Problems of the Modern Mathematical Physics — PMMP

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Quantum properties and exact results in supersymmetric theories revealed with the help of the higher derivative regularization

Although supersymmetry has not yet been discovered experimentally, at present there are some indirect evidences that it is really present in high energy physics. Namely,

- In supersymmetric theories running coupling constants are unified in agreement with the prediction of Grand Unified Theories, see PDG figure

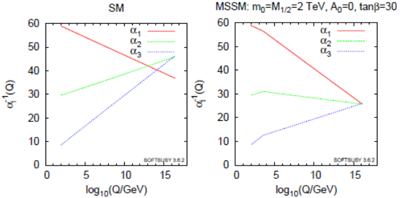


Figure 94.1: Running couplings in SM and MSSM using two-loop RG evolution. The SUSY threshold at 2 TeV is clearly visible on the MSSM side. (We thank Ben Allanach for providing the plots created using SOFTSUSY [61].)

- Supersymmetry forbids quadratically divergent quantum correction to the Higgs boson mass and does not require its fine tuning at the Grand Unification scale.
- Supersymmetry predicts existence of a light Higgs boson with a mass close to m_Z.

Supersymmetric theories at the quantum level

Investigation of quantum corrections in supersymmetric theories is very important both for theory and for phenomenology.

In supersymmetric theories possible ultraviolet divergences are restricted by some non-renormalization theorems. The most known of them are the following:

- $\mathcal{N}=1$ superpotential does not receive divergent quantum corrections.

  M. T. Grisaru, W. Siegel, M. Rocek, Nucl. Phys. B **159** (1979), 429.

- $\mathcal{N}=2$ supersymmetric gauge theories are finite starting from the two-loop approximation.

  M. T. Grisaru, W. Siegel, Nucl. Phys. B **201**, 292 (1982);
  P. S. Howe, K. S. Stelle, P. K. Townsend, Nucl. Phys. B **236**, 125 (1984);
  I. L. Buchbinder, S. M. Kuzenko, B. A. Ovrut, Phys. Lett. B **433**, 335 (1998).

- $\mathcal{N}=4$ supersymmetric Yang–Mills theory is finite in all loops.

  M. F. Sohnius, P. C. West, Phys. Lett. B **100**, 245 (1981);
  S. Mandelstam, Nucl. Phys. B **213**, 149 (1983);
  L. Brink, O. Lindgren, B. E. W. Nilsson, Nucl. Phys. B **212**, 401 (1983).

Other non-renormalization theorems will be discussed below.

The non-renormalization theorems allow constructing finite theories with $\mathcal{N} < 4$ supersymmetry. For $\mathcal{N} = 2$ supersymmetric theories it is made by a special choice of a gauge group and representations for the matter superfields

P. S. Howe, K. S. Stelle, P. C. West, Phys. Lett. B **124**, 55 (1983).

For constructing finite $\mathcal{N} = 1$ supersymmetric theories in a similar way it is also necessary to make a special tuning of a renormalization scheme,

A. Parkes, P. C. West, Phys. Lett. B **138**, 99 (1984);
D. I. Kazakov, Phys. Lett. B **179**, 352 (1986);
A. V. Ermushev, D. I. Kazakov, O. V. Tarasov, Nucl. Phys. B **281**, 72 (1987);
C. Lucchesi, O. Piguet, K. Sibold, Helv. Phys. Acta **61**, 321 (1988);
Phys. Lett. B **201**, 241 (1988).

Also it is possible to construct finite theories with softly broken supersymmetry,

I. Jack, D. R. T. Jones, A. Pickering, Phys. Lett. B **426**, 73 (1998);
D. I. Kazakov, Phys. Lett. B **421**, 211 (1998);
D. I. Kazakov, M. Y. Kalmykov, I. N. Kondrashuk, A. V. Gladyshev, Nucl. Phys. B **471**, 389 (1996).

Thus, supersymmetry is a step towards removing the ultraviolet divergences.

It is convenient to formulate $\mathcal{N}=1$ supersymmetric theories in $\mathcal{N}=1$ superspace, e.g.,

$$S = \frac{1}{2e_0^2}\,\mbox{Re}\,\mbox{tr}\int d^4x\, d^2\theta\, W^a W_a + \frac{1}{4}\int d^4x\, d^4\theta\, \phi^{*i}(e^{2V})_i{}^j\phi_j$$
$$+\Big\{\int d^4x\, d^2\theta\,\Big(\frac{1}{4}m_0^{ij}\phi_i\phi_j + \frac{1}{6}\lambda_0^{ijk}\phi_i\phi_j\phi_k\Big) + \mbox{c.c.}\Big\}.$$

Here V is the gauge superfield, ϕ_i are the chiral matter superfields in the representation R of the gauge group G, and

$$W_a = \frac{1}{8}\bar D^2\Big(e^{-2V}D_a e^{2V}\Big)$$

is the supersymmetric gauge field strength.

The gauge invariant theory is obtained if the Yukawa couplings and masses satisfy the constraints

$$m_0^{im}(T^A)_m{}^j + m_0^{mj}(T^A)_m{}^i = 0;$$
$$\lambda_0^{ijm}(T^A)_m{}^k + \lambda_0^{imk}(T^A)_m{}^j + \lambda_0^{mjk}(T^A)_m{}^i = 0,$$

where $(T^A)_i{}^j$ are the generators of the gauge group G in the representation R.

Most supersymmetric theories have ultraviolet divergences, although non-renormalization theorems impose some restrictions on the renormalization group functions (RGFs). For instance, due to the nonrenormalization of superpotential the renormalizations of masses and Yukawa couplings are related to the renormalization of the matter superfields

$$m^{ij} = m_0^{mn}\big(\sqrt{Z_\phi}\big)_m{}^i\big(\sqrt{Z_\phi}\big)_n{}^j;$$
$$\lambda^{ijk} = \lambda_0^{mnp}\big(\sqrt{Z_\phi}\big)_m{}^i\big(\sqrt{Z_\phi}\big)_n{}^j\big(\sqrt{Z_\phi}\big)_p{}^k,$$

where the renormalization constant for the chiral matter superfields is defined as

$$\phi_i = \big(\sqrt{Z_\phi}\big)_i{}^j\phi_{R,j}.$$

Consequently, the anomalous dimension of the matter superfields is related to the mass anomalous dimensions and, therefore, is gauge independent. Similarly, the Yukawa β-function is also related to the anomalous dimension of the chiral matter superfields by the equation

$$(\beta_\lambda)^{ijk} = \frac{1}{2}\Big((\gamma_\phi)_m{}^i\lambda^{mjk} + (\gamma_\phi)_m{}^j\lambda^{imk} + (\gamma_\phi)_m{}^k\lambda^{ijm}\Big) = \frac{3}{2}\,(\gamma_\phi)_m{}^{(i}\lambda^{jk)m}.$$

Wonderfully, the gauge β-function in supersymmetric theories can also be related to the anomalous dimension of the matter superfields.

The exact Novikov, Shifman, Vainshtein, and Zakharov (NSVZ) β-function can also be considered as a non-renormalization theorem.

V.Novikov, M.A.Shifman, A.Vainshtein, V.I.Zakharov, Nucl.Phys. **B 229** (1983) 381; Phys.Lett. **B 166** (1985) 329; M.A.Shifman, A.I.Vainshtein, Nucl.Phys. **B 277** (1986) 456; D.R.T.Jones, Phys.Lett. **B 123** (1983) 45.

It relates the β-function and the anomalous dimension of the matter superfields in $\mathcal{N}=1$ supersymmetric gauge theories,

$$\beta(\alpha,\lambda) = -\frac{\alpha^2\Big(3C_2 - T(R) + C(R)_i{}^j(\gamma_\phi)_j{}^i(\alpha,\lambda)/r\Big)}{2\pi(1-C_2\alpha/2\pi)}.$$

Here α and λ are the gauge and Yukawa coupling constants, respectively, and we use the notation

$$\mbox{tr}\,(T^A T^B) \equiv T(R)\,\delta^{AB}; \qquad (T^A)_i{}^k (T^A)_k{}^j \equiv C(R)_i{}^j;$$
$$f^{ACD} f^{BCD} \equiv C_2 \delta^{AB}; \qquad r \equiv \delta_{AA} = \mbox{dim}\, G.$$

Explicit calculation and the problem of constructing an NSVZ scheme

Three- and four-loop calculations in $\mathcal{N}=1$ supersymmetric theories made with dimensional reduction supplemented by modified minimal subtraction (i.e. in the so-called $\overline{\text{DR}}$-scheme)

L.V.Avdeev, O.V.Tarasov, Phys.Lett. **112 B** (1982) 356; I.Jack, D.R.T.Jones, C.G.North, Phys.Lett **B386** (1996) 138; Nucl.Phys. **B 486** (1997) 479; R.V.Harlander, D.R.T.Jones, P.Kant, L.Mihaila, M.Steinhauser, JHEP **0612** (2006) 024.

revealed that the NSVZ relation in the $\overline{\text{DR}}$-scheme holds only in the one- and two-loop approximations, where the β-function is scheme independent. (The NSVZ relation relates the two-loop β-function to the one-loop anomalous dimension, which is also scheme independent.)

However, in the three- and four-loop approximations it is possible to restore the NSVZ relation with the help of a specially tuned finite renormalization of the gauge coupling constant. Note that a possibility of making this finite renormalization is highly nontrivial.

This implies that the NSVZ relation holds only in some special renormalization schemes, which are usually called "NSVZ schemes", and the $\overline{\text{DR}}$-scheme is not NSVZ.

The higher covariant derivative regularization

The exact NSVZ β-function can be derived in all loops by direct summation of the perturbative series with the help of the higher covariant derivative regularization proposed by A.A.Slavnov

A.A.Slavnov, Nucl.Phys. **B31**, (1971), 301;
Theor.Math.Phys. **13** (1972) 1064.

By construction, it includes insertion of the Pauli–Villars determinants for removing residual one-loop divergencies

A.A.Slavnov, Theor.Math.Phys. **33**, (1977), 977.

Unlike dimensional reduction, this regularization is self-consistent. It can be formulated in a manifestly supersymmetric way in terms of $\mathcal{N} = 1$ superfields

V.K.Krivoshchekov, Theor.Math.Phys. **36** (1978) 745;
P.West, Nucl.Phys. B268, (1986), 113.

The use of the higher covariant derivative regularization also allows constructing all-loop renormalization prescriptions which give some NSVZ schemes.

