{12}=0,\quad s_{i0}-s_{23}=0,\quad s_{i0}-s_{34}=0,\quad s_{i0}-s_{14}=0,$$
$$s_{i0}-s_{24}=0,\quad s_{i0}-s_{13}=0,\quad s_{i0}\pm m_0^2=0,\quad s_{10}-s_{20}=0,\quad s_{10}-s_{30}=0,$$
$$s_{10}-s_{40}=0,\quad s_{20}-s_{30}=0,\quad s_{20}-s_{40}=0,\quad s_{30}-s_{40}=0,$$

111930 systems, each consisting of 4 equations. Only 29 systems have nontrivial solutions without radicals. It took 2 hours 19 min CPU time to analyze 111930 systems of equations.

One of these solutions gives FE expressing our initial integral depending on 6 variables in terms of integrals depending on 4 variables:

$$\boxed{\begin{aligned} &I_4^{(d)}(0,0,0,0;s_{12},s_{23},s_{34},s_{14},s_{24},s_{13})\\ &\quad=\phi_{1234}\ B_4^{(d)}(\mu_4;s_{23},s_{34},s_{24})+\phi_{2341}\ B_4^{(d)}(\mu_4;s_{34},s_{14},s_{13})\\ &\quad+\phi_{3412}\ B_4^{(d)}(\mu_4;s_{12},s_{14},s_{24})+\phi_{4123}\ B_4^{(d)}(\mu_4;s_{12},s_{23},s_{13}),\end{aligned}}$$

where

$$B_4^{(d)}(\mu_4;s_{ij},s_{jk},s_{ik})=I_4^{(d)}(0,0,0,\mu_4;s_{ij},s_{jk},-\mu_4,-\mu_4,-\mu_4,s_{ik}),$$

Box-type integral

where

$$\mu_4 = \left. \frac{\Delta_4(\{M\},\{S\})}{G_3(\{S\})} \right|_{m_1=m_2=m_3=m_4=0}$$

$$\phi_{ijkr} = -\frac{\partial}{\partial m_i^2} \left. \frac{\Delta_4(\{M\},\{S\})}{G_3(\{S\})} \right|_{m_i=m_j=m_k=m_r=0}$$

$$\{S\} = \{s_{ij}, s_{jk}, s_{kr}, s_{ir}, s_{jr}, s_{ik}\},$$
$$\{M\} = \{m_i^2, m_j^2, m_k^2, m_r^2\}.$$

$$\Delta_4(\{M\},\{S\}) = \begin{vmatrix} Y_{ii} & Y_{ij} & Y_{ik} & Y_{ir} \\ Y_{ij} & Y_{jj} & Y_{jk} & Y_{jr} \\ Y_{ik} & Y_{jk} & Y_{kk} & Y_{kr} \\ Y_{ir} & Y_{jr} & Y_{kr} & Y_{rr} \end{vmatrix}, \qquad Y_{nl} = -s_{nl} + m_n^2 + m_l^2,$$

$$G_3(\{S\}) = -2 \begin{vmatrix} S_{ii} & S_{ij} & S_{ik} \\ S_{ij} & S_{jj} & S_{jk} \\ S_{ik} & S_{jk} & S_{kk} \end{vmatrix}, \qquad S_{nl} = s_{ns} + s_{ls} - s_{nl},$$

Box-type integral

Further reduction is possible. Second step of functional reduction gives:

$$\boxed{\begin{aligned} &B_4^{(d)}(\mu_4; s_{ij}, s_{jk}, s_{ik}) \\ &\qquad = \phi_{jki}\ \xi_4^{(d)}(\mu_{ijk}, \mu_4; s_{jk}) + \phi_{ikj}\ \xi_4^{(d)}(\mu_{ijk}, \mu_4; s_{ik}) + \phi_{ijk}\ \xi_4^{(d)}(\mu_{ijk}, \mu_4; s_{ik}), \end{aligned}}$$

where

$$\xi_4^{(d)}(\mu_{ijk}, \mu_4; s_{ij}) = I_4^{(d)}(0, 0, \mu_{ijk}, \mu_4; s_{ij}, -\mu_{ijk}, \mu_{ijk} - \mu_4, -\mu_4, -\mu_4, -\mu_{ijk}).$$

The function with 4 variables was expressed in terms of function with 3 variables.

