On the divergences of $6D$, $\mathcal{N} = (1,0)$ supersymmetric four-derivative gauge theory

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Problems of the Modern Mathematical Physics
Dubna, 2024

Motivations

- The main motivation for studying the quantum structure of maximally extended supersymmetric gauge theories in dimensions larger then four is basically motivated by the connection with the low-energy limit of superstring theory and the interaction of D- and M-branes, which in principle allows the study of low-energy effects of superstring theory using quantum field theory methods.
- Although supersymmetric gauge theories in higher dimensions are not renormalizable by power counting, direct calculations sometimes show that some divergences are reduced by supersymmetries.
- Though models with higher derivatives usually have ghost states in their spectrum, they are often used for different purposes in classical and quantum field theory. There are various approaches to handle ghost fields in interacting higher derivative supersymmetric (and non-supersymmetric) theories, ensuring that they do not contribute to observable quantities. These models considered as an effective field theories on the low energy limit.

- The $6D$, $\mathcal{N} = (1, 0)$ supersymmetric higher-derivative gauge theory in harmonic superspace formulation was firstly constructed in [**Ivanov, Smilga and Zupnik(2005)**]
- Harmonic superspace was originally developed by [**Galperin, Ivanov, Ogievetsky, Sokatchev (86')**]
- The ordinary $6D$, $\mathcal{N} = (1, 0)$ supersymmetric Yang-Mills theory in has a dimensionful coupling constant and for this reason is non-renormalizable.
- The UV behavior of such a theory was studied by direct quantum calculations in the component approach, by using the gauge and supersymmetry methods, by applying the background field method in superspace [**Buchbinder, Ivanov, Merzlikin, Stepanyantz (86')**] and by the modern amplitude techniques [**Bork, Kazakov, Kompaniets, Tolkachev (2015)**]
- In contrast to the gauge theory with the standard kinetic term the higher-derivative model with four space-time derivatives possesses a dimensionless coupling constant and is renormalizable by power counting.

Goals and methods

- We review the higher-derivative $6D, \mathcal{N} = (1,0)$ supersymmetric gauge theory [**Ivanov, Smilga and Zupnik(2005)**] and its quantum structure on the one-loop level[**Buchbinder, Ivanov, Merzlikin, Stepanyantz(2020)**]
- As a further generalization we consider the higher-derivative $\mathcal{N} = (1,0)$ supersymmetric gauge theory in six dimensions coupled to the conventional $6D, \mathcal{N} = (1,0)$ gauge theory and to the hypermultiplet.
- The quantization procedure is carried out in the framework of the superfield background method that ensures the manifest $6D$, $\mathcal{N} = (1,0)$ supersymmetry and the classical gauge invariance of the quantum effective action.
- Using the dimensional regularization and minimal subtraction scheme we analyze the one-loop divergent contributions to the effective action.

Plan

1. Notations
2. The model
3. Background field method
4. Effective action
5. Calculation of the one-loop corrections
6. Generalization: HD coupled to conventional $6D, \mathcal{N} = (1, 0)$ SYM.
7. One loop divergences
8. Further perspectives: Two-loop calculations

Notations

The central basis coordinates of the $6D, \mathcal{N} = (1,0)$ harmonic superspace [**Galperin, Ivanov, Ogievetsky, Sokatchev - Harmonic Superspace (2001)**]

$$(z,u) = (x^M, \theta^a_i, u^{\pm i}), \qquad M = 0,..,5, \quad a = 1,2 \quad i = 1,2. \tag{1}$$

The analytic harmonic superspace coordinates

$$(\zeta,u) = (x^M_{\mathcal{A}}, \theta^{+a}, u^{\pm i}), \quad x^M_{\mathcal{A}} = x^M + \frac{i}{2}\theta^a_k \gamma^M_{ab} \theta^b_l u^{+k} u^{-l}, \quad \theta^{\pm a} = u^{\pm}_k \theta^{ak}. \tag{2}$$