NSVZ relation for $\mathcal{N}=1$ supersymmetric electrodynamics

The simplest example of an $\mathcal{N}=1$ supersymmetric gauge theory is $\mathcal{N}=1$ SQED with N_f flavors, which in the massless limit is described by the action

$$S=\frac{1}{4e_0^2}\mbox{Re}\int d^4x\,d^2\theta\,W^aW_a+\sum_{\alpha=1}^{N_f}\frac{1}{4}\int d^4x\,d^4\theta\,\Big(\phi_\alpha^*e^{2V}\phi_\alpha+\widetilde\phi_\alpha^*e^{-2V}\widetilde\phi_\alpha\Big),$$

where V is a real gauge superfield, ϕ_α and $\widetilde\phi_\alpha$ with $\alpha=1,\ldots,N_f$ are chiral matter superfields. The supersymmetric field strength in the Abelian case is defined as $W_a=\bar D^2D_aV/4$. For this theory $C_2=0,\quad C(R)=I,\quad T(R)=2N_f,\quad r=1$, where I is the $2N_f\times 2N_f$ identity matrix.

In this case the NSVZ β-function takes the form

$$\beta(\alpha)=\frac{\alpha^2N_f}{\pi}\Big(1-\gamma(\alpha)\Big).$$

M.A.Shifman, A.I.Vainshtein, V.I.Zakharov, JETP Lett. **42** (1985) 224; Phys.Lett. **B 166** (1986) 334.

This equation relates the L-loop β-function to the $(L-1)$-loop anomalous dimension of the matter superfields $\gamma(\alpha)$.

The higher derivative regularization for $\mathcal{N}=1$ supersymmetric electrodynamics

To regularize $\mathcal{N}=1$ SQED by higher derivatives, we first add to its action a term containing higher derivatives. Then the regularized action takes the form

$$S_{\text{reg}} = \frac{1}{4e_0^2}\text{Re}\int d^4x\, d^2\theta\, W^a R(\partial^2/\Lambda^2) W_a + \sum_{\alpha=1}^{N_f} \frac{1}{4}\int d^4x\, d^4\theta\, \Big(\phi_\alpha^* e^{2V}\phi_\alpha + \widetilde{\phi}_\alpha^* e^{-2V}\widetilde{\phi}_\alpha\Big),$$

where $R(\partial^2/\Lambda^2)$ is a regulator function, e.g., $R = 1 + \partial^{2n}/\Lambda^{2n}$.

After the adding of the higher derivative term divergences survive only in the one-loop approximation. To regularize the residual one-loop (sub)divergences, we insert the Pauli–Villars determinants into the generating functional,

$$Z[J,j,\widetilde{j}] = \int D\mu\, \Big(\det PV(V,M)\Big)^{N_f} \exp\Big\{iS_{\text{reg}} + iS_{\text{gf}} + S_{\text{sources}}\Big\}.$$

Masses of the Pauli–Villars superfields should satisfy the important condition $M = a\Lambda$ with $a \neq a(e_0)$.

Different definition of renormalization group functions

It is important to distinguish RGFs defined in terms of the bare coupling constant α_0,

$$\beta(\alpha_0) \equiv \frac{d\alpha_0(\alpha,\Lambda/\mu)}{d\ln\Lambda}\Big|_{\alpha=\text{const}}; \qquad \gamma(\alpha_0) \equiv -\frac{d\ln Z(\alpha,\Lambda/\mu)}{d\ln\Lambda}\Big|_{\alpha=\text{const}},$$

and RGFs standardly defined in terms of the renormalized coupling constant α,

$$\widetilde{\beta}(\alpha) \equiv \frac{d\alpha(\alpha_0,\Lambda/\mu)}{d\ln\mu}\Big|_{\alpha_0=\text{const}}; \qquad \widetilde{\gamma}(\alpha) \equiv \frac{d\ln Z(\alpha_0,\Lambda/\mu)}{d\ln\mu}\Big|_{\alpha_0=\text{const}}.$$

A.L.Kataev and K.S., Nucl.Phys. **B875** (2013) 459.

RGFs defined in terms of the bare coupling constant do not depend on a renormalization prescription for a fixed regularization, but depend on a regularization.

RGFs defrined in terms of the renormalized coupling constant depend both on regularization and on a renormalization prescription.

The HD+MSL scheme. Integrals of double total derivatives.

Both definitions of RGFs give the same functions in the HD+MSL-scheme, when a theory is regularized by Higher Derivatives, and divergences are removed by Minimal Subtractions of Logarithms. This means that the renormalization constants include only powers of $\ln\Lambda/\mu$, where μ is a renormalization point.

$$\widetilde{\beta}(\alpha)\Big|_{\text{HD+MSL}} = \beta(\alpha_0\to\alpha);\qquad \widetilde{\gamma}(\alpha)\Big|_{\text{HD+MSL}} = \gamma(\alpha_0\to\alpha).$$

A key observation needed for derivation of the NSVZ relation is that in the case of using the higher derivative regularization the integrals giving the β-function defined in terms of the bare coupling constant are integrals of double total derivatives in $\mathcal{N}=1$ supersymmetric gauge theories.

A.A.Soloshenko, K.S., ArXiv: hep-th/0304083v1 (the factorization into total derivatives);
A.V.Smilga, A.I.Vainshtein, Nucl.Phys. **B 704** (2005) 445 (the factorization into double total derivatives).

The three-loop β-function of $\mathcal{N}=1$ SQED as an integral of double total derivatives

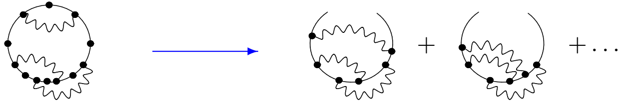
$$\frac{\beta(\alpha_0)}{\alpha_0^2} = N_f \frac{d}{d\ln\Lambda}\bigg\{2\pi\int\frac{d^4Q}{(2\pi)^4}\frac{\partial}{\partial Q^\mu}\frac{\partial}{\partial Q_\mu}\frac{\ln(Q^2+M^2)}{Q^2} + 4\pi\int\frac{d^4Q}{(2\pi)^4}\frac{d^4K}{(2\pi)^4}\frac{e^2}{K^2R_K^2}$$
$$\times\frac{\partial}{\partial Q^\mu}\frac{\partial}{\partial Q_\mu}\bigg(\frac{1}{Q^2(K+Q)^2} - \frac{1}{(Q^2+M^2)((K+Q)^2+M^2)}\bigg)\bigg[R_K\Big(1+\frac{e^2N_f}{4\pi^2}\ln\frac{\Lambda}{\mu}\Big)$$
$$-2e^2N_f\bigg(\int\frac{d^4L}{(2\pi)^4}\frac{1}{L^2(K+L)^2} - \int\frac{d^4L}{(2\pi)^4}\frac{1}{(L^2+M^2)((K+L)^2+M^2)}\bigg)\bigg]$$
$$+4\pi\int\frac{d^4Q}{(2\pi)^4}\frac{d^4K}{(2\pi)^4}\frac{d^4L}{(2\pi)^4}\frac{e^4}{K^2R_KL^2R_L}\frac{\partial}{\partial Q^\mu}\frac{\partial}{\partial Q_\mu}\bigg\{\bigg(-\frac{2K^2}{Q^2(Q+K)^2(Q+K+L)^2}$$
$$\times\frac{1}{(Q+L)^2} + \frac{2}{Q^2(Q+K)^2(Q+L)^2}\bigg) - \bigg(-\frac{2(K^2+M^2)}{((Q+K)^2+M^2)((Q+L)^2+M^2)}$$
$$\times\frac{1}{(Q^2+M^2)((Q+K+L)^2+M^2)} + \frac{2}{(Q^2+M^2)((Q+K)^2+M^2)((Q+L)^2+M^2)}$$
$$-\frac{4M^2}{(Q^2+M^2)^2((Q+K)^2+M^2)((Q+L)^2+M^2)}\bigg) + O(e^6)\bigg\}$$

Integrals of double total derivatives and a graphical interpretation of the NSVZ relation for $\mathcal{N}=1$ SQED

The integrals of double total derivatives do not vanish due to singularities of the integrands. Really, if $f(Q^2)$ is a non-singular function which rapidly decrease at infinity, then

$$\int \frac{d^4Q}{(2\pi)^4} \frac{\partial}{\partial Q^\mu} \frac{\partial}{\partial Q_\mu} \Big(\frac{f(Q^2)}{Q^2}\Big) = \int\limits_{S^3_\varepsilon} \frac{dS^\mu}{(2\pi)^4} \Big(- \frac{2Q_\mu}{Q^4} f(Q^2) + \frac{2Q_\mu}{Q^2} f'(Q^2) \Big)$$
$$= \frac{1}{4\pi^2} f(0) \ne 0.$$

Due to similar equations the double total derivatives effectively cut a loop of the matter superfields. As a result we obtain diagrams contributing to the anomalous dimension of the matter superfields, in which a number of loops is less by 1.

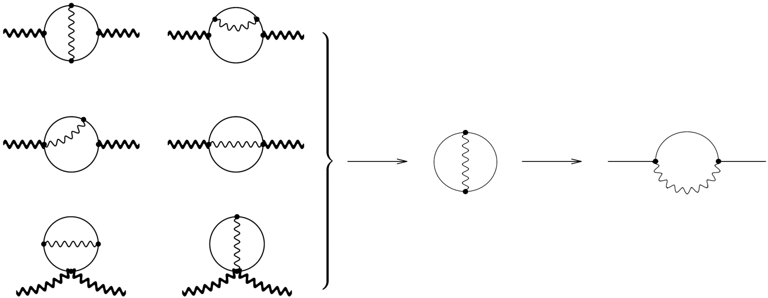


This allows to give a simple graphical interpretation of the NSVZ relation for the considered Abelian case.

Graphical interpretation of the NSVZ relation for $\mathcal{N}=1$ SQED

For each vacuum supergraph the NSVZ equation relates a contribution to the β-function obtained by attaching two external lines of the gauge superfield to the corresponding contribution to the anomalous dimension of matter superfields obtained by cuts of the matter line:

A.V.Smilga, A.I.Vainshtein, Nucl.Phys. **B 704** (2005) 445.



RGFS defined in terms of the bare coupling constant obtained after calculating the integrals of double total derivatives and the integrals which determine the two-loop anomalous dimension are given by the expressions

$$\frac{\beta(\alpha_0)}{\alpha_0^2} = \frac{N_f}{\pi} + \frac{\alpha_0 N_f}{\pi^2} - \frac{\alpha_0^2 N_f}{\pi^3}\Big(N_f \ln a + N_f + \frac{N_f A}{2} + \frac{1}{2}\Big) + O(\alpha_0^3);$$

$$\gamma(\alpha_0) = -\frac{\alpha_0}{\pi} + \frac{\alpha_0^2}{\pi^2}\Big(N_f \ln a + N_f + \frac{N_f A}{2} + \frac{1}{2}\Big) + O(\alpha_0^3),$$

where

$$A \equiv \int\limits_0^\infty dx\, \ln x\, \frac{d}{dx}\frac{1}{R(x)}; \qquad a = \frac{M}{\Lambda}.$$

They do not depend on finite constants b_i and g_i, which specify the renormalization scheme and satisfy the NSVZ relation.