One-loop box type integral with massless propagators depending on 6 variables is a combination of 12 integrals depending on 3 variables.

Box-type integral

Dimensional recurrence relation for $\xi_4^{(d)}$

$$(d-3)\xi_4^{(d+2)}(\mu_3,\mu_4,s_{ij}) = -2\widetilde{\mu}_4\xi_4^{(d)}(\mu_3,\mu_4,s_{ij}) - \xi_3^{(d)}(\mu_3,s_{ij}).$$

Solution of this relation is

$$\xi_4^{(d)}(\mu_3,\mu_4,s_{ij}) = \frac{1}{2}\sum_{r=0}^{\infty}\frac{\left(\frac{d-3}{2}\right)_r}{(-\widetilde{\mu}_4)^{r+1}}\xi_3^{(d+2r)}(\mu_3,s_{ij}) + \frac{(-\widetilde{\mu}_4)^{d/2}}{\Gamma\left(\frac{d-3}{2}\right)}C_4(s_{ij},d).$$

Periodic function $C_4(s_{ij},d+2) = C_4(s_{ij},d)$ can be found from differential equation

$$s_{ij}(s_{ij}+4\mu_3)\frac{\partial C_4(s_{ij},d)}{\partial s_{ij}} + (s_{ij}+2\mu_3)C_4(s_{ij},d) + K_4.$$

$$C_4(s_{ij},d) = -\frac{K_4}{R}\ln\left(2\mu_3+s_{ij}+R\right) + \frac{\kappa_4}{R}.$$

Since $\xi_4^{(d)}(\mu_3,\mu_4,s_{ij})$ is finite at $s_{ij}=0$, then $C_4(0,d)$ also must be finite. Taking the limit $s_{ij}\to 0$ we get

$$\kappa_4 = K_4\ln 2\mu_3.$$

Box-type integral

After simplifications the final result reads

$$\xi_4^{(d)}(\mu_3,\mu_4,s_{ij}) = -\frac{(d-2)}{2\widetilde{\mu}_3\widetilde{\mu}_4 R}\mathrm{Arth}\left(\frac{s_{ij}}{R}\right)\xi_1^{(d)}(\widetilde{\mu}_3)\,{}_2F_1\left[\begin{array}{cc} 1, \frac{d-3}{2}; & \frac{\widetilde{\mu}_3}{\widetilde{\mu}_4} \\ \frac{d-2}{2}; & \end{array}\right]$$
$$-\frac{\pi^{3/2}\,\widetilde{\mu}_4^{d/2-3}}{\Gamma\left(\frac{d-3}{2}\right)KR\,\sin\frac{\pi d}{2}}\left[\mathrm{Arth}\left(\frac{s_{ij}}{R}\right)-\mathrm{Arth}\left(\frac{s_{ij}}{R}K\right)\right]$$
$$+\frac{1}{2\widetilde{\mu}_3\widetilde{\mu}_4}\left(\frac{\widetilde{\mu}_3}{\widetilde{s}_{ij}+4\widetilde{\mu}_3}\right)^{\frac{1}{2}}\xi_2^{(d)}(s_{ij})F_1\left(\frac{d-3}{2},\frac{1}{2},1,\frac{d-1}{2};\frac{-\widetilde{s}_{ij}}{4\widetilde{\mu}_3},\frac{-\widetilde{s}_{ij}}{4\widetilde{\mu}_4}\right),$$

where

$$R = \sqrt{s_{ij}(s_{ij}+4\mu_3)}$$
$$K = \left(1-\frac{\widetilde{\mu}_3}{\widetilde{\mu}_4}\right)^{1/2}.$$

Numerical comparison of our result for $I_4^{(d)}(s_{12},s_{23},s_{34},s_{14},s_{24},s_{13})$ with the result obtained by using package SecDec reveals perfect agreement.

