Harmonic Superspace: why it is unavoidable

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Minkowski space and $\mathcal{N} = 1, 4D$ superspaces

Poincaré invariance plays the central role in the modern physics: all theories of interest are formulated in terms of Poincaré - covariant fields (scalars, vectors, spinors,...) defined on 4-dimensional Minkowski space $\{x^m \sim x^{\alpha\dot{\alpha}}\}, m = 0, 1, 2, 3; \alpha, \dot{\beta} = 1, 2$:

$$(P_m\,, L_{[mn]}), \quad [P_m, P_n] = 0\,,\ [P_m, L_{[ns]}] \sim P_q\,,\ [L_{[mn]}, L_{[pq]}] \sim L_{[st]}\,,$$

$$x^m \propto \frac{\mathcal{P}_4}{SO(1,3)}\,, \qquad x^{m\prime} = x^m + a^m$$

$\mathcal{N} = 1, 4D$ supersymmetry is an extension of the Poincaré symmetry by spinor generators $Q_\alpha, \bar{Q}_{\dot{\alpha}}$

$$\{Q_\alpha, \bar{Q}_{\dot{\beta}}\} = 4\,P_{\alpha\dot{\beta}}\,,\ \{Q_\alpha, Q_\beta\} = \{\bar{Q}_{\dot{\alpha}}, \bar{Q}_{\dot{\beta}}\} = 0\,,\ [P_m, Q_\alpha] = [P_m, \bar{Q}_{\dot{\alpha}}] = 0$$

$\mathcal{N} = 1, 4D$ superspace is an extension of Minkowski space by a doublet of Grassmann anticommuting spinorial coordinates $\theta^\gamma, \bar{\theta}^{\dot{\mu}}$

$$x^{\alpha\dot{\alpha}} \Rightarrow (x^{\alpha\dot{\alpha}}, \theta^\gamma, \bar{\theta}^{\dot{\mu}})\,, \theta^{\gamma\prime} = \theta^\gamma + \epsilon^\gamma\,, x^{\alpha\dot{\alpha}\prime} = x^{\alpha\dot{\alpha}} - 2i(\theta^\alpha\epsilon^{\dot{\alpha}} - \epsilon^\alpha\bar{\theta}^{\dot{\alpha}})$$

$\mathcal{N} = 1, 4D$ superfields are functions on $\mathcal{N} = 1$ superspace, $\Phi(x, \theta, \bar{\theta})$, with the scalar transformation law,

$$\Phi'(x', \theta', \bar{\theta}') = \Phi(x, \theta, \bar{\theta})$$

There is one very essential difference between Minkowski space and $\mathcal{N}=1$ superspace. While the former does not include any subspace where the whole $4D$ Poincaré symmetry could be linearly realized, the latter contains such smaller supermanifolds, $\mathcal{N}=1, 4D$ chiral superspaces $(x_L, \theta), (x_R, \bar\theta)$ with twice as less Grassmann coordinates:

$$x_L^{\alpha\dot\beta} = x^{\alpha\dot\beta} + 2i\theta^\alpha\bar\theta^{\dot\beta}\,, \quad \delta x_L^{\alpha\dot\beta} = -4i\theta^\alpha\bar\epsilon^{\dot\beta}\,, \quad x_R^{\alpha\dot\beta} = (x_L^{\alpha\dot\beta})^\dagger$$

Just the chiral superfields are carriers of the basic matter $\mathcal{N}=1$ multiplet, the chiral one

$$\varphi(x_L,\theta) = \phi(x_L) + \theta^\alpha\psi_\alpha(x_L) + (\theta)^2F(x_L)\,, S_{free} \sim \int d^4xd^2\theta d^2\bar\theta\,\varphi(x_L,\theta)\bar\varphi(x_R,\bar\theta)$$

The chiral superfields can be looked upon as complex general $\mathcal{N}=1$ superfields subject to the covariant Grassmann analyticity condition (A. Galperin, E.I., V. Ogievetsky, 1981)

$$\varphi(x_L,\theta) = \Phi_L(x_L,\theta,\bar\theta)\,, \quad \frac{\partial}{\partial\bar\theta^{\dot\gamma}}\Phi_L = 0$$

The same constraint can be rewritten in the basis $(x,\theta,\bar\theta)$ in terms of spinor covariant derivatives

$$\bar D_{\dot\gamma}\Phi_L(x,\theta,\bar\theta) = 0\,, \quad D_\alpha = \frac{\partial}{\partial\theta^\alpha} + 2i\bar\theta^{\dot\alpha}\partial_{\alpha\dot\alpha}\,, \bar D_{\dot\gamma} = -\frac{\partial}{\partial\bar\theta^{\dot\gamma}} - 2i\theta^\alpha\partial_{\alpha\dot\gamma}\,,$$

$$\{D_\gamma, D_\beta\} = \{\bar D_{\dot\gamma}, \bar D_{\dot\beta}\} = 0\,, \quad \{D_\gamma, \bar D_{\dot\beta}\} = -4i\partial_{\gamma\dot\beta}$$