RGFs defined in terms of the renormalized coupling constant are written as

$$\frac{\widetilde{\beta}(\alpha)}{\alpha^2} = \frac{N_f}{\pi} + \frac{\alpha N_f}{\pi^2} - \frac{\alpha^2 N_f}{\pi^3}\Big(N_f \ln a + N_f + \frac{N_f A}{2} + \frac{1}{2} + N_f(b_2 - b_1)\Big) + O(\alpha^3)$$

$$\widetilde{\gamma}(\alpha) = -\frac{\alpha}{\pi} + \frac{\alpha^2}{\pi^2}\Big(N_f \ln a + N_f + \frac{N_f A}{2} + \frac{1}{2} - N_f b_1 + N_f g_1\Big) + O(\alpha^3).$$

Finite constants determining a renormalization scheme

We see that RGFs defined in terms of the renormalized coupling constant depend on a renormalization scheme due to the dependence of the finite constants b_i and g_i. The constants b_i in the considered three-loop approximation are defined by the equation

$$\frac{1}{\alpha_0} = \frac{1}{\alpha} - \frac{N_f}{\pi}\Big(\ln\frac{\Lambda}{\mu} + b_1\Big) - \frac{\alpha N_f}{\pi^2}\Big(\ln\frac{\Lambda}{\mu} + b_2\Big) - \frac{\alpha^2 N_f}{\pi^3}\Big(\frac{N_f}{2}\ln^2\frac{\Lambda}{\mu} - \ln\frac{\Lambda}{\mu}\Big(N_f\ln a + N_f + \frac{N_f A}{2} + \frac{1}{2} - N_f b_1\Big) + b_3\Big) + O(\alpha^3).$$

Similarly, the finite constants g_i appear in the two-loop expression for the renormalization constant of the matter superfields Z, which is not also uniquely defined,

$$Z = 1 + \frac{\alpha}{\pi}\Big(\ln\frac{\Lambda}{\mu} + g_1\Big) + \frac{\alpha^2(N_f+1)}{2\pi^2}\ln^2\frac{\Lambda}{\mu} - \frac{\alpha^2}{\pi^2}\ln\frac{\Lambda}{\mu}\Big(N_f\ln a - N_f b_1 + N_f + \frac{N_f A}{2} + \frac{1}{2} - g_1\Big) + \frac{\alpha^2 g_2}{\pi^2} + O(\alpha^3).$$

The choice of the constants b_i and g_i fixes a renormalization scheme in the considered approximation.

The HD+MSL scheme

In the HD+MSL scheme all these finite constants vanish,

$$g_2 = b_1 = b_2 = b_3 = 0,$$

and both definition of RGFs give the same functions up to the formal replacing of arguments. In particular, in the considered approximation

$$\frac{\widetilde{\beta}(\alpha)}{\alpha^2} = \frac{N_f}{\pi} + \frac{\alpha N_f}{\pi^2} - \frac{\alpha^2 N_f}{\pi^3}\Big(N_f \ln a + N_f + \frac{N_f A}{2} + \frac{1}{2}\Big) + O(\alpha^3) = \frac{\beta(\alpha)}{\alpha^2};$$

$$\widetilde{\gamma}(\alpha) = \frac{d \ln Z}{d \ln \mu} = -\frac{\alpha}{\pi} + \frac{\alpha^2}{\pi^2}\Big(N_f + \frac{N_f A}{2} + \frac{1}{2} + N_f \ln a\Big) + O(\alpha^3) = \gamma(\alpha).$$

That is why in this scheme the NSVZ equation is valid. It turns out that it is so in all orders of the perturbation theory.

Below we will compare explicit expressions for RGFs for some special renormalization schemes. For the HD+MSL and MOM schemes they were obtained in

A.L.Kataev, K.S., Phys.Lett. **B730** (2014) 184; Theor.Math.Phys. **181** (2014) 1531; A.E.Kazantsev, K.S., JHEP **06** (2020) 108.

Renormalization group functions for different renormalization prescriptions

The HD+MSL-scheme

$$\widetilde{\gamma}_{\text{HD+MSL}}(\alpha) = -\frac{\alpha}{\pi} + \frac{\alpha^2}{\pi^2}\Big(\frac{1}{2} + N_f \ln a + N_f + \frac{N_f A}{2}\Big) + O(\alpha^3);$$

$$\widetilde{\beta}_{\text{HD+MSL}}(\alpha) = \frac{\alpha^2 N_f}{\pi}\Big(1 + \frac{\alpha}{\pi} - \frac{\alpha^2}{\pi^2}\Big(\frac{1}{2} + N_f \ln a + N_f + \frac{N_f A}{2}\Big) + O(\alpha^3)\Big).$$

The MOM-scheme (The result is the same of dimensional reduction and the higher derivative regularization.)

$$\widetilde{\gamma}_{\text{MOM}}(\alpha) = -\frac{\alpha}{\pi} + \frac{\alpha^2(1+N_f)}{2\pi^2} + O(\alpha^3);$$

$$\widetilde{\beta}_{\text{MOM}}(\alpha) = \frac{\alpha^2 N_f}{\pi}\Big(1 + \frac{\alpha}{\pi} - \frac{\alpha^2}{2\pi^2}\Big(1 + 3N_f\,(1-\zeta(3))\Big) + O(\alpha^3)\Big).$$

The $\overline{\text{DR}}$-scheme

I. Jack, D.R.T. Jones and C.G. North, Phys. Lett. **B386** (1996) 138.

$$\widetilde{\gamma}_{\overline{\text{DR}}}(\alpha) = -\frac{\alpha}{\pi} + \frac{\alpha^2(2+2N_f)}{4\pi^2} + O(\alpha^3);$$

$$\widetilde{\beta}_{\overline{\text{DR}}}(\alpha) = \frac{\alpha^2 N_f}{\pi}\Big(1 + \frac{\alpha}{\pi} - \frac{\alpha^2(2+3N_f)}{4\pi^2} + O(\alpha^3)\Big).$$

Similarly, it is possible to obtain the four-loop β-function of $\mathcal{N}=1$ SQED with N_f flavors and the three-loop anomalous dimension

I. Shirokov and K.S., JHEP 04 (2022) 108;
I. Shirokov, V. Shirokova, arXiv:2310.13109 [hep-th].

The result (for RGFs defined in terms of the renormalized coupling constant) is

$$\widetilde{\gamma}(\alpha) = -\frac{\alpha}{\pi} + \frac{\alpha^2}{2\pi^2} + \frac{\alpha^2 N_f}{\pi^2}\Big(\ln a + 1 + \frac{A_1}{2} + g_{1,0} - b_{1,0}\Big) - \frac{\alpha^3}{2\pi^3} + \frac{\alpha^3 N_f}{\pi^3} \times\Big(-\ln a - \frac{3}{4} - C - b_{2,0} + b_{1,0} - g_{2,0} + g_{1,0}\Big) + \frac{\alpha^3 (N_f)^2}{\pi^3}\Big\{-\Big(\ln a + 1 - b_{1,0}\Big)^2 + \frac{A_2}{4} - D_1 \ln a - D_2 + b_{1,0}A_1 - g_{2,1}\Big\} + O(\alpha^4);$$

$$\frac{\widetilde{\beta}(\alpha)}{\alpha^2} = \frac{N_f}{\pi} + \frac{\alpha N_f}{\pi^2} - \frac{\alpha^2 N_f}{2\pi^3} - \frac{\alpha^2 (N_f)^2}{\pi^3}\Big(\ln a + 1 + \frac{A_1}{2} + b_{2,0} - b_{1,0}\Big) + \frac{\alpha^3 N_f}{2\pi^4} + \frac{\alpha^3 (N_f)^2}{\pi^4}\Big(\ln a + \frac{3}{4} + C + b_{3,0} - b_{1,0}\Big) + \frac{\alpha^3 (N_f)^3}{\pi^4}\Big\{\Big(\ln a + 1 - b_{1,0}\Big)^2 - \frac{A_2}{4} + D_1 \ln a + D_2 - b_{1,0}A_1 + b_{3,1}\Big\} + O(\alpha^4).$$

Notations

Here the notations are

$$A_1 \equiv \int\limits_0^\infty dx\, \ln x\, \frac{d}{dx}\Big(\frac{1}{R(x)}\Big); \qquad A_2 \equiv \int\limits_0^\infty dx\, \ln^2 x\, \frac{d}{dx}\Big(\frac{1}{R(x)}\Big);$$

$$C \equiv \int\limits_0^1 dx \int\limits_0^\infty dy\, x\, \ln y\, \frac{d}{dy}\Big(\frac{1}{R(y)R(x^2 y)}\Big); \qquad D_1 \equiv \int\limits_0^\infty dx\, \ln x\, \frac{d}{dx}\Big(\frac{1}{R^2(x)}\Big);$$

$$D_2 \equiv \int\limits_0^\infty dx\, \ln x\, \frac{d}{dx}\left\{\frac{1}{R^2(x)}\Big[-\frac{1}{2}\Big(1-R(x)\Big)\ln x + \sqrt{1+\frac{4a^2}{x}}\,\mathrm{arctanh}\sqrt{\frac{x}{x+4a^2}}\,\Big]\right\}.$$

and the finite constants are defined by the equations

$$\ln Z = \frac{\alpha}{\pi}\Big(\ln\frac{\Lambda}{\mu} + g_{1,0}\Big) - \frac{\alpha^2}{2\pi^2}\Big(\ln\frac{\Lambda}{\mu} + g_{2,0} + N_f g_{2,1}\Big) - \frac{\alpha^2 N_f}{\pi^2}\Big(\ln a + 1 + \frac{A_1}{2} - b_{1,0}\Big)\ln\frac{\Lambda}{\mu} + \frac{\alpha^2 N_f}{2\pi^2}\ln^2\frac{\Lambda}{\mu} + O(\alpha^3).$$

$$\frac{1}{\alpha_0} = \frac{1}{\alpha} - \frac{N_f}{\pi}\Big(\ln\frac{\Lambda}{\mu} + b_{1,0}\Big) - \frac{\alpha N_f}{\pi^2}\Big(\ln\frac{\Lambda}{\mu} + b_{2,0}\Big) + \frac{\alpha^2 N_f}{2\pi^3}\Big(\ln\frac{\Lambda}{\mu} + b_{3,0} + N_f b_{3,1}\Big) + \frac{\alpha^2 (N_f)^2}{\pi^3}\Big(\ln a + 1 + \frac{A_1}{2} - b_{1,0}\Big)\ln\frac{\Lambda}{\mu} - \frac{\alpha^2 (N_f)^2}{2\pi^3}\ln^2\frac{\Lambda}{\mu} + O(\alpha^3).$$

We see that the terms in the anomalous dimension without N_f and terms in the β-function proportional to $(N_f)^1$ are scheme-independent in agreement with the general all-loop statement proved in

A.L.Kataev and K.S., Phys. Lett. B **730** (2014), 184;
Theor. Math. Phys. **181** (2014), 1531.