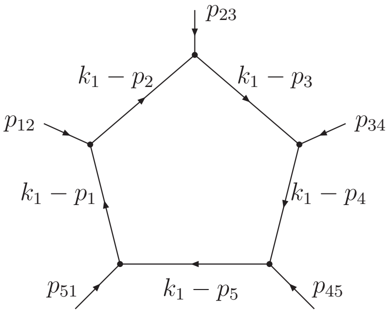
Definitions

Now let's consider pentagon type integral:

$$I_5^{(d)}(m_1^2, m_2^2, m_3^2, m_4^2, m_5^2; s_{12}, s_{23}, s_{34}, s_{45}, s_{15}, s_{13}, s_{14}, s_{24}, s_{25}, s_{35})$$
$$= \int \frac{d^d k_1}{i\pi^{d/2}} \frac{1}{P_1 P_2 P_3 P_4 P_5}.$$

where

$$P_i = (k_1 - p_i)^2 - m_i^2 + i\eta, \qquad s_{ij} = p_{ij}^2, \qquad p_{ij} = p_i - p_j.$$



Notations

To obtain FE we use relation for product of 5 propagators:

$$\frac{1}{P_1P_2P_3P_4P_5} = \frac{x_1}{P_0P_2P_3P_4P_5} + \frac{x_2}{P_1P_0P_3P_4P_5} + \frac{x_3}{P_1P_2P_0P_4P_5} + \frac{x_4}{P_1P_2P_3P_0P_5} + \frac{x_5}{P_1P_2P_3P_4P_0}$$

where $p_0,\ x_j,\ m_0$ obey the following conditions:

$$x_1+x_2+x_3+x_4+x_5=1, \qquad p_0 = x_1p_1+x_2p_2+x_3p_3+x_4p_4+x_5p_5,$$

$$\begin{aligned}&x_1x_2s_{12}+x_1x_3s_{13}+x_1x_4s_{14}+x_1x_5s_{15}\\ &+x_2x_3s_{23}+x_2x_4s_{24}+x_2x_5s_{25}+x_3x_4s_{34}+x_3x_5s_{35}+x_4x_5s_{45}\\ &-x_1m_1^2-x_2m_2^2-x_3m_3^2-x_4m_4^2-x_5m_5^2+m_0^2=0.\end{aligned}$$

Notations

Integration of this relationship with respect to k_1 yields FE:

$$
\begin{aligned}
& I_5^{(d)}(\{m_k\}; s_{12}, s_{23}, s_{34}, s_{45}, s_{15}; s_{13}, s_{14}, s_{24}, s_{25}, s_{35}) \\
& = x_1 I_5^{(d)}(\{m_k\}; s_{20}, s_{23}, s_{34}, s_{45}, s_{50}, s_{30}, s_{40}, s_{24}, s_{25}, s_{35}) \\
& + x_2 I_5^{(d)}(\{m_k\}; s_{10}, s_{30}, s_{34}, s_{45}, s_{15}; s_{13}, s_{14}, s_{40}, s_{50}, s_{35}) \\
& + x_3 I_5^{(d)}(\{m_k\}; s_{12}, s_{20}, s_{40}, s_{45}, s_{15}, s_{10}, s_{14}, s_{24}, s_{25}, s_{50}) \\
& + x_4 I_5^{(d)}(\{m_k\}; s_{12}, s_{23}, s_{30}, s_{50}, s_{15}; s_{13}, s_{10}, s_{20}, s_{25}, s_{35}) \\
& + x_5 I_5^{(d)}(\{m_k\}; s_{12}, s_{23}, s_{34}, s_{40}, s_{10}, s_{13}, s_{14}, s_{24}, s_{20}, s_{30}).
\end{aligned}
$$