The vanishing of anticommutators of the same chirality spinor derivatives is just the integrability conditions for $\mathcal{N} = 1$ chirality. This chirality underlies all the gauge and supergravity $\mathcal{N} = 1$ theories: the interacting case just corresponds to replacing all covariant derivatives by the gauge-covariant ones through adding proper superfield gauge connections

$$D_\gamma \;\Rightarrow\; \mathcal{D}_\gamma\,, \quad \bar{D}_{\dot\gamma} \;\Rightarrow\; \bar{\mathcal{D}}_{\dot\gamma}\,, \quad \partial_{\gamma\dot\beta} \;\Rightarrow\; \mathcal{D}_{\gamma\dot\beta}\,,$$

still preserving the flat integrability constraints

$$\{\mathcal{D}_\gamma, \mathcal{D}_\beta\} = \{\bar{\mathcal{D}}_{\dot\gamma}, \bar{\mathcal{D}}_{\dot\beta}\} = 0$$

The general $\mathcal{N} = 1$ matter is also described by chiral superfields, implying a general Kähler target geometry for bosonic fields (Zumino, 1979).

For extended supersymmetries (with few sorts of Q generators) new kinds of Grassmann analyticities (different from chirality) can be defined and they just form the basis of the Harmonic Superspace approach.

$SO(2)$ vs $SU(2)$ Grassmann analyticities

The simplest extended $4D$ supersymmetry is $\mathcal{N}=2$ one, with two independent supercharges:

$$\{Q^i_\alpha, \bar{Q}_{\dot\beta k}\} = 4\delta^i_k P_{\alpha\dot\beta}\,, \; \{Q^i_\alpha, Q^k_\beta\} = \{\bar{Q}_{\dot\alpha i}, \bar{Q}_{\dot\beta k}\} = 0\,,$$

where $i,k=1,2$. This superalgebra has the automorphism $SU(2)_R$ which rotates supercharges as doublets. Equally, one can consider a theory in which $SU(2)_R$ is reduced to $SO(2)_R \subset SU(2)_R$.

One can define the standard $\mathcal{N}=2$ superspace, $(x^{\alpha\dot\beta}, \theta^{\alpha i}, \bar\theta^{\dot\alpha}_i) =: Z$, as well as chiral superspace $(x_L^{\alpha\dot\beta}, \theta^{\alpha i}) =: \zeta_L$ and its conjugate ζ_R. However, there occur new possibilities for the invariant subspaces. Namely, one can define

$$(x_{an}^{\alpha\dot\beta}, \theta^{\alpha 1} + i\theta^{\alpha 2}) \quad \text{or} \quad (\tilde{x}_{an}^{\alpha\dot\beta}, \theta^{\alpha 1} - i\theta^{\alpha 2})$$

Clearly, $SU(2)_R$ is broken to $SO(2)_R$ with such a definition. Indeed, these subspaces are closed under $\mathcal{N}=2$ supersymmetry and internal $SO(2)_R$ symmetry, but not under $SU(2)_R$. Just such subspaces were called analytic by Galperin, Ivanov and Ogievetsky, 1981 in order to distinguish them from the simplest analytic subspaces, the chiral ones.

On the other hand, the constraints defining the gauge $\mathcal{N}=2$ theory and $\mathcal{N}=2$ matter (as well as $\mathcal{N}=2$ supergravity) in the standard $\mathcal{N}=2$ superspace are known to be covariant under the whole $SU(2)_R$ symmetry

$$\{\mathcal{D}^{(i}_{\alpha}, \mathcal{D}^{k)}_{\beta}\} = \{\bar{\mathcal{D}}^{(i}_{\dot\alpha}, \bar{\mathcal{D}}^{k)}_{\dot\beta}\} = \{\mathcal{D}^{(i}_{\alpha}, \bar{\mathcal{D}}^{k)}_{\dot\beta}\} = 0\,,$$

$$D^{(i}_{\alpha} q^{k)}(Z) = \bar{D}^{(i}_{\dot\alpha} q^{k)}(Z) = 0,$$

where $\mathcal{D}^{i}_{\alpha}, \bar{\mathcal{D}}^{i}_{\dot\alpha}$ are properly gauge-covariantized spinor derivatives and $q^i(Z)$ are hypermultiplet superfields in the "central basis".

As compared to $\mathcal{N}=1$ SYM constraints, their $\mathcal{N}=2$ counterparts have more complicated structure and include spinor derivatives of different chiralities. Moreover, while the $\mathcal{N}=1$ chirality constraints on the matter superfield Φ do not impose any dynamical restrictions on the component fields, $\mathcal{N}=2$ constraints give rise to the free equations of motion for the hypermultiplet superfield components.