From the explicit above expressions for RGFs we see that by a special choice of the finite constants b_i and g_i it is possible to remove all terms proportional to $(N_f)^k$ with $k\geq 1$ in the anomalous dimension and all terms proportional to $(N_f)^k$ with $k\geq 2$ in the β-function. Then we obtain the simplest, so-called minimal scheme, in which

$$\widetilde{\gamma}(\alpha) = -\frac{\alpha}{\pi} + \frac{\alpha^2}{2\pi^2} - \frac{\alpha^3}{2\pi^3} + O(\alpha^4);$$
$$\widetilde{\beta}(\alpha) = \frac{\alpha^2 N_f}{\pi} + \frac{\alpha^3 N_f}{\pi^2} - \frac{\alpha^4 N_f}{2\pi^3} + \frac{\alpha^5 N_f}{2\pi^4} + O(\alpha^6).$$

The minimal renormalization scheme for the considered theory can be chosen in all orders of the perturbation theory. This scheme is NSVZ in all orders.

The NSVZ β-function for $\mathcal{N}=1$ supersymmetric electrodynamics as a sum of singularities

In all loops the expression for the β-function determined by singular contributions has been calculated in

K.S., Nucl.Phys. **B 852** (2011) 71.

The result is the NSVZ relation

$$\frac{\beta(\alpha_0)}{\alpha_0^2} = \frac{N_f}{\pi}\Big(1-\gamma(\alpha_0)\Big).$$

Therefore, RGFs defined in terms of the bare coupling constant satisfy the NSVZ relation in all orders for an arbitrary ξ-gauge and for an arbitrary renormalization prescription which supplements the higher derivative regularization.

Consequently, for RGFs defined in terms of the renormalized coupling constant the HD+MSL prescription gives some NSVZ schemes, so that

$$\frac{\widetilde\beta(\alpha)}{\alpha^2} = \frac{N_f}{\pi}\Big(1-\widetilde\gamma(\alpha)\Big).$$

A class of the NSVZ schemes for $\mathcal{N}=1$ supersymmetric electrodynamics

Different subtraction schemes are related by finite renormalizations

$$\alpha' = \alpha'(\alpha); \qquad Z' = z(\alpha) Z,$$

where $\alpha'(\alpha)$ and $z(\alpha)$ are finite functions of the coupling constant. Then RGFs change as

$$\widetilde\beta(\alpha') = \frac{d\alpha'}{d\alpha}\widetilde\beta(\alpha); \qquad \widetilde\gamma(\alpha') = \widetilde\beta(\alpha)\frac{d\ln z}{d\alpha} + \widetilde\gamma(\alpha).$$

A.A.Vladimirov, Sov.J.Nucl.Phys. **31** (1980) 558; Theor.Math.Phys. **25** (1976) 1170.

The finite renormalizations relating various NSVZ schemes should satisfy the constraint

$$\frac{1}{\alpha'(\alpha)} - \frac{1}{\alpha} - \frac{N_f}{\pi}\ln z(\alpha) = B,$$

where B is a constant.

I. O. Goriachuk, A. L. Kataev, K.S., Phys. Lett. B **785** (2018), 561.

The on-shell scheme is also NSVZ in all orders

A. L. Kataev, A. E. Kazantsev, K.S., Eur. Phys. J. C **79** (2019) 477.

Renormalizable non-Abelian $\mathcal{N}=1$ supersymmetric gauge theories with matter superfields at the classical level are described by the action

$$S = \frac{1}{2e_0^2}\,\mbox{Re}\,\mbox{tr}\int d^4x\, d^2\theta\, W^a W_a + \frac{1}{4}\int d^4x\, d^4\theta\, \phi^{*i}(e^{2V})_i{}^j\phi_j$$
$$+\Big\{\int d^4x\, d^2\theta\, \Big(\frac{1}{4}m_0^{ij}\phi_i\phi_j + \frac{1}{6}\lambda_0^{ijk}\phi_i\phi_j\phi_k\Big) + \mbox{c.c.}\Big\}.$$

We assume that the gauge group is simple, and the chiral matter superfields ϕ_i lie in its representation R.

For quantizing the theory it is convenient to use the background field method. Moreover, it is necessary to take into account nonlinear renormalization of the quantum gauge superfield

O. Piguet and K. Sibold, Nucl.Phys. **B197** (1982) 257; 272;
I.V.Tyutin, Yad.Fiz. **37** (1983) 761.

This can be done with the help of the replacement $e^{2V} \to e^{2\mathcal{F}(V)}e^{2\boldsymbol{V}}$, where $\boldsymbol{V}$ and V are the background and quantum gauge superfields, respectively, and the function $\mathcal{F}(V)$ includes an infinite set of parameters needed for describing the nonlinear renormalization.

The background superfield method and the nonlinear renormalization

In the lowest order the function describing the nonlinear renormalization is given by the expression

J.W.Juer and D.Storey, Phys.Lett. **119B** (1982) 125; Nucl. Phys. **B216** (1983) 185.

$$\mathcal{F}(V)^A = V^A + e_0^2\, y_0\, G^{ABCD}\, V^B V^C V^D + \ldots,$$

where y_0 is one of the constants entering this set, and G^{ABCD} is a certain function of the structure constants.

In the case of using the background (super)field method the original gauge invariance produces two invariances, namely, the background gauge invariance and the quantum gauge invariance.

The background gauge invariance

$$\phi_i \to (e^A)_i{}^j \phi_j; \qquad V \to e^{-A^+} V e^{A^+}; \qquad e^{2\boldsymbol{V}} \to e^{-A^+} e^{2\boldsymbol{V}} e^{-A}$$

parameterized by a chiral superfield A remains a manifest symmetry of the effective action.

The quantum gauge invariance is broken down to the BRST symmetry after the gauge fixing procedure.

For constructing the regularized theory we first add to its action terms with higher derivatives,

$$S_{\text{reg}} = \frac{1}{2e_0^2}\,\text{Re}\,\text{tr}\int d^4x\, d^2\theta\, W^a \Big(e^{-2\boldsymbol{V}} e^{-2\mathcal{F}(V)}\Big)_{Adj} R\Big(-\frac{\bar\nabla^2\nabla^2}{16\Lambda^2}\Big)_{Adj}$$
$$\times\Big(e^{2\mathcal{F}(V)}e^{2\boldsymbol{V}}\Big)_{Adj} W_a + \frac{1}{4}\int d^4x\, d^4\theta\, \phi^{*i}\Big[F\Big(-\frac{\bar\nabla^2\nabla^2}{16\Lambda^2}\Big)e^{2\mathcal{F}(V)}e^{2\boldsymbol{V}}\Big]_i{}^j\phi_j$$
$$+\Big[\int d^4x\, d^2\theta\,\Big(\frac{1}{4}m_0^{ij}\phi_i\phi_j + \frac{1}{6}\lambda_0^{ijk}\phi_i\phi_j\phi_k\Big) + \text{c.c.}\Big],$$

where the covariant derivatives are defined as

$$\nabla_a = D_a; \qquad \bar\nabla_{\dot a} = e^{2\mathcal{F}(V)}e^{2\boldsymbol{V}}\,\bar D_{\dot a} e^{-2\boldsymbol{V}}e^{-2\mathcal{F}(V)}.$$

Gauge is fixed by adding the term

$$S_{\text{gf}} = -\frac{1}{16\xi_0 e_0^2}\,\text{tr}\int d^4x\, d^4\theta\, \boldsymbol{\nabla}^2 V K\Big(-\frac{\bar{\boldsymbol{\nabla}}^2\boldsymbol{\nabla}^2}{16\Lambda^2}\Big)_{Adj}\bar{\boldsymbol{\nabla}}^2 V.$$

Also it is necessary to introduce the Faddeev-Popov and Nielsen–Kalosh ghosts. The regulator functions $R(x)$, $F(x)$, and $K(x)$ should rapidly increase at infinity and satisfy the condition $R(0) = F(0) = K(0) = 1$.

For regularizing residual one-loop divergences we insert into the generating functional two Pauli–Villars determinants,

$$Z = \int D\mu\, \mathsf{Det}(PV, M_\varphi)^{-1} \mathsf{Det}(PV, M)^c \times \exp\Big\{ i\Big(S_{\text{reg}} + S_{\text{gf}} + S_{\text{FP}} + S_{\text{NK}} + S_{\text{sources}} \Big) \Big\},$$

where $D\mu$ is the functional integration measure, and

$$\mathsf{Det}(PV, M_\varphi)^{-1} \equiv \int D\varphi_1\, D\varphi_2\, D\varphi_3\, \exp(iS_\varphi);$$

$$\mathsf{Det}(PV, M)^{-1} \equiv \int D\Phi\, \exp(iS_\Phi).$$

Here we use chiral commuting Pauli–Villars superfields.
The superfields $\varphi_{1,2,3}$ belong to the adjoint representation and cancel one-loop divergences coming from gauge and ghost loops. The superfields Φ_i lie in a representation R_{PV} and cancel divergences coming from a loop of the matter superfields if $c = T(R)/T(R_{\text{PV}})$. The masses of these superfields are

$$M_\varphi = a_\varphi \Lambda; \qquad M = a\Lambda,$$

where the coefficients a_φ and a do not depend on couplings.

The all-loop derivation of the NSVZ equation: the main steps

1. First, one proves the ultraviolet finiteness of triple vertices with two external lines of the Faddeev–Popov ghosts and one external line of the quantum gauge superfield.

2. Next, it is necessary to rewrite the NSVZ relation in the equivalent form

$$\frac{\beta(\alpha_0,\lambda_0)}{\alpha_0^2} = -\frac{1}{2\pi}\Big(3C_2 - T(R) - 2C_2\gamma_c(\alpha_0,\lambda_0) - 2C_2\gamma_V(\alpha_0,\lambda_0) + C(R)_i{}^j(\gamma_\phi)_j{}^i(\alpha_0,\lambda_0)/r\Big).$$

K.S., Nucl.Phys. **B909** (2016) 316.

3. After that, we prove that the β-function is determined by integrals of double total derivatives with respect to loop momenta and present a method for constructing this integrals.

K.S., JHEP **10** (2019) 011.

4. Then the NSVZ equation is obtained by summing singular contributions.

5. Finally, an NSVZ scheme is constructed.

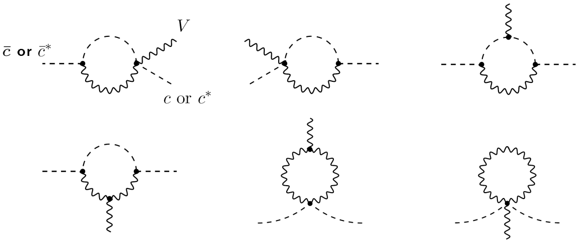
K.S., Eur.Phys.J. **C80** (2020) 10, 911.