The spinor and harmonic derivatives

$$D^+_a \;=\; \partial_{-a}, \quad D^-_a = -\partial_{+a} - 2i\theta^{-b}\partial_{ab}, \tag{3}$$

where $\partial_{\pm a}\theta^{\pm b} = \delta^b_a, u^{+i}u_{-i} = 1$, and $(\gamma^M)_{ab}$ are the antisymmetric $6D$ Weyl γ-matrices, $(\gamma^M)_{ab} = -(\gamma^M)_{ba}\,, (\widetilde{\gamma}^M)^{ab} = \frac{1}{2}\varepsilon^{abcd}(\gamma^M)_{cd}\,$, with ε^{abcd} being the totally antisymmetric Levi-Civita tensor.

The harmonic derivatives

$$\begin{aligned} D^0 &= u^{+i}\frac{\partial}{\partial u^{+i}} - u^{-i}\frac{\partial}{\partial u^{-i}} + \theta^{+a}\partial_{+a} - \theta^{-a}\partial_{-a}, \\ D^{\pm\pm} &= \partial^{\pm\pm} + i\theta^{\pm a}\theta^{\pm b}\partial_{ab} + \theta^{\pm a}\partial_{\mp a}, \quad \partial\pm\pm = u^{\pm i}\frac{\partial}{\partial u^{\mp i}}. \end{aligned} \tag{4}$$

The algebra of harmonic and spinor derivatives

$$\begin{aligned} \{D^+_a, D^-_b\} &= i(\gamma^M)_{ab}\partial_M\,, \qquad [D^{++}, D^{--}] = D^0, \\ [D^{\pm\pm}, D^\pm_a] &= 0\,, \qquad [D^{\pm\pm}, D^\mp_a] = D^\pm_a\,. \end{aligned} \tag{5}$$

The full and analytic superspace integration measures are

$$d^{14}z \equiv d^6x_{\mathcal{A}}\,(D^-)^4(D^+)^4, \quad d\zeta^{(-4)} \equiv d^6x_{\mathcal{A}}\,du\,(D^-)^4. \tag{6}$$

where we have assumed the notation

$$(D^\pm)^4 = -\frac{1}{24}\varepsilon^{abcd}D^\pm_a D^\pm_b D^\pm_c D^\pm_d. \tag{7}$$

A necessary ingredient is also a non-analytic harmonic connection V^{--} obtained as a solution of the harmonic zero-curvature condition

$$D^{++}V^{--} - D^{--}V^{++} + i[V^{++}, V^{--}] = 0\,. \tag{8}$$

Using these superfields one can construct the gauge covariant harmonic derivative $\nabla^{\pm\pm} = D^{\pm\pm} + iV^{\pm\pm}$. The superfield V^{--} is also used to define the spinor and vector connections in the gauge-covariant derivatives

$$\nabla^+_a = D^+_a, \qquad \nabla^-_a = D^-_a + i\mathcal{A}^-_a, \qquad \nabla_{ab} = \partial_{ab} + i\mathcal{A}_{ab}\,, \tag{9}$$

where $\nabla_{ab} = \frac{1}{2}(\gamma^M)_{ab}\nabla_M$ and $\nabla_M = \partial_M - iA_M$, with the superfield connections defined as

$$\mathcal{A}^-_a = iD^+_a V^{--}\,, \mathcal{A}_{ab} = \frac{1}{2}D^+_a D^+_b V^{--}. \tag{10}$$

The covariant derivatives (9) satisfy the algebra

$$\{\nabla^+_a, \nabla^-_b\} = 2i\nabla_{ab}\,, \qquad [\nabla^\pm_c, \nabla_{ab}] = \frac{i}{2}\varepsilon_{abcd}W^{\pm\, d}, \qquad [\nabla_M, \nabla_N] = iF_{MN}\,. \tag{11}$$

where the superfield strength of the gauge multiplet are defined

$$W^{+a} = -\frac{i}{6}\varepsilon^{abcd}D^+_b D^+_c D^+_d V^{--} \tag{12}$$

The classical action of the higher-derivative $6D$, $\mathcal{N}=(1,0)$ supersymmetric gauge theory [**Ivanov, Smilga, Zupnik (2005)**] is written in the harmonic superspace as