What is the geometric meaning of these $\mathcal{N}=2$ constraints and how to solve them in full generality? Mezincescu was first to find the explicit solution of the $\mathcal{N}=2$ SYM constraints through an unconstrained superfield gauge prepotential (1979) but only for abelian case. No any geometric meaning can be ascribed to this prepotential. Also it remained unclear how to generalize the hypermultiplet constraints to the general case, how to ensure their off-shell realization and how to construct the invariant superfield actions.

$SU(2)$ harmonic superspace

All these questions were answered and all problems were solved after invention of the Harmonic Superspace in (1984) by Galperin, Ivanov, Kalitzin, Ogievetsky, Sokatchev.

$\mathcal{N}=2$ harmonic superspace (HSS) is the product

$$(x^m, \theta_{\alpha\, i}, \bar{\theta}^k_{\dot{\beta}}) \otimes S^2$$

Here, the internal two-sphere $S^2 \sim SU(2)_R/U(1)_R$ is represented, in a parametrization-independent way, by the lowest (isospinor) $SU(2)_R$ harmonics

$$S^2 \in (u^+_i, u^-_k), \quad u^{+i}u^-_i = 1, \quad u^\pm_i \to e^{\pm i\lambda}u^\pm_i$$

The superfields given on HSS (harmonic $\mathcal{N}=2$ superfields) admit the harmonic expansions on S^2, with the set of all symmetrized products of u^+_i, u^-_i as the basis. Such an expansion is fully specified by the harmonic $U(1)$ charge of the given superfield.

The main advantage of HSS is that it contains an invariant subspace, the $\mathcal{N}=2$ analytic HSS, involving only half of the original Grassmann coordinates.

One can pass to the **analytic basis** in HSS

$$\{(x^m_A, \theta^+_\alpha, \bar\theta^+_{\dot\alpha}, u^\pm_i)\, \theta^-_\alpha, \bar\theta^-_{\dot\alpha}\} \equiv \{\left(\zeta^M, u^\pm_i\right), \theta^-_\alpha, \bar\theta^-_{\dot\alpha}\},$$
$$x^m_A = x^m - 2i\theta^{(i}\sigma^m\bar\theta^{k)}u^+_i u^-_k\ , \quad \theta^\pm_\alpha = \theta^i_\alpha u^\pm_i\ , \ \bar\theta^\pm_{\dot\alpha} = \bar\theta^i_{\dot\alpha} u^\pm_i$$

Then the set of coordinates

$$(x^m_A, \theta^+_\alpha, \bar\theta^+_{\dot\alpha}, u^\pm_i) \equiv \left(\zeta^M, u^\pm_i\right),$$

is just $SU(2)$ covariantization of the $O(2)$ analytic superspace. It is closed under both $\mathcal{N}=2$ supersymmetry transformations and $SU(2)_R$ and it is real with respect to the special involution defined as the product of the ordinary complex conjugation and the antipodal map (Weyl reflection) of S^2.

The superfields given on the analytic subspace can be defined in the basis-independent way by the new Grassmann-analyticity conditions

$$D^+_\alpha\Phi(Z,u) = \bar D^+_{\dot\alpha}\Phi(Z,u) = 0 \ \leftrightarrow \left(\frac{\partial}{\partial\theta^-_\alpha}, \frac{\partial}{\partial\bar\theta^-_{\dot\alpha}}\right)\Phi(\zeta^M, u^\pm_i, \theta^-_\alpha, \bar\theta^-_{\dot\alpha}) = 0$$
$$D^+_\alpha = u^+_i D^i_\alpha\ , \ \bar D^+_{\dot\alpha} = u^+_i \bar D^i_{\dot\alpha}\ , \quad \Phi = \varphi(\zeta^M, u^\pm_i)$$

All $\mathcal{N}=2$ theories of interest **(SYM, matter, supergravities, superextension of higher spins,...)** are underlain by analytic superfields $\varphi^{+q}(\zeta^M, u^\pm_i)$ as the fundamental geometric objects.

$\mathcal{N}=2$ **SYM constraints** are solved as follows. One projects

$$(\mathcal{D}^i_\alpha, \bar{\mathcal{D}}^i_{\dot\alpha}) \Leftrightarrow (\mathcal{D}^\pm_\alpha, \bar{\mathcal{D}}^\pm_{\dot\alpha}), \qquad \mathcal{D}^\pm_\alpha = u^\pm_i \mathcal{D}^i_\alpha\,,\ \bar{\mathcal{D}}^\pm_{\dot\alpha} = u^\pm_i \bar{\mathcal{D}}^i_{\dot\alpha} \tag{1}$$

Then the constraints are rewritten as

$$\{\mathcal{D}^+_\alpha, \mathcal{D}^+_\beta\} = \{\bar{\mathcal{D}}^+_{\dot\alpha}, \bar{\mathcal{D}}^+_{\dot\beta}\} = \{\mathcal{D}^+_\alpha, \bar{\mathcal{D}}^+_{\dot\beta}\} = 0 \tag{2}$$