If $m_1 = m_2 = m_3 = m_4 = m_5 = 0$ then

$$
\begin{aligned}
s_{10} &= m_0^2 + s_{15} - s_{15}x_1 + (s_{12} - s_{15})x_2 + (s_{13} - s_{15})x_3 + (s_{14} - s_{15})x_4, \\
s_{20} &= m_0^2 + s_{25} + (s_{12} - s_{25})x_1 - x_2 s_{25} + (s_{23} - s_{25})x_3 + (s_{24} - s_{25})x_4, \\
s_{30} &= m_0^2 + s_{35} + (s_{13} - s_{35})x_1 + (s_{23} - s_{35})x_2 - x_3 s_{35} + (s_{34} - s_{35})x_4, \\
s_{40} &= m_0^2 + s_{45} + (s_{14} - s_{45})x_1 + (s_{24} - s_{45})x_2 + (s_{34} - s_{45})x_3 - x_4 s_{45}, \\
s_{50} &= m_0^2 + s_{15}x_1 + s_{25}x_2 + s_{35}x_3 + s_{45}x_4.
\end{aligned}
$$

Pentagon type integral

We combined out of 20 equations:

$$
\begin{array}{lllll}
s_{10} = 0, & s_{10} - s_{20} = 0, & s_{10} - s_{30} = 0, & s_{10} - s_{40} = 0, & s_{10} - s_{50} = 0, \\
s_{10} + m_0^2 = 0, & s_{20} = 0, & s_{20} - s_{30} = 0, & s_{20} - s_{40} = 0, & s_{20} - s_{50} = 0, \\
s_{20} + m_0^2 = 0, & s_{30} = 0, & s_{30} - s_{40} = 0, & s_{30} - s_{50} = 0, & s_{30} + m_0^2 = 0, \\
s_{40} = 0, & s_{40} - s_{50} = 0, & s_{40} + m_0^2 = 0, & s_{50} = 0, & s_{50} + m_0^2 = 0.
\end{array}
$$

15504 systems, each consisting of 5 equations and found 36 solutions without square roots of Gram determinants.

Pentagon type integral

The most compact FE was found when $s_{10} = s_{20} = s_{30} = s_{40} = s_{50} = -m_0^2$:

$$\boxed{\begin{aligned} & I_5^{(d)}(s_{12}, s_{23}, s_{34}, s_{45}, s_{15}, s_{13}, s_{14}, s_{24}, s_{25}, s_{35}) \\ & \quad = \phi_{12345} F^{(d)}(\mu_5; s_{23}, s_{34}, s_{45}, s_{24}, s_{25}, s_{35}) \\ & \quad + \phi_{23451} F^{(d)}(\mu_5; s_{13}, s_{34}, s_{45}, s_{14}, s_{15}, s_{35}) \\ & \quad + \phi_{34512} F^{(d)}(\mu_5; s_{12}, s_{24}, s_{45}, s_{14}, s_{15}, s_{25}) \\ & \quad + \phi_{45123} F^{(d)}(\mu_5; s_{12}, s_{23}, s_{35}, s_{13}, s_{15}, s_{25}) \\ & \quad + \phi_{51234} F^{(d)}(\mu_5; s_{12}, s_{23}, s_{34}, s_{13}, s_{14}, s_{24}) \end{aligned}}$$

where

$$\begin{aligned} & F^{(d)}(\mu_5; s_{ij}, s_{jk}, s_{kr}, s_{ik}, s_{ir}, s_{jr}) \\ & \quad = I_5^{(d)}(-\mu_5, 0, 0, 0, 0; \mu_5, s_{ij}, s_{jk}, s_{kr}, \mu_5, \mu_5, \mu_5, s_{ik}, s_{ir}, s_{jr}), \end{aligned}$$

$$\mu_5 = \left. \frac{\Delta_5(\{M\}, \{S\})}{G_4(\{S\})} \right|_{m_i = m_j = m_k = m_r = m_s = 0},$$