Non-renormalization of the three-point gauge-ghost vertices

An important statement needed for proving the NSVZ equation in the non-Abelian case is the all-order finiteness of triple vertices in which two external lines correspond to the Faddeev–Popov ghosts and one external line corresponds to the quantum gauge superfield.

K.S., Nucl.Phys. **B909** (2016) 316.

The one-loop contribution to these vertices comes from the superdiagrams presented below. The ultraviolet finiteness of their sum has been verified by an explicit calculation



Example: a part of the one-loop expression for one of the triple gauge-ghost vertices

A part of the effective action corresponding to the $\bar{c}^+ V c$ vertex is written as

$$\frac{ie_0}{4} f^{ABC} \int d^4\theta\, \frac{d^4p}{(2\pi)^4} \frac{d^4q}{(2\pi)^4} \bar{c}^{*A}(\theta, p+q) \Big(f(p,q) \partial^2 \Pi_{1/2} V^B(\theta,-p)$$
$$+ F_\mu(p,q) (\gamma^\mu)_{\dot{a}}{}^b D_b \bar{D}^{\dot{a}} V^B(\theta,-p) + F(p,q) V^B(\theta,-p) \Big) c^C(\theta,-q).$$

After the Wick rotation the sum of the tree and one-loop contributions to the function F is given by

$$F(P,Q) = 1 + \frac{e_0^2 C_2}{4} \int \frac{d^4K}{(2\pi)^4} \Big\{ - \frac{(Q+P)^2}{R_K K^2 (K+P)^2 (K-Q)^2} - \frac{\xi_0\, P^2}{K_K K^2 (K+Q)^2}$$
$$\times \frac{1}{(K+P+Q)^2} + \frac{\xi_0\, Q^2}{K_K K^2 (K+P)^2 (K+Q+P)^2} + \Big(\frac{\xi_0}{K_K} - \frac{1}{R_K} \Big)$$
$$\times \Big(- \frac{2(Q+P)^2}{K^4 (K+Q+P)^2} + \frac{2}{K^2 (K+Q+P)^2} - \frac{1}{K^2 (K+Q)^2} - \frac{1}{K^2 (K+P)^2} \Big) \Big\}$$
$$+ O(\alpha_0^2, \alpha_0 \lambda_0^2).$$

We see that this expression is finite in the ultraviolet region.

The all-loop proof is based on the superfield Feynman rules and the Slavnov–Taylor identities. It is valid in the case of using the superfield quantization for an arbitrary ξ-gauge.

There are 4 vertices of the considered structure, $\bar{c}Vc$, $\bar{c}^+Vc$, $\bar{c}Vc^+$, and $\bar{c}^+Vc^+$. All of them have renormalization constant $Z_\alpha^{-1/2}Z_cZ_V$. Therefore, due to their finiteness

$$\frac{d}{d\ln\Lambda}(Z_\alpha^{-1/2}Z_cZ_V) = 0,$$

where the renormalization constants are defined by the equations

$$\frac{1}{\alpha_0} = \frac{Z_\alpha}{\alpha}; \qquad \boldsymbol{V} = \boldsymbol{V}_R; \qquad V = Z_V Z_\alpha^{-1/2} V_R; \qquad \bar{c}c = Z_c Z_\alpha^{-1}\bar{c}_R c_R.$$

The explicit two-loop verification of the finiteness of the triple gauge-ghost vertices has been done in

M. Kuzmichev, N. Meshcheriakov, S. Novgorodtsev, I. Shirokov, K.S., Phys. Rev. D **104** (2021) 025008;
M. Kuzmichev, N. Meshcheriakov, S. Novgorodtsev, V. Shatalova, I. Shirokov, K.S., Eur. Phys. J. C **82** (2022) 69.

Non-renormalization of the triple gauge-ghost vertices and the new form of the NSVZ β-function

The non-Abelian NSVZ equation can be equivalently rewritten as

$$\frac{\beta(\alpha_0,\lambda_0)}{\alpha_0^2} = -\frac{3C_2 - T(R) + C(R)_i{}^j(\gamma_\phi)_j{}^i(\alpha_0,\lambda_0)/r}{2\pi} + \frac{C_2}{2\pi}\cdot\frac{\beta(\alpha_0,\lambda_0)}{\alpha_0}.$$

The β-function in the right hand side can be expressed in terms of the charge renormalization constant Z_α:

$$\beta(\alpha_0,\lambda_0) = \frac{d\alpha_0(\alpha,\lambda,\Lambda/\mu)}{d\ln\Lambda}\Big|_{\alpha,\lambda=\mathrm{const}} = -\alpha_0\frac{d\ln Z_\alpha}{d\ln\Lambda}\Big|_{\alpha,\lambda=\mathrm{const}}.$$

Using the finiteness of the gauge-ghost vertices we obtain

$$\beta(\alpha_0,\lambda_0) = -2\alpha_0\frac{d\ln(Z_c Z_V)}{d\ln\Lambda}\Big|_{\alpha,\lambda=\mathrm{const}} = 2\alpha_0\Big(\gamma_c(\alpha_0,\lambda_0) + \gamma_V(\alpha_0,\lambda_0)\Big),$$

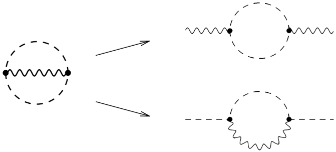
where γ_c and γ_V are the anomalous dimensions of the Faddeev–Popov ghosts and of the quantum gauge superfield (defined in terms of the bare couplings), respectively.

Substituting this expression into the the right hand side we obtain the equivalent form of the NSVZ equation

$$\frac{\beta(\alpha_0,\lambda_0)}{\alpha_0^2} = -\frac{1}{2\pi}\Big(3C_2 - T(R) - 2C_2\gamma_c(\alpha_0,\lambda_0)$$
$$-2C_2\gamma_V(\alpha_0,\lambda_0) + C(R)_i{}^j(\gamma_\phi)_j{}^i(\alpha_0,\lambda_0)/r\Big).$$

It relates the β-function in a certain loop to the anomalous dimensions of quantum superfields in the previous loop, because the right hand side does not contain a denominator depending on couplings.

The new form of the NSVZ equation has a graphical interpretation similar to the Abelian case:



The β-function of $\mathcal{N}=1$ supersymmetric gauge theories as an integral of double total derivatives

In the non-Abelian case the integrals giving the β-function are also integrals of double total derivatives if a supersymmetric theory is regularized by higher covariant derivatives. For instance, the expression for three-loop terms quartic in the Yukawa couplings is

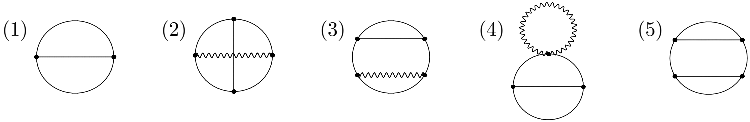
$$\frac{\Delta\beta(\alpha_0,\lambda_0)}{\alpha_0^2} = -\frac{2\pi}{r}C(R)_i{}^j\frac{d}{d\ln\Lambda}\int\frac{d^4K}{(2\pi)^4}\frac{d^4Q}{(2\pi)^4}\lambda_0^{imn}\lambda^*_{0jmn}\frac{\partial}{\partial Q_\mu}\frac{\partial}{\partial Q^\mu}\Big(\frac{1}{K^2}$$
$$\times\frac{1}{F_K\,Q^2F_Q\,(Q+K)^2F_{Q+K}}\Big) + \frac{4\pi}{r}C(R)_i{}^j\frac{d}{d\ln\Lambda}\int\frac{d^4K}{(2\pi)^4}\frac{d^4L}{(2\pi)^4}\frac{d^4Q}{(2\pi)^4}\Big[\lambda_0^{iab}$$
$$\times\lambda^*_{0kab}\lambda_0^{kcd}\lambda^*_{0jcd}\Big(\frac{\partial}{\partial K_\mu}\frac{\partial}{\partial K^\mu} - \frac{\partial}{\partial Q_\mu}\frac{\partial}{\partial Q^\mu}\Big) + 2\lambda_0^{iab}\lambda^*_{0jac}\lambda_0^{cde}\lambda^*_{0bde}\frac{\partial}{\partial Q_\mu}\frac{\partial}{\partial Q^\mu}\Big]$$
$$\times\frac{1}{K^2F_K^2\,Q^2F_Q\,(Q+K)^2F_{Q+K}\,L^2F_L\,(L+K)^2F_{L+K}} = -\frac{1}{2\pi r}C(R)_i{}^j(\Delta\gamma_\phi)_j{}^i.$$

In all loops the factorization into integrals of double total derivatives has been proved in

K.S., JHEP **10** (2019) 011.

An example: a three-loop contribution to the β-function containing the Yukawa couplings

Moreover, this proof demonstrated that the results for various contributions to the β-function can be obtained by calculating only a specially modefied vacuum supergraphs. For example, the above result is produced by the supergraphs



The standard calculation of the corresponding superdiagrams with two external lines of the background gauge superfield was done in

V.Yu.Shakhmanov, K.S., Nucl.Phys., **B920**, (2017), 345;
A.E.Kazantsev, V.Yu.Shakhmanov, K.S., JHEP 1804 (2018) 130.

Subsequently, a similar calculation was done with the help of a new method. It allowed to verify if the new method correctly reproduces the results of the above calculation of the β-function.

1. We consider a vacuum sipergraph. A contribution coming from all superdiagrams obtained from it by adding two external lines of the background gauge superfield to the function

$$\frac{1}{\alpha_0^2}\Big(\beta(\alpha_0,\lambda_0)-\beta_{\text{1-loop}}(\alpha_0)\Big)$$

can be obtained with the help of the following formal operations:

2. We insert a factor $\theta^4(v^B)^2$ to an arbitrary point of the supergraph.

3. The resulting expression is calculated. Terms in which derivatives act on v^B should be omitted.

4. We mark L propagators with the momenta Q_i^μ which are considered as independent. Their product is proportional to $\prod_{i=1}^L \delta_{a_i}^{b_i}$.

5. In the integrand we make the formal substitution

$$\prod_{i=1}^{L}\delta_{a_i}^{b_i}\ \to\ \sum_{k,l=1}^{L}\prod_{i\ne k,l}\delta_{a_i}^{b_i}\,(T^A)_{a_k}{}^{b_k}(T^A)_{a_l}{}^{b_l}\,\frac{\partial^2}{\partial Q_k^\mu\,\partial Q_l^\mu}.$$

6. To the resulting expression we apply the operator

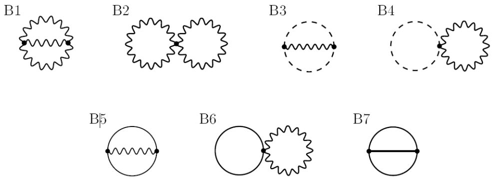
$$-\frac{2\pi}{r\mathcal{V}_4}\frac{d}{d\ln\Lambda}.$$

By construction, the result has the form of an integral of double total derivatives.