After taking off u^{+i}, the standard constraints are recovered in view of arbitrariness of u^+_i. How to reduce the procedure of converting with harmonics to a new kind of differential constraint?
This can be done with the new differential operators, the harmonic derivatives

$$\partial^{\pm\pm} = u^{\pm i}\frac{\partial}{\partial u^{\mp i}},\ \partial^0 = u^{+i}\frac{\partial}{\partial u^{+i}} - u^{-i}\frac{\partial}{\partial u^{-i}}\,,$$
$$[\partial^{++}, \partial^{--}] = \partial^0\,, \quad [\partial^0, \partial^{\pm\pm}] = \pm 2\partial^{\pm\pm}\,, \tag{3}$$

One adds to (2) new constraints

$$[\partial^{++}, \mathcal{D}^+_\alpha] = [\partial^{++}, \bar{\mathcal{D}}^+_{\dot\alpha}] = 0 \Rightarrow \mathcal{D}^+_{\alpha,\dot\alpha} = u^+_i \mathcal{D}^i_{\alpha,\dot\alpha}\,, \tag{4}$$

using $\partial^{++}u^+_i = 0, \partial^{++}u^-_i = u^+_i$. Now one can treat $\mathcal{D}^+_{\alpha,\dot\alpha}$ to have an unconstrained dependence on $u^\pm$ while the linear dependence appears as a result of imposing extra constraints (4).

The extended set of constraints (2) and (4), besides the standard SYM constraints, admits a new solution. By making some similarity gauge-like transformation (with a general harmonic suiperfield as a parameter, so called “bridge”) and simultaneously passing to the analytic basis in HSS, one can solve the integrability conditions (2) and (4) as

$$(\mathcal{D}^+_\alpha, \bar{\mathcal{D}}^+_{\dot\alpha}) \Rightarrow (\partial^+_\alpha, \bar\partial^+_{\dot\alpha}), \quad \partial^{++} \Rightarrow D^{++} + iV^{++}(\zeta, u), \qquad (5)$$
$$D^{++} = \partial^{++} - 2i\theta^+\sigma^a\bar\theta^+\partial_a + \theta^{+\alpha}\partial^+_\alpha + \bar\theta^{+\dot\alpha}\bar\partial^+_{\dot\alpha}$$

In other words, (2) implies the spinorial connections in $\mathcal{D}^+_{\alpha,\dot\alpha}$ to be a “pure gauge” in the full HSS, while D^{++} in the new frame acquires some harmonic connection V^{++} which is unconstrained **analytic** by the constraint (4), $V^{++} \Rightarrow V^{++}(\zeta, u)$. It is just the fundamental gauge prepotential of $\mathcal{N} = 2$ SYM theory. It carries the analytic gauge freedom,

$$\delta V^{++} = D^{++}\Lambda + i[V^{++}, \Lambda],$$

which can be used to reduce V^{++} to the Wess-Zumino form

$$V^{++} = (\theta^+)^2\phi + (\bar\theta^+)^2\bar\phi + i\theta^+\sigma^m\bar\theta A_m + [(\theta^+)^2\bar\theta^+_{\dot\alpha}\bar\psi^{\dot\alpha i}u^-_i + \text{c.c.}] + (\theta^+)^4 D^{(ik)}u^-_i u^-_k$$

These fields form off-shell $\mathcal{N} = 2$ vector supermultiplet.

The off-shell action can be constructed using the second (non-analytic) harmonic connection V^{--} related to V^{++} by the harmonic flatness condition

$$D^{++}V^{--} - D^{--}V^{++} + i[V^{++}, V^{--}] = 0$$

The action for the abelian case is

$$S_v \sim \int d^{12}Zdu\ V^{++}V^{--}$$

For non-abelian case the action looks more complicated since it contains non-locality in harmonics (B. Zupnik, 1987). Its variation is much simpler

$$\delta S_v \sim \int d^{12}Zdu\ \mathrm{Tr}(\delta V^{++}V^{--})$$

Hypermultiplet. The on-shell constraints on the hypermultiplet are rewritten in HSS as

$$D^{(i}_{\alpha,\dot\alpha}q^{k)} = 0\,, \Leftrightarrow\ (a)\, D^{+}_{\alpha,\dot\alpha}q^{+} = 0,\ (b)\, D^{++}q^{+} = 0 \tag{6}$$

Indeed, eq. (6)(b) implies $q^{+} = u^{+}_{k}q^{k}$, then eq (6)(a) yields the standard constraints for $\mathcal{N} = 2$ superfield q^{k}.