Pentagon type integral

$$\phi_{ijkrs} = -\frac{\partial}{\partial m_i^2} \left. \frac{\Delta_5(\{M\},\{S\})}{G_4(\{S\})} \right|_{m_i=m_j=m_k=m_r=m_s=0}$$

where

$$\{S\} = \{s_{ij}, s_{jk}, s_{kr}, s_{rs}, s_{is}\},$$
$$\{M\} = \{m_i^2, m_j^2, m_k^2, m_r^2, m_s^2\}.$$

$$\Delta_5(\{M\},\{S\}) = \begin{vmatrix} Y_{ii} & Y_{ij} & Y_{ik} & Y_{ir} & Y_{is} \\ Y_{ij} & Y_{jj} & Y_{jk} & Y_{jr} & Y_{js} \\ Y_{ik} & Y_{jk} & Y_{kk} & Y_{kr} & Y_{ks} \\ Y_{ir} & Y_{jr} & Y_{kr} & Y_{rr} & Y_{rs} \\ Y_{is} & Y_{js} & Y_{ks} & Y_{rs} & Y_{ss} \end{vmatrix}, \qquad Y_{nl} = -s_{nl} + m_n^2 + m_l^2,$$

$$G_4(\{S\}) = -2 \begin{vmatrix} S_{ii} & S_{ij} & S_{ik} & S_{ir} \\ S_{ij} & S_{jj} & S_{jk} & S_{jr} \\ S_{ik} & S_{jk} & S_{kk} & S_{kr} \\ S_{ir} & S_{jr} & S_{kr} & S_{rr} \end{vmatrix}, \qquad S_{nl} = s_{ns} + s_{ls} - s_{nl},$$

Pentagon type integral

The obtained FE represent integral depending on 10 variables in terms of integrals depending on 7 variables.

Let's consider for simplicity the on-shell case when

$$s_{12}=0, \quad s_{23}=0, \quad s_{34}=0, \quad s_{45}=0, \quad s_{15}=0$$

Substituting these values into previous FE yields:

$$\begin{aligned}
&I_5^{(d)}(0,0,0,0,0,s_{13},s_{14},s_{24},s_{25},s_{35}) \\
&\quad=\phi_{12345}F^{(d)}(\mu_5;0,0,0,s_{24},s_{25},s_{35}) \\
&\quad+\phi_{23451}F^{(d)}(\mu_5;0,0,0,s_{14},s_{13},s_{35}) \\
&\quad+\phi_{34512}F^{(d)}(\mu_5;0,0,0,s_{14},s_{24},s_{25}) \\
&\quad+\phi_{45123}F^{(d)}(\mu_5;0,0,0,s_{13},s_{35},s_{25}) \\
&\quad+\phi_{51234}F^{(d)}(\mu_5;0,0,0,s_{13},s_{14},s_{24})
\end{aligned}$$

Integral with 5 variables is a combination of integrals with 4 variables.

Pentagon-type integral

The result of the second step of functional reduction is:

$$\begin{aligned}
&F^{(d)}(\mu_5; 0,0,0,s_{13},s_{14},s_{24}) \\
&= -\frac{s_{14}}{s_{13}-s_{14}+s_{24}} I_5^{(d)}(0,0,0,0,-\mu_5;\ 0,0,0,\mu_5,\mu_5,s_{14},\mu_6,\mu_6,\mu_5,\mu_5) \\
&\quad + \frac{s_{24}}{s_{13}-s_{14}+s_{24}} I_5^{(d)}(0,0,0,0,-\mu_5;\ 0,0,0,\mu_5,\mu_5,s_{24},\mu_6,\mu_6,\mu_5,\mu_5) \\
&\quad + \frac{s_{13}}{s_{13}-s_{14}+s_{24}} I_5^{(d)}(0,0,0,0,-\mu_5;\ 0,0,0,\mu_5,\mu_5,s_{13},\mu_6,\mu_6,\mu_5,\mu_5),
\end{aligned}$$