The method described above simplifies explicit calculations of the β-function in a great extent. For instance, the total two-loop contribution to the β-function of $\mathcal{N}=1$ supersymmetric Yang–Mills theory with matter superfields in an arbitrary ξ-gauge has been calculated in

K.S., Proceedings of the Steklov Institute of Mathematics **309** (2020) 284.

It is generated by the supergraphs



To obtain usual superdiagrams which determine the β-function, we need attach two external lines of the background gauge superfield in all possible ways. However, the new method allows to calculate only (specially modified) vacuum supergraphs.

The result (for the β-function defined in terms of the are couplings) is given by the expression

$$\frac{\beta(\alpha_0,\lambda_0)}{\alpha_0^2} = -\frac{1}{2\pi}\Big(3C_2 - T(R)\Big) + \frac{\alpha_0}{(2\pi)^2}\Big[-3C_2^2 + \frac{1}{r}C_2\,\mbox{tr}\,C(R) + \frac{2}{r}\,\mbox{tr}\,\Big(C(R)^2\Big)\Big] - \frac{1}{8\pi^3 r}C(R)_i{}^j\lambda^*_{0jmn}\lambda_0^{imn} + O(\alpha_0^2,\alpha_0\lambda_0^2,\lambda_0^4).$$

The gauge dependence disappears, and the result agrees with the one found earlier with the help of the dimensional technique. Moreover, it turns out that the NSVZ equations

$$\frac{\beta(\alpha_0,\lambda_0)}{\alpha_0^2} = -\frac{1}{2\pi}\Big(3C_2 - T(R) - 2C_2\gamma_c(\alpha_0,\lambda_0) - 2C_2\gamma_V(\alpha_0,\lambda_0) + \frac{1}{r}C(R)_i{}^j\left(\gamma_\phi\right)_j{}^i(\alpha_0,\lambda_0)\Big) + O(\alpha_0^2,\alpha_0\lambda_0^2,\lambda_0^4);$$

$$\frac{\beta(\alpha_0,\lambda_0)}{\alpha_0^2} = -\frac{3C_2 - T(R) + C(R)_i{}^j\left(\gamma_\phi\right)_j{}^i(\alpha_0,\lambda_0)/r}{2\pi(1 - C_2\alpha_0/2\pi)} + O(\alpha_0^2,\alpha_0\lambda_0^2,\lambda_0^4).$$

are valid even for the loop integrals. However, in this approximation the scheme dependence does not manifest itself.

Deriving the new form of the exact NSVZ β-function by summing singular contributions

The method for constructing integrals of double total derivatives described above can be used for deriving the exact NSVZ β-function in all orders of the perturbation theory

K.S., Eur.Phys.J. **C80** (2020) 10, 911.

As a result of similar calculations we obtained

$$\frac{\beta(\alpha_0,\lambda_0)}{\alpha_0^2} - \frac{\beta_{\textbf{1-loop}}(\alpha_0)}{\alpha_0^2}$$
$$= \frac{1}{\pi}C_2\gamma_V(\alpha_0,\lambda_0) + \frac{1}{\pi}C_2\gamma_c(\alpha_0,\lambda_0) - \frac{1}{2\pi r}C(R)_i{}^j(\gamma_\phi)_j{}^i(\alpha_0,\lambda_0).$$

gauge propagators

Faddeev–Popov ghost propagators

matter propagators

Thus, we obtain:

The NSVZ relation

$$\frac{\beta(\alpha_0,\lambda_0)}{\alpha_0^2} = -\frac{1}{2\pi}\Big(3C_2 - T(R) - 2C_2\gamma_c(\alpha_0,\lambda_0) - 2C_2\gamma_V(\alpha_0,\lambda_0) + C(R)_i{}^j(\gamma_\phi)_j{}^i(\alpha_0,\lambda_0)/r\Big),$$

and, therefore, the NSVZ relation

$$\beta(\alpha_0,\lambda_0) = -\frac{\alpha_0^2\Big(3C_2 - T(R) + C(R)_i{}^j(\gamma_\phi)_j{}^i(\alpha_0,\lambda_0)/r\Big)}{2\pi(1 - C_2\alpha_0/2\pi)}.$$

are valid in all orders of the perturbation theory for RGFs defined in terms of the bare couplings if a theory is regularized by higher covariant derivatives.

Consequently, for RGFs defined in terms of the renormalized couplings, similar equations hold in the HD+MSL scheme in all orders of the perturbation theory.

Knowing the conditions under which the NSVZ equation is valid it is possible to essentially simplify various multiloop calculations.

The two-loop anomalous dimension of the matter superfields with the higher derivative regularization

The two-loop anomalous dimension defined in terms of the bare coupling constant for $\mathcal{N}=1$ supersymmetric theories regularized by higher derivatives has been calculated in

A.E.Kazantsev, K.S., JHEP **2006** (2020) 108.

$$(\gamma_\phi)_i{}^j(\alpha_0,\lambda_0) = -\frac{\alpha_0}{\pi}C(R)_i{}^j + \frac{1}{4\pi^2}\lambda^*_{0imn}\lambda_0^{jmn} + \frac{\alpha_0^2}{2\pi^2}\left[C(R)^2\right]_i{}^j - \frac{1}{16\pi^4}$$
$$\times\lambda^*_{0iac}\lambda_0^{jab}\lambda^*_{0bde}\lambda_0^{cde} - \frac{3\alpha_0^2}{2\pi^2}C_2 C(R)_i{}^j\Big(\ln a_\varphi + 1 + \frac{A}{2}\Big) + \frac{\alpha_0^2}{2\pi^2}T(R)C(R)_i{}^j$$
$$\times\Big(\ln a + 1 + \frac{A}{2}\Big) - \frac{\alpha_0}{8\pi^3}\lambda^*_{0lmn}\lambda_0^{jmn}C(R)_i{}^l(1-B+A) + \frac{\alpha_0}{4\pi^3}\lambda^*_{0imn}\lambda_0^{jml}$$
$$\times C(R)_l{}^n(1-A+B) + O\Big(\alpha_0^3, \alpha_0^2\lambda_0^2, \alpha_0\lambda_0^4, \lambda_0^6\Big),$$

where

$$A = \int\limits_0^\infty dx\,\ln x\,\frac{d}{dx}\frac{1}{R(x)};\qquad B = \int\limits_0^\infty dx\,\ln x\,\frac{d}{dx}\frac{1}{F^2(x)}\qquad a = \frac{M}{\Lambda};\qquad a_\varphi = \frac{M_\varphi}{\Lambda}.$$

Obtaining the three-loop β-function from the NSVZ equation

If the anomalous dimension of the matter superfields defined in terms of the bare couplings has been calculated in L-loops with the higher derivative regularization, then it is possible to construct the $(L+1)$-loop β-function from the NSVZ equation without loop calculations. For example, in the three-loop approximation

$$\frac{\beta(\alpha_0,\lambda_0)}{\alpha_0^2} = -\frac{1}{2\pi}\Big(3C_2 - T(R)\Big) + \frac{\alpha_0}{4\pi^2}\Big\{-3C_2^2 + \frac{1}{r}C_2\,\mbox{tr}\,C(R) + \frac{2}{r}\,\mbox{tr}\left[C(R)^2\right]\Big\}$$
$$-\frac{1}{8\pi^3 r}C(R)_j{}^i\lambda^*_{0imn}\lambda_0^{jmn} + \frac{\alpha_0^2}{8\pi^3}\Big\{-3C_2^3 + \frac{1}{r}C_2^2\,\mbox{tr}\,C(R) - \frac{2}{r}\,\mbox{tr}\left[C(R)^3\right] + \frac{2}{r}$$
$$\times C_2\,\mbox{tr}\left[C(R)^2\right]\Big(3\ln a_\varphi + 4 + \frac{3A}{2}\Big) - \frac{2}{r^2}\,\mbox{tr}\,C(R)\,\mbox{tr}\left[C(R)^2\right]\Big(\ln a + 1 + \frac{A}{2}\Big)\Big\}$$
$$-\frac{\alpha_0 C_2}{16\pi^4 r}C(R)_j{}^i\lambda^*_{0imn}\lambda_0^{jmn} + \frac{\alpha_0}{16\pi^4 r}\left[C(R)^2\right]_j{}^i\lambda^*_{0imn}\lambda_0^{jmn}\Big(1 + A - B\Big) - \frac{\alpha_0}{8\pi^4 r}$$
$$\times C(R)_j{}^i C(R)_l{}^n\lambda^*_{0imn}\lambda_0^{jml}\Big(1 - A + B\Big) + \frac{1}{32\pi^5 r}C(R)_j{}^i\lambda^*_{0iac}\lambda_0^{jab}\lambda^*_{0bde}\lambda_0^{cde}$$
$$+O\Big(\alpha_0^3,\,\alpha_0^2\lambda_0^2,\,\alpha_0\lambda_0^4,\,\lambda_0^6\Big).$$

Certainly, RGFs defined in terms of the renormalized couplings can also be calculated for an arbitrary renormalization prescription.

Obtaining RGFs defined in terms of the renormalized couplings

To calculate RGFs defined in terms of the renormalized couplings, first, we integrate the equations

$$\beta(\alpha_0,\lambda_0) \equiv \frac{d\alpha_0}{d\ln\Lambda}\Big|_{\alpha,\lambda=\text{const}}; \qquad (\gamma_\phi)_i{}^j(\alpha_0,\lambda_0) \equiv -\frac{d(\ln Z_\phi)_i{}^j}{d\ln\Lambda}\Big|_{\alpha,\lambda=\text{const}},$$

and obtain the expressions for the renormalized gauge coupling constant and $(\ln Z_\phi)_i{}^j$. They depend on a set of finite constants which determine a subtraction scheme in the considered approximation. Next, we substitute the expressions obtained in this way into the equations

$$\widetilde\beta(\alpha,\lambda) \equiv \frac{d\alpha}{d\ln\mu}\Big|_{\alpha_0,\lambda_0=\text{const}}; \qquad (\widetilde\gamma_\phi)_i{}^j(\alpha,\lambda) \equiv \frac{d(\ln Z_\phi)_i{}^j}{d\ln\mu}\Big|_{\alpha_0,\lambda_0=\text{const}}.$$

These RGFs will nontrivially depend on the finite constants due to the scheme dependence.

Here (at the next slide) we only present the result for one particular case, namely, for one-loop finite $\mathcal{N}=1$ supersymmetric theories, see

P.West, Phys.Lett. **B 137** (1984) 371;
A.Parkes, P.West, Phys.Lett. **B 138** (1984) 99.