Once the standard hypermultiplet constraints have been rewritten as differential conditions in HSS, one can pass to the analytic basis where $D^{+}_{\alpha,\dot\alpha}$ become "short" and (6)(a) simply imply that q^{+} is analytic in this basis:

$$(6)(a) \Rightarrow q^{+} = q^{+}(\zeta, u) \tag{7}$$

As any analytic $\mathcal{N} = 2$ superfield, $q^{+}(\zeta, u)$ is off-shell and involves the ∞ number of fields. The whole dynamics proves to be concentrated in (6)(b)

$$(6)(b) \Rightarrow D^{++}q^{+} = (\partial^{++} - 2i\theta^{+}\sigma^{a}\bar\theta^{+}\partial_{a})q^{+} = 0 \tag{8}$$

This equation nullifies all fields in $q^{+}(\zeta, u)$ except for those entering at zero and first degrees of $\theta^{+}_{\alpha}, \bar\theta^{+}_{\dot\alpha}$ and at first and zero powers in harmonics:

$$q^{+} \Rightarrow q^{i}(x)u^{+}_{i} + \theta^{+}_{\alpha}\psi^{\alpha}(x) + \bar\theta^{+}_{\dot\alpha}\chi^{\dot\alpha}(x) \tag{9}$$

These are just physical bosonic and fermionic fields and for them (8) implies the standard free massless equations of motion.

The splitting of the hypermultiplet constraints into the kinematical and dynamical parts in the analytic basis of HSS entails a remarkable consequence. Now the dynamical constraint (6)(b) can ne derived as an equation of motion from the off-shell action:

$$D^{++}q^{+} = 0 \quad \text{from} \quad S_q^{free} \sim \int d\zeta^{(-4)} du \left(q^{+} D^{++} \bar{q}^{+} - \bar{q}^{+} D^{++} q^{+}\right) \tag{10}$$

Thus the Grassmann harmonic analyticity allowed to construct off-shell action for the hypermultiplet which was not possible in the framework of standard superspaces. It became possible just at cost of admitting an infinite number of auxiliary fields, the cardinally new feature brought about by the harmonic superspace formalism.

Now it is straightforward to write the most general action for interacting hypermultiplets:

$$S_q^{gen} \sim \int d\zeta^{(-4)} du \left(q^{+A} D^{++} \bar{q}^{+}_A - \bar{q}^{+}_A D^{++} q^{+A} + \mathcal{L}^{+4}(q^{+}, \bar{q}^{+}, u^{\pm})\right) \tag{11}$$

where $\mathcal{L}^{+4}(q^{+}, \bar{q}^{+}, u^{\pm})$ is an arbitrary function of its arguments, the "hyper Kähler potential", the true analog of Kähler potential of $\mathcal{N} = 1$ supersymmetric matter. It was proven in Alvarez-Gaume, Freedman, 1980, 1981 that any $\mathcal{N} = 2, 4D$ supersymmetric matter Lagrangian contains as its bosonic "core" just sigma model with hyper-Kähler target space. So any Lagrangian (11) provides an efficient way of the **explicit** construction of HK metrics (Galperin, Ivanov, Ogievetsky, Sokatchev, 1986).

$SU(3)$ harmonic superspace

One more striking example of how HSS works is the HSS formulation of $\mathcal{N}=3$ SYM theory (which is equivalent to the renowned $\mathcal{N}=4$ SYM theory on shell). The $\mathcal{N}=3$ superspace constraints of this theory read

$$\{\mathcal{D}^{(i}_{\alpha}, \mathcal{D}^{j)}_{\beta}\} = \{\bar{\mathcal{D}}_{(i\dot{\alpha}}, \bar{\mathcal{D}}_{j)\dot{\beta}}\} = 0\,, \quad \{\mathcal{D}^{(i}_{\alpha}, \bar{\mathcal{D}}_{j)\dot{\beta}}\} = 0 \qquad (12)$$

Here lower and upper indices $i, j, ...$ refer to the fundamental and co-fundamental representations of $\mathcal{N}=3$ R-symmetry group $SU(3)_R$. The interpretation of these constraints as integrability conditions for a sort of Grassmann analyticity was proposed by Rosly, 1983. Later, these were realized as a natural generalization of $\mathcal{N}=2$ HSS Grassmann analyticity in Galperin, Ivanov, Kalitzin, Ogievetsky, Sokatchev, 1985. The $\mathcal{N}=3$ HSS with the $SU(3)/[U(1)\times U(1)]$ harmonic part was introduced there. It involves the analytic subspace with two independent complex Grassmann coordinates compared to three such coordinates in the full $\mathcal{N}=3$ HSS. The relevant Grassmann analyticity conditions read

$$\mathcal{D}^{(1,0)}_{\alpha}\Phi = 0\,, \quad \bar{\mathcal{D}}^{(0,1)}_{\dot{\alpha}}\Phi = 0\,, \quad \{\mathcal{D}^{(1,0)}_{\alpha}, \bar{\mathcal{D}}^{(0,1)}_{\dot{\alpha}}\} = 0\,, \qquad (13)$$

where $\mathcal{D}^{(a,b)}$ are the proper projections of the original spinor covariant derivatives. These projections commuute with the set of three harmonic derivatives

$$(D^{(2,-1)}, D^{(-1,2)}, D^{(1,1)})\,, \qquad [D^{(2,-1)}, D^{(-1,2)}] = D^{(1,1)}\,, \qquad (14)$$

which is $\mathcal{N}=3$ analog of the $\mathcal{N}=2$ analyticity-preserving derivative D^{++}.