$$S_0 = \pm \frac{1}{2g_0^2}\mathrm{tr}\int d\zeta^{(-4)}du\,(F^{++})^2\,, \tag{13}$$

where g_0 is a dimensionless coupling constant. The covariant strength of the analytic gauge superfield V^{++} is defined by the expression

$$F^{++} = (D^+)^4 V^{--} = -\frac{1}{24}\varepsilon^{abcd}D_a^+D_b^+D_c^+D_d^+V^{--}, \tag{14}$$

where

$$V^{--}(z,u) = \sum_{n=1}^{\infty}(-i)^{n+1}\int du_1\ldots du_n\,\frac{V^{++}(z,u_1)\ldots V^{++}(z,u_n)}{(u^+u_1^+)(u_1^+u_2^+)\ldots(u_n^+u^+)}. \tag{15}$$

Following the background field method, we split the superfields V^{++} and q^+_A into the sums of the background superfields V^{++}, Q^+_A and the quantum ones v^{++}, q^+_A,

$$V^{++} \to V^{++} + g_0 v^{++}, \quad q^+_A \to Q^+_A + q^+_A, \tag{16}$$

In what follows we will focus only on the gauge multiplet dependent sector of the effective action and put $Q^+_A = 0$.

Full action S_{total} is constructed in the standard way as the sum of the classical action, the action for the fermionic Faddeev-Popov ghosts S_{fp}, the action for the bosonic real analytic Nielsen-Kallosh ghost S_{nk} and gauge fixing term S_{gf}

$$\begin{aligned} S_{\text{gf}}[v^{++}, V^{++}] &= \frac{1}{2\xi}\text{tr}\int d^{14}z du\, v^{++}_{\tau,1}\, \widehat{\Box}^{\,2} v^{++}_{\tau,2} \\ &- \frac{1}{2\xi}\text{tr}\int d^{14}z \frac{du_1 du_2}{(u^+_1 u^+_2)^2}\Big\{ v^{++}_{\tau,1}(\widehat{\Box}\, v^{++})_{\tau,2} \\ &+ \frac{i}{2} v^{++}_{\tau,1}\nabla^{--}_2 [F^{++}, v^{++}]_{\tau,2}\Big\}, \end{aligned} \tag{17}$$

where ξ is the arbitrary real parameter.

Using the expression for gauge fixing term (17), taking into account that $e^{-ib}\nabla^{--}e^{ib} = D^{--}$ we can write

$$
\begin{aligned}
S^{(2)}_{\text{total}} \;=\;& \pm\frac{1}{2g_0^2\xi}\mathrm{tr}\int d\zeta^{(-4)}du\, v^{++}\,\widehat{\Box}^{\,2}v^{++} \\
& \pm\frac{1}{2g_0^2}\mathrm{tr}\int d^{14}z\frac{du_1du_2}{(u_1^+u_2^+)^2}\Big\{\Big(1-\frac{1}{\xi}\Big)v^{++}_{\tau,1}(\widehat{\Box}\, v^{++})_{\tau,2} \\
& + iv^{++}_{\tau,1}[(\nabla^{--}F^{++}), v^{++}]_{\tau,2}\Big\} \\
& \mp\frac{i}{2g_0^2}\Big(1+\frac{1}{\xi}\Big)\mathrm{tr}\int d^{14}z du_1 du_2\frac{(u_1^+u_2^-)}{(u_1^+u_2^+)^3}v^{++}_{\tau,1}[F^{++},v^{++}]_{\tau,2} \\
& + \mathrm{tr}\int d\zeta^{(-4)}du\, b(\nabla^{++})^2\mathbf{c} + \frac{1}{2}\mathrm{tr}\int d\zeta^{(-4)}du\,\varphi(\nabla^{++})^2\varphi\,, \qquad (18)
\end{aligned}
$$

where the covariant super d'Alembertian is presented in the form

$$
\widehat{\Box} = \Box + iW^{+a}\nabla^-_a + iF^{++}\nabla^{--} - \frac{i}{2}(\nabla^{--}F^{++}). \qquad (19)
$$