An important particular case is theories finite in the one-loop approximation which satisfy the conditions

$$T(R) = 3C_2; \qquad \lambda^*_{imn}\lambda^{jmn} = 4\pi\alpha C(R)_i{}^j.$$

In this case the two-loop anomalous dimension and the three-loop β-function defined in terms of the renormalized couplings have the form

$$(\widetilde{\gamma}_\phi)_i{}^j(\alpha,\lambda) = -\frac{3\alpha^2}{2\pi^2}C_2 C(R)_i{}^j\Big(\ln\frac{a_\varphi}{a} - b_{11} + b_{12}\Big) - \frac{\alpha}{4\pi^2}\Big(\frac{1}{\pi}\lambda^*_{imn}\lambda^{jml}C(R)_l{}^n$$
$$+2\alpha\left[C(R)^2\right]_i{}^j\Big)\Big(A - B - 2g_{12} + 2g_{11}\Big) + O\Big(\alpha^3,\alpha^2\lambda^2,\alpha\lambda^4,\lambda^6\Big);$$
$$\frac{\widetilde{\beta}(\alpha,\lambda)}{\alpha^2} = \frac{3\alpha^2}{4\pi^3 r}C_2\,\mathrm{tr}\left[C(R)^2\right]\Big(\ln\frac{a_\varphi}{a} - b_{11} + b_{12}\Big) + \frac{\alpha}{8\pi^3 r}\Big(\frac{1}{\pi}C(R)_j{}^i C(R)_l{}^n$$
$$\times\lambda^*_{imn}\lambda^{jml} + 2\alpha\,\mathrm{tr}\left[C(R)^3\right]\Big)\Big(A - B - 2g_{12} + 2g_{11}\Big) + O\Big(\alpha^3,\alpha^2\lambda^2,\alpha\lambda^4,\lambda^6\Big).$$

We see that in this case the NSVZ equation is satisfied in the lowest nontrivial approximation for an arbitrary renormalization presription,

$$\frac{\beta(\alpha,\lambda)}{\alpha^2} = -\frac{1}{2\pi r}C(R)_i{}^j(\gamma_\phi)_j{}^i(\alpha,\lambda) + O(\alpha^3,\alpha^2\lambda^2,\alpha\lambda^4,\lambda^6).$$

The NSVZ equation for theories finite in the lowest loops

For $\mathcal{N} = 1$ supersymmetric theories finite in the one-loop approximation it is possible to tune a subtraction scheme so that the theory will be all-loop finite

D.I.Kazakov, Phys. Lett. B **179** (1986) 352; A.V.Ermushev, D.I.Kazakov, O.V.Tarasov, Nucl.Phys. B **281** (1987) 72; C.Lucchesi, O.Piguet, K.Sibold, Helv.Phys.Acta **61** (1988) 321; Phys.Lett. B **201** (1988) 241.

If a subtraction scheme is tuned in such a way that the β-function vanishes in the first L loops and the anomalous dimension for the matter superfields vanishes in the first $(L-1)$ loops, then

K.S., Eur.Phys.J. C **81** (2021) 571.

for an arbitrary renormalization prescription the $(L+1)$-loop gauge β-function satisfies the equation

$$\frac{\beta_{L+1}(\alpha,\lambda)}{\alpha^2} = -\frac{1}{2\pi r} C(R)_i{}^j (\gamma_{\phi,L})_j{}^i(\alpha,\lambda),$$

Therefore, if a theory is finite in a certain approximation, its β-function vanishes in the next order. This exactly agrees with the earlier known result of

A.J.Parkes, P.West, Nucl.Phys. B **256** (1985) 340;
M.T.Grisaru, B.Milewski and D.Zanon, Phys.Lett. **155B** (1985) 357.

The NSVZ equations can also be written for theories with multiple gauge couplings. In this case a number of the NSVZ equations is equal to a number of (simple or $U(1)$) factors in the gauge group $G = G_1 \times G_2 \times \ldots \times G_n$. They can be written in the form

D.Korneev, D.Plotnikov, K.S., N.Tereshina, JHEP **10** (2021) 046.

$$\frac{\beta_K(\alpha,\lambda)}{\alpha_K^2} = -\frac{1}{2\pi(1-C_2(G_K)\alpha_K/2\pi)}\Big[3C_2(G_K) - \sum_{\mathsf{a}} \boldsymbol{T}_{\mathsf{a}K}\Big(1-\gamma_{\mathsf{a}}(\alpha,\lambda)\Big)\Big],$$

where the subscript a numerates chiral matter superfields in irreducible representations of simple G_I,

$$\boldsymbol{T}_K(R) = \sum_{\mathsf{a}} \boldsymbol{T}_{\mathsf{a}K},$$

and we use the notation

$$\boldsymbol{T}_{\mathsf{a}K} = \begin{cases} \delta_{i_1}{}^{i_1}\ldots\delta_{i_{K-1}}{}^{i_{K-1}}\, T_K(R_{\mathsf{a}K})\,\delta_{i_{K+1}}{}^{i_{K+1}}\ldots\delta_{i_n}{}^{i_n} & \text{if } G_K \text{ is simple;}\\ \delta_{i_1}{}^{i_1}\ldots\delta_{i_{K-1}}{}^{i_{K-1}}\, q_{\mathsf{a}K}^2\,\delta_{i_{K+1}}{}^{i_{K+1}}\ldots\delta_{i_n}{}^{i_n} & \text{if } G_K = U(1). \end{cases}$$

For MSSM the all-order exact NSVZ β-functions are given by the equations

$$\frac{\beta_3(\alpha,\lambda)}{\alpha_3^2} = -\frac{1}{2\pi(1-3\alpha_3/2\pi)}\Big[3+\mbox{tr}\Big(\gamma_Q(\alpha,\lambda)+\frac{1}{2}\gamma_U(\alpha,\lambda)+\frac{1}{2}\gamma_D(\alpha,\lambda)\Big)\Big];$$

$$\frac{\beta_2(\alpha,\lambda)}{\alpha_2^2} = -\frac{1}{2\pi(1-\alpha_2/\pi)}\Big[-1+\mbox{tr}\Big(\frac{3}{2}\gamma_Q(\alpha,\lambda)+\frac{1}{2}\gamma_L(\alpha,\lambda)\Big)+\frac{1}{2}\gamma_{H_u}(\alpha,\lambda) +\frac{1}{2}\gamma_{H_d}(\alpha,\lambda)\Big];$$

$$\frac{\beta_1(\alpha,\lambda)}{\alpha_1^2} = -\frac{3}{5}\cdot\frac{1}{2\pi}\Big[-11+\mbox{tr}\Big(\frac{1}{6}\gamma_Q(\alpha,\lambda)+\frac{4}{3}\gamma_U(\alpha,\lambda)+\frac{1}{3}\gamma_D(\alpha,\lambda) +\frac{1}{2}\gamma_L(\alpha,\lambda)+\gamma_E(\alpha,\lambda)\Big)+\frac{1}{2}\gamma_{H_u}(\alpha,\lambda)+\frac{1}{2}\gamma_{H_d}(\alpha,\lambda)\Big],$$

where the traces are taken with respect to the generation indices.

(In a different form) they were first presented in

M. A. Shifman, Int. J. Mod. Phys. A **11** (1996), 5761.

and correctly reproduce the (scheme-independent) two-loop contributions.

Three-loop MSSM β-functions for an arbitrary supersymmetric renormalization prescription

Starting from the two-loop expressions for the anomalous dimensions of the matter superfields it is possible to find the three-loop MSSM β-functions for an arbitrary supersymmetric renormalization prescription supplementing the higher covariant derivative regularization

O.Haneychuk, V.Shirokova, K.S., JHEP **09** (2022), 189.

The result is very large and depends on both regularization parameters and finite constants fixing a subtraction scheme. For certain values of these finite constants it reproduces the $\overline{\text{DR}}$ result obtained earlier.

I.Jack, D.R.T.Jones, A.F.Kord, Annals Phys. **316** (2005), 213.

As an example, at the next slide we present the three-loop expression for the function $\widetilde{\beta}_3$.

Therefore, the higher covariant derivative regularization can really be used for making very complicated explicit multiloop calculations.

$$\frac{\widetilde{\beta}_3(\alpha,Y)}{\alpha_3^2} = -\frac{1}{2\pi}\Big\{3 - \frac{11\alpha_1}{20\pi} - \frac{9\alpha_2}{4\pi} - \frac{7\alpha_3}{2\pi} + \frac{1}{8\pi^2}\,\mbox{tr}\Big(2\,Y_U^+ Y_U + 2\,Y_D^+ Y_D\Big) + \frac{1}{2\pi^2}\Big[\frac{137\alpha_1^2}{1200}$$
$$+\frac{27\alpha_2^2}{16} + \frac{\alpha_3^2}{6} + \frac{3\alpha_1\alpha_2}{40} - \frac{11\alpha_1\alpha_3}{60} - \frac{3\alpha_2\alpha_3}{4} + \frac{363\alpha_1^2}{100}\Big(\ln a_1 + 1 + \frac{A}{2} + b_{2,31} - b_{1,1}\Big) + \frac{9\alpha_2^2}{4}$$
$$\times\Big(-6\ln a_{\varphi,2} + 7\ln a_2 + 1 + \frac{A}{2} + b_{2,32} - b_{1,2}\Big) - 24\alpha_3^2\Big(3\ln a_{\varphi,3} - 2\ln a_3 + 1 + \frac{A}{2} + \frac{7}{16} b_{2,33}$$
$$-\frac{7}{16} b_{1,3}\Big)\Big] + \frac{1}{8\pi^3}\mbox{tr}\Big(Y_U Y_U^+\Big)\Big[\frac{3\alpha_1}{20} + \frac{3\alpha_2}{4} + 3\alpha_3 + \frac{13\alpha_1}{30}\Big(B - A + 2b_{2,3U} - 2j_{U1}\Big) + \frac{3\alpha_2}{2}$$
$$\times\Big(B - A + 2b_{2,3U} - 2j_{U2}\Big) + \frac{8\alpha_3}{3}\Big(B - A + 2b_{2,3U} - 2j_{U3}\Big)\Big] + \frac{1}{8\pi^3}\mbox{tr}\Big(Y_D Y_D^+\Big)\Big[\frac{3\alpha_1}{20} + \frac{3\alpha_2}{4}$$
$$+3\alpha_3 + \frac{7\alpha_1}{30}\Big(B - A + 2b_{2,3D} - 2j_{D1}\Big) + \frac{3\alpha_2}{2}\Big(B - A + 2b_{2,3D} - 2j_{D2}\Big) + \frac{8\alpha_3}{3}\Big(B - A$$
$$+2b_{2,3D} - 2j_{D3}\Big)\Big] - \frac{1}{(8\pi^2)^2}\Big[\frac{3}{2}\mbox{tr}\Big((Y_U Y_U^+)^2\Big)\Big(1 + 4b_{2,3U} - 4j_{UU}\Big) + \frac{3}{2}\mbox{tr}\Big((Y_D Y_D^+)^2\Big)\Big(1$$
$$+4b_{2,3D} - 4j_{DD}\Big) + 3\Big(\mbox{tr}(Y_U Y_U^+)\Big)^2\Big(1 + 2b_{2,3U} - 2j_{UtU}\Big) + 3\Big(\mbox{tr}(Y_D Y_D^+)\Big)^2\Big(1 + 2b_{2,3D}$$
$$-2j_{DtD}\Big) + \mbox{tr}\Big(Y_E Y_E^+\Big)\,\mbox{tr}\Big(Y_D Y_D^+\Big)\Big(1 + 2b_{2,3D} - 2j_{DtE}\Big) + \mbox{tr}\Big(Y_D Y_D^+ Y_U Y_U^+\Big)\Big(1 + 2b_{2,3U}$$
$$+2b_{2,3D} - 2j_{UD} - 2j_{DU}\Big)\Big]\Big\} + O(\alpha^3, \alpha^2 Y^2, \alpha Y^4, Y^6).$$