After passing to the analytic basis, the spinorial Grassmann analyticity conditions imply that the harmonic potentials appearing in the rotated harmonic derivatives $\mathcal{D}^{(a,b)}$ are $\mathcal{N}=3$ analytic superfields $V^{(2,-1)}(\zeta,u)$, $V^{(-1,2)}(\zeta,u)$, $V^{(1,1)}(\zeta,u)$, while the vanishing of the superfield strengths appearing in the commutation relations between these covariantized harmonic derivatives become just the equations of motion for $\mathcal{N}=3$ SYM:

$$[\mathcal{D}^{(2,-1)},\mathcal{D}^{(-1,2)}]:=\mathcal{D}^{(1,1)}+iF^{(1,1)}\,,\ [\mathcal{D}^{(1,1)},\mathcal{D}^{(2,-1)}]:=iF^{(3,0)}\,,$$
$$[\mathcal{D}^{(-1,2)},\mathcal{D}^{(1,1)}]:=iF^{(0,3)}\,,\qquad F^{(1,1)}=F^{(3,0)}=F^{(0,3)}=0$$

It is rather surprising that these equations of motion are derivable from the Chern-Simons type off-shell action

$$S_{N=3}=\int dud\zeta^{(-2,-2)}\mathrm{Tr}\,\{V^{(2,-1)}F^{(0,3)}+V^{(-1,2)}F^{(3,0)}$$
$$+V^{(1,1)}(F^{(1,1)}-i[V^{(2,-1)},V^{(-1,2)}])\}$$

It is highly non-trivial and unique peculiarity that the $U(1)$ charges and the dimension of the measure ($(-2,-2)$ and 0) precisely match those of the Lagrangian ($(2,2)$ and 0)! The action is analogous to $\mathcal{N}=2$ action of q^{+} as the gauge potentials involve infinite numbers of auxiliary fields off shell.

$\mathcal{N} = 2$ supergravity in HSS

The fundamental gauge group of Einstein $\mathcal{N} = 2$ supergravity in HSS is general diffeomorphisms of the analytic superspace, such that the harmonics themselves remain untouched

$$\delta\zeta^M = \Lambda^M(\zeta, u)\,, \;\delta u_i^{\pm} = 0\,, \quad M := (\alpha\dot{\beta}, 5\,, \hat{\alpha}+), \;\hat{\alpha} := (\alpha, \dot{\alpha})$$

We added one more coordinate x^5, which is necessary for description of *massive* q^{+a} hypermultiplets. Nothing depends on x^5 while the x^5 dependence of q^{+a} is assumed to be trivial, $q^{+a} \to (e^{im\,x^5\sigma^3})_b^a q^{+b}$.
To ensure the gauge invariance of the q^+ Lagrangian, we need to gauge-covariantize the harmonic derivative D^{++}:

$$\frac{1}{2}q^{+a}D^{++}q_a^+ \;\Rightarrow\; \frac{1}{2}q^{+a}\mathcal{D}^{++}q_a^+\,,\; \mathcal{D}^{++} = \partial^{++} + H^{++M}\partial_M\,, \tag{15}$$

The gauge transformation of H^{++M} reads

$$\delta H^{++M} = \mathcal{D}^{++}\lambda^M - \lambda^N\partial_N H^{++M} \tag{16}$$

It can be checked that the q^+ Lagrangian $\frac{1}{2}q^{+a}\mathcal{D}^{++}q_a^+$ is transformed by a total derivative under (16) and the transformation $\delta q^{+a} = -\frac{1}{2}(\partial_{\alpha\dot{\alpha}}\lambda^{\alpha\dot{\alpha}} - \partial_{\hat{\alpha}}\lambda^{\hat{\alpha}})q^{+a} - \lambda^N\partial_N q^{+a}$, so the action is invariant.

Using these gauge transformations of H^{++M}, one can pass to WZ gauge with $(40+40)$ off-shell degrees of freedom, which is just the off-shell content of the so called “minimal” $\mathcal{N}=2$ supergravity Fradkin, Vasiliev, 1979; de Wit, van Holten, 1979. The invariant superfield action for H^{++M} can be also constructed. In the linearized approximation:

$$S_Y \sim \int dud^4xd^8\theta[G^{++\alpha\dot{\alpha}}G^{--}_{\alpha\dot{\alpha}} + G^{++5}G^{--5}]\,,$$