where

$$\mu_6 = \frac{s_{13}s_{24}}{s_{13}-s_{14}+s_{24}}$$

$$\mu_5 = \frac{\delta_5}{g_4}$$

and $\delta_5,\ g_4$ are the on-shell values of $\Delta_5,\ G_4$:

$$\begin{aligned}
\delta_5 &= -2s_{13}s_{14}s_{24}s_{25}s_{35}, \\
g_4 &= 2s_{24}s_{35}(2s_{13}s_{24}-2s_{13}s_{25}-s_{24}s_{35}) - 4(s_{25}+s_{35})s_{13}s_{14}s_{24} \\
&\quad - 2s_{13}^2(s_{24}-s_{25})^2 + 2(s_{25}-s_{35})s_{14}[2s_{25}s_{13}-2s_{24}s_{35}-(s_{25}-s_{35})s_{14}].
\end{aligned}$$

Pentagon type integral

After second step of functional reduction integrals with 4 variables are expressed in terms of integrals with 3 variables.

This means that *the on-shell massless pentagon type integral depending on 5 variables is a combination of 15 integrals depending on 3 variables*.

Pentagon type integral

Analytical result for the most elementary integral

$$I_5^{(d)}(s_{13},s_{14},m^2) \equiv I_5^{(d)}(0,0,0,0,m^2;0,0,0,-m^2,-m^2,s_{13},s_{14},s_{14},-m^2,-m^2),$$

can be obtained as a solution of a simple dimensional recurrence relation:

$$(d-4)I_5^{(d+2)}(s_{13},s_{14},m^2) = -2\widetilde{m}^2 I_5^{(d)}(s_{13},s_{14},m^2) - I_4^{(d)}(s_{13},s_{14}),$$

where

$$I_4^{(d)}(s_{13},s_{14}) \equiv I_4^{(d)}(0,0,0,0;0,0,0,s_{14},s_{14},s_{13})$$
$$= -\frac{4(d-3)}{s_{13}s_{14}(d-4)} I_2^{(d)}(s_{13})\, {}_2F_1\left[\begin{matrix} 1, \frac{d-4}{2}; \\ \frac{d-2}{2}; \end{matrix} \frac{\widetilde{s}_{13}}{\widetilde{s}_{14}}\right] - \frac{(d-3)}{s_{13}s_{14}} I_2^{(d)}(s_{14}) \ln\left(1-\frac{\widetilde{s}_{13}}{\widetilde{s}_{14}}\right).$$

was obtained as a solution of dimensional recurrence relation:

$$I_4^{(d+2)}(s_{13},s_{14}) = \frac{\widetilde{s}_{14}}{2(d-3)} I_4^{(d)}(s_{13},s_{14}) + \frac{2}{\widetilde{s}_{13}(d-4)} I_2^{(d)}(s_{13}).$$

Pentagon type integral

In order to solve dimensional recurrence relation we redefine $I_5^{(d)}(s_{13}, s_{14}, m^2)$

$$I_5^{(d)}(s_{13}, s_{14}, m^2) = \frac{(-\widetilde{m}^2)^{d/2-2}}{\Gamma\left(\frac{d-4}{2}\right)} \widetilde{I}_5^{(d)}(s_{13}, s_{14}, m^2) + b(d) I_4^{(d)}(s_{13}, s_{14}) + A_{13}(d) I_2^{(d)}(s_{13}).$$

By choosing arbitrary $b(d)$ and $A_{13}(d)$ we will obtain homogeneous equation for $\widetilde{I}_5^{(d)}(s_{13}, s_{14}, m^2)$.
Substituting this Ansatz into dimensional recurrence relation, equating to zero coefficient in front of $I_2^{(d)}(s_{13})$ yields equation for $b(d)$:

$$I_2^{(d)}(s_{13}): \qquad \widetilde{s}_{14}(d-4)b(d+2) + 4\widetilde{m}^2(d-3)b(d) + 2d - 6 = 0.$$