After integration over quantum superfields we obtain the one-loop quantum correction to the effective action

$$\begin{aligned}\Delta\Gamma^{(1)}[V^{++}] \quad &= \frac{i}{2}\,\mathsf{Tr}_{(2,2)}\ln\Big\{\frac{1}{\xi}(\widehat{\Box}_1)^2\,(D_1^+)^4\delta^{(-2,2)}(u_1,u_2)\\
&+\Big(1-\frac{1}{\xi}\Big)\frac{(D_1^+)^4\,\widehat{\Box}_2\,(D_2^+)^4}{(u_1^+u_2^+)^2}e^{ib_1}e^{-ib_2}\\
&+\frac{(D_1^+)^4(D_2^+)^4}{(u_1^+u_2^+)^2}e^{ib_1}e^{-ib_2}\Big[i(\nabla^{--}F^{++})-\frac{i(u_1^+u_2^-)}{(u_1^+u_2^+)}\Big(1+\frac{1}{\xi_0}\Big)F^{++}\Big]_2\Big\}_{Adj}\\
&-i\mathsf{Tr}\ln\nabla^{++}_{Adj}\,.\end{aligned} \tag{20}$$

where $\delta^{14}(z_1-z_2)=\delta^8(\theta_1-\theta_2)\delta^6(x_1-x_2)$. The functional trace over harmonic superspace is defined as

$$\mathrm{Tr}_{(q,4-q)}\mathcal{O}=\mathrm{tr}\int d\zeta_1^{(-4)}d\zeta_2^{(-4)}\,\delta_{\mathcal{A}}^{(q,4-q)}(2|1)\,\mathcal{O}^{(q,4-q)}(1|2), \tag{21}$$

where $\delta_{\mathcal{A}}^{(q,4-q)}(2|1)$ is an analytic delta-function and $\mathcal{O}^{(q,4-q)}(\zeta_1,u_1|\zeta_2,u_2)$ is the kernel of some operator $\mathcal{O}$ acting in the space of analytic superfields possessing the harmonic $U(1)$ charge q.

We rewrite the first term of the expression (20) as the sum of two logarithms and obtain

$$\frac{i}{2}\mathsf{Tr}_{(2,2)}\ln\Big\{\frac{1}{\xi}(\widehat{\Box}_1)^2(D_1^+)^4\delta^{(-2,2)}(u_1,u_2)\delta^{14}(z_1-z_2)\Big\}$$
$$+\frac{i}{2}\mathsf{Tr}_{(2,2)}\ln\Big\{(D_1^+)^4\delta^{(-2,2)}(u_1,u_2)\delta^{14}(z_1-z_2)$$
$$+\frac{1}{(\widehat{\Box}_1)^2_{Adj}}\Big\{(\xi-1)\frac{(D_1^+)^4\ \widehat{\Box}_2\ (D_2^+)^4}{(u_1^+u_2^+)^2}e^{ib_1}e^{-ib_2}$$
$$+\frac{(D_1^+)^4(D_2^+)^4}{(u_1^+u_2^+)^2}e^{ib_1}e^{-ib_2}\Big[i\xi(\nabla^{--}F^{++})$$
$$-(\xi+1)\frac{i(u_1^+u_2^-)}{(u_1^+u_2^+)}F^{++}\Big]_2\Big\}_{Adj}\delta^{14}(z_1-z_2)\Big\}. \qquad (22)$$

According to [**Buchbinder, Ivanov, Merzlikin, Stepanyantz(2017)**], the first term in this expression vanishes. To calculate the divergent part of the second term in the lowest order in $(\xi-1)$, we need to expand the logarithm up to a linear term only. As a result, we arrive at the calculation of the divergent part of the expression

$$\Gamma^{(1)}_{\text{div}}[V^{++}] = (\Gamma_1+\Gamma_2+\Gamma_3+\Gamma_4+\Gamma)_{\text{div}}. \qquad (23)$$

Using the dimensional regularization scheme we calculate each of this terms.