$$G^{\pm\pm\alpha\dot{\alpha}} = h^{\pm\pm\alpha\dot{\alpha}} + 2i(h^{\pm\pm\alpha+}\bar{\theta}^{-\dot{\alpha}} + \theta^{-\alpha}h^{\pm\pm\dot{\alpha}+}),\ G^{\pm\pm5} = h^{\pm\pm5} - 2ih^{\pm\pm\hat{\alpha}+}\theta^{-}_{\hat{\alpha}}\,,$$

$$H^{++\alpha\dot{\alpha}} = h^{++\alpha\dot{\alpha}} - 4i\theta^{+\alpha}\bar{\theta}^{+\dot{\alpha}}\,,\ H^{++5} = h^{++5} + i(\theta^{\hat{+}})^2\,,$$

$$D^{++}G^{--\alpha\dot{\alpha},5} = D^{--}G^{++\alpha\dot{\alpha},5}\,.$$

Also, conformal $\mathcal{N}=2$ supergravity was formulated in HSS and various versions of $\mathcal{N}=2$ Einstein supergravities through the compensating procedure by the appropriate compensating superfields were reproduced.

It was rather surprising that the unconstrained superfield formulations of the higher-spin $\mathcal{N}=2$ supergravities could be formulated in HSS (for the first time!) as a direct generalization of the HSS formulation of Einstein $\mathcal{N}=2$ supergravity (Buchbinder, Ivanov, Zaigraev, 2021 - 2024)

$\mathcal{N} = 2$ higher spins

I will limit my presentation by the $\mathcal{N} = 2$ spin **3**. Like in $\mathcal{N} = 2$ supergravity, the basic analytic superfields form a triad, with additional spinorial indices:

$$h^{++(\alpha\beta)(\dot\alpha\dot\beta)}(\zeta)\,,\; h^{++\alpha\dot\alpha}(\zeta),\; h^{++(\alpha\beta)\dot\alpha+}(\zeta),\; h^{++(\dot\alpha\dot\beta)\alpha+}(\zeta)\,,$$

$$\begin{aligned}
&\delta h^{++(\alpha\beta)(\dot\alpha\dot\beta)} = D^{++}\lambda^{(\alpha\beta)(\dot\alpha\dot\beta)} + 2i\big[\lambda^{+(\alpha\beta)(\dot\alpha}\bar\theta^{+\dot\beta)} + \theta^{+(\alpha}\bar\lambda^{+\beta)(\dot\alpha\dot\beta)}\big],\\
&\delta h^{++\alpha\dot\alpha} = D^{++}\lambda^{\alpha\dot\alpha} - 2i\big[\lambda^{+(\alpha\beta)\dot\alpha}\theta^{+}_{\beta} + \bar\lambda^{+(\dot\alpha\dot\beta)\alpha}\bar\theta^{+}_{\dot\beta}\big],\\
&\delta h^{++(\alpha\beta)\dot\alpha+} = D^{++}\lambda^{+(\alpha\beta)\dot\alpha}\,,\; \delta h^{++(\dot\alpha\dot\beta)\alpha+} = D^{++}\lambda^{+(\dot\alpha\dot\beta)\alpha}
\end{aligned}$$

The bosonic physical fields in the WZ gauge are collected in

$$\begin{aligned}
h^{++(\alpha\beta)(\dot\alpha\dot\beta)} &= -2i\theta^{+\rho}\bar\theta^{+\dot\rho}\Phi^{(\alpha\beta)(\dot\alpha\dot\beta)}_{\rho\dot\rho} + \ldots\\
h^{++\alpha\dot\alpha} &= -2i\theta^{+\rho}\bar\theta^{+\dot\rho}C^{\alpha\dot\alpha}_{\rho\dot\rho} + \ldots
\end{aligned}$$

The physical gauge fields are $\Phi^{(\alpha\beta)(\dot\alpha\dot\beta)}_{\rho\dot\rho}$ (spin 3), $C^{\alpha\dot\alpha}_{\rho\dot\rho}$ (spin 2) and $\psi^{(\alpha\beta)(\dot\alpha\dot\beta)i}_{\gamma}$ (spin 5/2). Other fields are auxiliary. On shell, $\mathbf{(3, 5/2, 5/2, 2)}$.

The linearized gauge action has the form quite similar to the spin **2** action

$$S_{s=3} = \int d^4x d^8\theta du \left\{ G^{++(\alpha\beta)(\dot\alpha\dot\beta)} G^{--}_{(\alpha\beta)(\dot\alpha\dot\beta)} + G^{++\alpha\dot\beta} G^{--}_{\alpha\dot\beta} \right\},$$

$$G^{++(\alpha\beta)(\dot\alpha\dot\beta)} = h^{++(\alpha\beta)(\dot\alpha\dot\beta)} + 2i\big[h^{++(\alpha\beta)(\dot\alpha+}\bar\theta^{-\dot\beta)} - h^{++(\dot\alpha\dot\beta)(\alpha+}\theta^{-\beta)}\big]\,,$$

$$G^{++\alpha\dot\beta} = h^{++\alpha\dot\beta} - 2i\big[h^{++(\alpha\beta)\dot\beta+}\theta^-_\beta - \bar\theta^-_{\dot\alpha} h^{++(\dot\alpha\dot\beta)\alpha+}\big],$$

$$D^{++}G^{--(\alpha\beta)(\dot\alpha\dot\beta)} - D^{--}G^{++(\alpha\beta)(\dot\alpha\dot\beta)} = 0\,,\ D^{++}G^{--\alpha\dot\beta} - D^{--}G^{++\alpha\dot\beta} = 0$$