It's particular solution is:

$$b(d) = -\frac{1}{2m^2}\, {}_2F_1\left[\begin{array}{c} 1, \frac{d}{2}-2; \\ \frac{d-3}{2}; \end{array} \frac{-\widetilde{s}_{14}}{4m^2}\right].$$

Pentagon type integral

Equating to zero coefficient in front of $I_4^{(d)}$ gives equation for A_{13}:

$$I_4^{(d)}: \qquad A_{13}(d+2) + \frac{4(d-1)\widetilde{m}^2}{(d-4)\widetilde{s}_{13}} A_{13}(d) + \frac{4(d-1)}{\widetilde{s}_{13}^2(d-4)} b(d+2) = 0.$$

Solution of the equation for $A_{13}(d)$:

$$A_{13}(d) = \frac{2}{\widetilde{s}_{13}\widetilde{m}^2(\widetilde{s}_{14}+4\widetilde{m}^2)} F_3\left(1,1,\frac{1}{2},\frac{d-4}{2},\frac{d-1}{2};\frac{\widetilde{s}_{14}}{\widetilde{s}_{14}+4\widetilde{m}^2},\frac{-\widetilde{s}_{13}}{4\widetilde{m}^2}\right),$$

where F_3 is Appell function:

$$F_3\left(1,1,\frac{1}{2},\frac{d-4}{2},\frac{d-1}{2};w,z\right) = \frac{\Gamma\left(\frac{d-1}{2}\right)}{\Gamma\left(\frac{d-4}{2}\right)} \sum_{j,k=0}^{\infty} \frac{\Gamma\left(\frac{d-4}{2}+j\right)\left(\frac{1}{2}\right)_k}{\Gamma\left(\frac{d-1}{2}+j+k\right)} w^k z^j.$$

It can be expressed in terms of one-fold integral

$$F_3(\ldots,w,z) \to {}_2F_1 + \int_0^1 \frac{dv\ v^{\frac{d-4}{2}}}{(1-vz)} \ \ln\frac{1+\sqrt{w(1-v)}}{1+\sqrt{w(1-v)}},$$

convenient for the $\varepsilon = (4-d)/2$ expansion.

Pentagon type integral

Solution for homogeneous part was obtained as a solution of a first order differential equation:

$$(1-t)^2 m^6 R_{14}\, \widetilde{I}_5^{(d)}(s_{13}, s_{14}) = f(R_{14}) - f(-R_{14})$$

$$+ \frac{t}{4}\left[\ln(m^2t^2 - (2m^2 + s_{14})t + m^2) - \ln t - \ln(-s_{14})\right],$$

where

$$f(R_{14}) = \frac{t}{4}\ln t \;\; \ln\left(1 - \frac{2m^2 t}{2m^2 + s_{14} + R_{14}}\right)$$
$$+ \mathrm{Li}_2\left(\frac{2m^2}{2m^2 + s_{14} - R_{14}}\right) - \mathrm{Li}_2\left(\frac{2m^2 t}{2m^2 + s_{14} - R_{14}}\right)$$

and

$$t = 1 + \frac{s_{13} + R_{13}}{2m^2},$$
$$R_{13} = \sqrt{s_{13}(s_{13} + 4m^2)},$$
$$R_{14} = \sqrt{s_{14}(s_{14} + 4m^2)},$$

Conclusions and outlook

- Master integrals are not the simplest integrals. By applying method of functional reduction master integrals can be reduced to elementary integrals.
- Functional reduction of six and seven point one-loop integrals even with massive propagartors to elementary integrals is straightforward.
- The next direction of investigation will be functional reduction of multiloop integrals.
- Another direction of investigation will be computer search of functional equations for analytic continuation of integrals into specific kinematic regions.