The terms above are given as

$$\Gamma_1 = \frac{1}{2}\mathsf{tr}\int d\zeta_1^{(-4)} du_1 \,\frac{(\xi+1)}{(\widehat{\Box}_1)^2}(D_1^+)^4(D_2^+)^4\frac{(u_1^+u_2^-)}{(u_1^+u_2^+)^3}e^{ib_1}e^{-ib_2}F_2^{++}\delta^{14}(z_1-z_2),$$

$$\Gamma_2 = -\frac{1}{2}\mathsf{tr}\int d\zeta_1^{(-4)} du_1 \,\frac{\xi}{(\widehat{\Box}_1)^2}\frac{(D_1^+)^4(D_2^+)^4}{(u_1^+u_2^+)^2}e^{ib_1}e^{-ib_2}(\nabla^{--}F^{++})_2\,\delta^{14}(z_1-z_2),$$

$$\Gamma_3 = \frac{i}{2}\mathsf{tr}\int d\zeta_1^{(-4)} du_1 \,\frac{(\xi-1)}{(\widehat{\Box}_1)^2}(D_1^+)^4\,\widehat{\Box}_2\,(D_2^+)^4\frac{1}{(u_1^+u_2^+)^2}e^{ib_1}e^{-ib_2}\delta^{14}(z_1-z_2)$$

$$\Gamma_4 = -i\mathsf{Tr}\ln\nabla^{++},$$

in the limit of coincident z points. Here tr stands for the usual matrix trace. We briefly reproduce calculation of the first term as an example.

Let us calculate the divergent contribution

$$\Gamma_1 = \frac{(\xi+1)}{2}\int d\zeta_1^{(-4)}du_1\,((\widehat{\Box}_1)^{-2})^{IJ}(D_1^+)^4(D_2^+)^4\times \frac{(u_1^+u_2^-)}{(u_1^+u_2^+)^3}(e^{ib_1}e^{-ib_2})^{JK}(F_2^{++})^{KI}\delta^{14}(z_1-z_2)\Big|_{2\to1}. \quad (28)$$

In this expression, we recommute the operator $(\widehat{\Box})^{-2}$ to the right to delta-function. Taking into account that the covariant d'Alembertian (19) acting on harmonics only yields terms proportional to $i(F^{++})_{IJ}D^{--}$, we conclude that the this divergent contribution is proportional to the third inverse power of the operator $\partial^2=\partial^M\partial_M$ acting on the space-time delta-function $\delta^6(x_1-x_2)$ where the following identity is used,

$$\frac{1}{(\partial^2)^3}\delta^6(x_1-x_2)\Big|_{2\to1} = \frac{i}{(4\pi)^3\varepsilon}\,, \quad \varepsilon\to0\,. \quad (29)$$

Integrating over the Grassmann variables and taking into account (29), after some transformations one obtains

$$\Gamma_{1,\,\mathrm{div}} = -\frac{2(\xi+1)C_2}{(4\pi)^3\varepsilon}\mathrm{tr}\int d\zeta^{(-4)}du\,(F^{++})^2\,, \quad (30)$$

Similarly we calculate the divergent parts for the rest of the terms and get

$$\Delta\Gamma^{(1)}_{\rm div} = -\frac{11C_2}{3(4\pi)^3\varepsilon}{\rm tr}\int d\zeta^{(-4)}du\,(F^{++})^2\,. \qquad (31)$$

- For the arbitrary gauge $\xi \neq 0$ all divergent contributions depending on the gauge-fixing parameter ξ cancel each other. This agrees with the general statement that the renormalization of dimensionless coupling constants in multiplicatively renormalizable gauge theories does not depend on the gauge choice.