The actions for higher spins are constructed quite analogously. The on-shell spin contents of $\mathcal{N} = 2$ higher-spin multiplets can be summarized as

$\underline{spin\ 1}$: $1,\ (1/2)^2,\ (0)^2$

$\underline{spin\ 2}$: $2,\ (3/2)^2,\ 1$

$\underline{spin\ 3}$: $3,\ (5/2)^2,\ 2$

....

$\underline{spin\ s}$: $s,\ (s-1/2)^2,\ s-1$

Each spin enters the direct sum of these multiplets twice, in accord with the general Vasiliev theory of $4D$ higher spins. The off-shell contents of the spin $\mathbf{s}$ multiplet is described by the formula $8[\mathbf{s}^2 + (\mathbf{s}-\mathbf{1})^2]_B + 8[\mathbf{s}^2 + (\mathbf{s}-\mathbf{1})^2]_F$.

Bosonic analogs

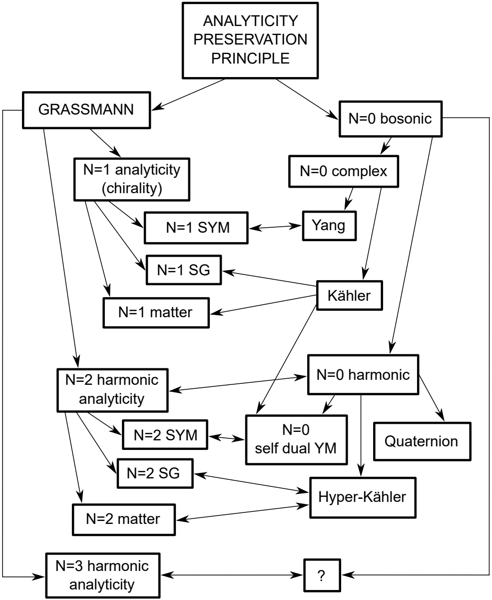
It is notable that there are bosonic analogs of the principle of preserving Grassman analyticity as the basis of gauge supersymmetric theories. This astonishing affinity was found out in Galperin, Ivanov, Ogievetsky, Sokatchev,1988. In particular, it was shown that the general solution of self-duality equation for Yang-Mills theory in the Euclidean $\mathbf{R}^4 \sim x^{\mu\, i}, \mu, i = 1, 2$ is given by an analog of $\mathcal{N} = 2$ potential $V^{++}(\zeta, u)$:

$$V^{++}(x^{\mu +}, u_i^{\pm}),\ x^{\mu +} = x^{\mu\, i} u_i^{+}\,, \quad u_i^{\pm} \in SU(2)_R/U(1)_R\,,\ SO(4) \sim SU(2)_L \times SU(2)_R$$

Also, the general hyper-Kähler geometry in $\mathbf{R}^{4n} \sim x^{\mu\, i}, \mu = 1, ...2n; i = 1, 2;$ was solved in terms of general hyper-Kähler potential

$$\mathcal{L}^{+4}(x^{\mu +}, u_i^{\pm})\,, \quad SO(4n) \rightarrow Sp(n) \times SU(2)$$

Links between bosonic and supersymmetric avatars of the generic Analyticity Preservation Principle are depicted in the Table.



The Analyticity Preservation Principle reveals deep relationships between theories that are seemingly very different

Further prospects

- One of the unsolved important problems is construction of $\mathcal{N} = 3$ supergravity in $\mathcal{N} = 3$ HSS. The main question is as to how to describe the relevant super Weyl multiplet in terms of unconstrained $\mathcal{N} = 3$ superfields, like this has been done in HSS for conformal $\mathcal{N} = 2$ supergravity. For the time being, we have no answer.

- A wide circle of problems arises in connection with the HSS description of $\mathcal{N} = 2$ supersymmetric higher spins. Some urgent ones are generalizing the theory to AdS type backgrounds and developing the appropriate quantization methods in $\mathcal{N} = 2$ HSS, like those existing in the case of $\mathcal{N} = 2$ SYM theories.

- How to extend the linearized theory of $\mathcal{N} = 2$ higher spins to its full nonlinear version? The latter is known only for $s \leq 2$ ($\mathcal{N} = 2$ super Yang - Mills and $\mathcal{N} = 2$ supergravities). This problem will seemingly require accounting for ALL higher $\mathcal{N} = 2$ superspins simultaneously. New supergeometries?

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THANK YOU FOR YOUR ATTENTION!