We can add the hypermultipllet term. Corresponding divergent contribution

$$\widetilde{\Gamma}^{(1)}_{\infty}[\boldsymbol{V}^{++}] = -\frac{11C_2 + T_R}{3(4\pi)^3\varepsilon}{\rm tr}\int d\zeta^{(-4)}du\,(F^{++})^2\,. \qquad (32)$$

The classical action for the 6D $\mathcal{N} = (1,0)$ SYM HD theory is represented as the sum of the 6D $\mathcal{N} = (1,0)$ SYM theory [**B. M. Zupnik (1986)**] and the term with higher derivatives introduced in [**Ivanov, Smilga, Zupnik (2005)**]

$$S_0[V^{++}] = \frac{1}{f_0^2}\sum_{n=2}^{\infty}\frac{(-i)^n}{n}\mathrm{tr}\int\frac{d^{14}z\,du_1\ldots du_n}{(u_1^+u_2^+)\ldots(u_n^+u_1^+)}V^{++}(z,u_1)\ldots V^{++}(z,u_n)$$
$$+\frac{1}{2g_0^2}\mathrm{tr}\int d\zeta^{(-4)}(F^{++})^2 - \frac{1}{2}\mathrm{tr}\int d\zeta^{(-4)}\,q^{+A}\nabla^{++}q^+_A, \qquad (33)$$

where g_0 is a dimensionless constant, f_0 is the second coupling constant of the inverse dimension of mass. The indices of the Pauli-Gursey group $SU(2)$ are transformed according to the rule $q^+_A = \epsilon_{AB}q^{+B}$, where $\epsilon_{12} = 1$. The analytic superfields V^{++} and q^+ belong to the adjoint representation of the gauge group G.

Similarly we calculate the divergent parts for the rest of the terms and get

$$\Delta\Gamma^{(1)}_{\rm div} = -\frac{11C_2 + T_R}{3(4\pi)^3\varepsilon}\,{\rm tr}\int d\zeta^{(-4)} du\,(F^{++})^2\,. \tag{34}$$

- In the case when the hypermultiplet belongs to adjoint representation the ghost field contribution is annihilated by the hypermultiplet contributions.
- We pay attention that the one-loop divergences in the theory (33) do not depend on presence or absence of the conventional Yang-Mills term in the classical action, higher derivatives suppress lower ones in ultraviolet domain.

Summary

- We studied the quantum divergence structure of the higher-derivative $\mathcal{N} = (1,0)$ supersymmetric non-abelian gauge theory in six dimensions.
- The model is formulated in harmonic $6D$, $\mathcal{N} = (1,0)$ superspace ensuring manifest $\mathcal{N} = (1,0)$ supersymmetry. The quantization was accomplished in the framework of the background superfield method with a one-parameter family of the quantum gauge-fixing conditions.
- The corresponding gauge invariant and manifestly supersymmetric quantum effective action was introduced and all possible divergent terms in such an action were identified.
- A manifestly supersymmetric and gauge invariant procedure to calculate the one-loop divergences was developed and applied for the explicit calculation of these divergencies.
- As a further generalization we considered the higher-derivative $\mathcal{N} = (1,0)$ supersymmetric gauge theory in six dimensions coupled to the conventional $6D, \mathcal{N} = (1,0)$ supergauge theory and to the hypermultiplet.
- The one-loop divergences of this theory was calculated.

- The multiloop calculations in theories formulated in harmonic superspace face the certain difficulties. The harmonic supergraphs contain the harmonic dependent propagators and integrals over harmonics, the number of which increases with the number of loops.
- At this point in time known, in the two-loop approximation there are two types of supergraphs with different topologies. The first has topology of 'Θ' and the second one possesses the '∞' topology.
- The explicit calculation of such integrals corresponding to these diagrams requires using the various non-trivial identities for harmonic distributions which are not evident in each concrete case.
- In some recent papers [**Buchbinder, Ivanov, Merzlikin, Stepanyantz (2021,2023)**] a method which allows calculating the leading $1/\varepsilon^2$ divergences was developed. It is hoped that by using this method, it will be possible to calculate two-loop divergences.

Thank you for your attention!