The above results can be applied to the particular case of $\mathcal{N}=2$ supersymmetric theories which can certainly be formulated in $\mathcal{N}=1$ superspace. In this formulation one supersymmetry is manifest, while the other is hidden. The action in the massless limit is written as

$$S=\frac{1}{2e_0^2}\mathrm{Re}\,\mathrm{tr}\int d^4x\,d^2\theta\,W^aW_a+\frac{1}{2e_0^2}\mathrm{tr}\int d^4x\,d^4\theta\,\Phi^+e^{2V}\Phi e^{-2V}$$
$$+\frac{1}{4}\int d^4x\,d^4\theta\Big(\phi^+e^{2V}\phi+\widetilde{\phi}^+e^{-2V^T}\widetilde{\phi}\Big)+\Big(\frac{i}{\sqrt{2}}\int d^4x\,d^2\theta\,\widetilde{\phi}^T\Phi\phi+\mathrm{c.c.}\Big).$$

The chiral matter superfields $\boldsymbol{\phi_i}=\Big(\Phi^A,\,\phi_i,\,\widetilde{\phi}^i\Big)$ belong to the reducible representation $R=Adj+R_0+\bar{R}_0$. Taking into account that

$$\frac{i}{\sqrt{2}}\int d^4x\,d^2\theta\,\widetilde{\phi}^T\Phi\phi=\frac{ie_0}{\sqrt{2}}(T^A)_i{}^j\int d^4x\,d^2\theta\,\widetilde{\phi}^i\Phi^A\phi_j$$

we see that the Yukawa couplings are related to the gauge coupling constant by the equations

$$(\lambda_0)_i{}^{jA}=(\lambda_0)_i{}^{Aj}=(\lambda_0)^j{}_i{}^A=(\lambda_0)^A{}_i{}^j=(\lambda_0)^{jA}{}_i=(\lambda_0)^{Aj}{}_i=\frac{ie_0}{\sqrt{2}}(T^A)_i{}^j,$$

where $(T^A)_i{}^j$ are the generators of the representation R_0.

$\mathcal{N}=2$ nonrenormalization theorems in $\mathcal{N}=1$ superspace

It appears that for an arbitrary $\mathcal{N}=1$ renormalization prescription the anomalous dimensions and the higher order contributions to the β-function do not vanish starting from the two- and three-loop approximations, respectively.

S.S.Aleshin, K.S., Phys. Rev. D **107** (2023) no.10, 105006.

However, the anomalous dimensions of chiral matter superfields vanish for such renormalization prescriptions that

1. The renormalization prescription does not break the $\mathcal{N}=2$ relation between the gauge and Yukawa couplings,

$$\frac{d}{d\ln\mu}\Big(\frac{\lambda^A{}_i{}^j}{e}\Big)=0.$$

2. The renormalization prescription is compatible with the structure of quantum corrections.

3. Moreover, all contributions to the β-function beyond the one-loop approximation vanish if the conditions 1 and 2 are satisfied and the renormalization prescription is NSVZ. Then

$$\gamma_\Phi=0;\qquad (\gamma_\phi)_i{}^j=0;\qquad \frac{\beta(\alpha)}{\alpha^2}=-\frac{1}{\pi}\Big(C_2-T(R_0)\Big).$$

Note that for $\mathcal{N}=2$ supersymmetric theories $\overline{\text{DR}}$-scheme is NSVZ, at least, in the lowest loops.

A higher derivative term S_Λ invariant under both supersymmetries has been constructed in

I.L.Buchbinder and K.S., Nucl.Phys. **B883** (2014) 20.

However, with the help of the $\mathcal{N}=1$ superfield technique it is impossible to quantize a theory in the $\mathcal{N}=2$ supersymmetric way. Therefore, in this case quantum corrections can break the hidden supersymmetry. That is why it is convenient to use the formulation of $\mathcal{N}=2$ supersymmetric theories in the harmonic superspace

A.Galperin, E.Ivanov, S.Kalitzin, V.Ogievetsky and E.Sokatchev, Class.Quant.Grav. **1** (1984) 469.

with the coordinates $(x^\mu, \theta^i_a, \bar\theta_{i\dot a}, u^\pm_i)$, where $u^-_i = (u^{+i})^*$ and $u^{+i}u^-_i = 1$.
With the help of the harmonic superspace one can quantize the theory in a manifestly $\mathcal{N}=2$ supersymmetric way. That is why the harmonic superspace technique together with the background superfield method allow having manifest $\mathcal{N}=2$ supersymmetry and gauge invariance at all steps of calculating quantum corrections.

A.S.Galperin, E.A.Ivanov, V.I.Ogievetsky and E.S.Sokatchev, Harmonic superspace. Cambridge University Press (2001) 306p.

The higher covariant derivative regularization can also be formulated in the harmonic superspace

I.L.Buchbinder, N.G.Pletnev and K.S., Phys.Lett. **B751** (2015) 434.

It allows to prove simply the $\mathcal{N}=2$ non-renormalization theorem starting from the NSVZ β-function.

The degree of divergence (for the non-regularized theory) in the harmonic superspace is written as

I.L.Buchbinder, S.M.Kuzenko and B.A.Ovrut, Phys.Lett. **B433** (1998) 335.

$$\omega=-N_\phi-N_c-\frac{1}{2}N_D,$$

where N_ϕ is a number of external hypermultiplet lines, N_c is a number of external ghost lines, and N_D is a number of spinor derivatives acting on external lines. Therefore, all superdiagrams containing hypermultiplet external lines are finite, so that $\gamma_\phi(\alpha_0)=0$. Consequently, from the NSVZ equation we obtain

$$\frac{\beta(\alpha_0)}{\alpha_0^2}=-\frac{1}{\pi}\Big(C_2-T(R)\Big).$$

This implies that the β-function is non-trivial only in the one-loop approximation.

Exact (?) results for the $P = \frac{1}{3}Q$ theories

Some interesting properties of quantum corrections exist in the so-called $P = \frac{1}{3}Q$ theories, which by definition satisfy the relation

$$\lambda^*_{imn}\lambda^{jmn} - 4\pi\alpha C(R)_i{}^j = \frac{2\pi\alpha}{3}Q\delta_i^j,$$

where $Q \equiv T(R) - 3C_2$. Really, it was demonstrated

I.Jack, D.R.T.Jones, C.G.North, Nucl. Phys. B **473** (1996), 308.

that in these theories in the first two orders of the perturbation theory the ratio of the Yukawa couplings to the gauge coupling is RG invariant,

$$\frac{d}{d\ln\mu}\Big(\frac{\lambda^{ijk}}{e}\Big) = 0,$$

exactly as in $\mathcal{N} = 2$ supersymmetric theories.

This presumably allows to reduce a number of couplings and is very interesting for the phenomenology,

S. Heinemeyer, M. Mondragon, N. Tracas, G. Zoupanos, Phys. Rept. **814** (2019) 1.

Exact (?) results for the $P = \frac{1}{3}Q$ theories

The renormalization group invariance of the ratio λ^{ijk}/e is equivalent to the equation relating the β-function to the anomalous dimension,

$$(\gamma_\phi)_i{}^j = \frac{\beta}{3\alpha}\,\delta_i^j.$$

Together with the NSVZ equation it produces exact equations for the β-function and for the anomalous dimension of the matter superfields

$$\beta(\alpha) = \frac{\alpha^2 Q}{2\pi(1+\alpha Q/6\pi)};\qquad (\gamma_\phi)_i{}^j(\alpha) = \frac{\alpha Q}{6\pi(1+\alpha Q/6\pi)}\delta_i^j \equiv \gamma_\phi \delta_i^j.$$

Using the finiteness of the triple gauge-ghost vertices the renormalization group invariance of the ratio λ^{ijk}/e can equivalently be rewritten as a relation between anomalous dimensions of quantum superfields in each order of the perturbation theory

M.D.Kuzmichev, K.S., Phys.Lett. **844** (2023), 138094.

Really, (as we saw above) the non-renormalization of the triple gauge-ghost vertices leads to the relation

$$\beta = 2\alpha(\gamma_c + \gamma_V).$$

Exact (?) results for the $P = \frac{1}{3}Q$ theories

Therefore, the above condition is equivalent to the all-loop relation between the anomalous dimensions of quantum superfields in the $P = \frac{1}{3}Q$ theories

$$2\big(\gamma_c + \gamma_V\big)\,\delta_i^j = 3\big(\gamma_\phi\big)_i{}^j.$$

The one-loop expressions for the anomalous dimensions entering the above equation are written as

$$\gamma_c^{(1)} = -\frac{\alpha C_2(1-\xi)}{6\pi}; \qquad \gamma_V^{(1)} = \frac{\alpha C_2(1-\xi)}{6\pi} + \frac{Q\alpha}{4\pi};$$
$$\big(\gamma_\phi^{(1)}\big)_i{}^j = -\frac{\alpha}{\pi}C(R)_i{}^j + \frac{1}{4\pi^2}\lambda^*_{imn}\lambda^{jmn} = \frac{Q\alpha}{6\pi}\delta_i^j.$$

We see that they really satisfy the above relation, which is therefore valid in the one-loop approximation. Also it is possible to demonstrate that for a certain renormalization prescriptions the above relation is valid in the two-loop approximation. In particular, it is valid in the $\overline{\text{DR}}$-scheme. However, in the three-loop approximation some terms proportional to $\zeta(3)$ do not satisfy the above relation

I.Jack, D.R.T.Jones, C.G.North, Nucl. Phys. B **473** (1996), 308.

Conclusion

- The higher covariant derivative regularization allows revealing some interesting features of supersymmetric theories and deriving some all-loop results.
- The β-function of $\mathcal{N}=1$ supersymmetric gauge theories is determined by integrals of double total derivatives in the momentum space.
- The triple gauge-ghost vertices are UV finite in all orders. This allows to rewrite the NSVZ relation in an equivalent form, which relates the β-function to the anomalous dimensions of the quantum superfields.
- RGFs defined in terms of the bare couplings satisfy the NSVZ relation in theories regularized by higher derivatives in all loops.
- Some all-order NSVZ schemes are given by the HD+MSL prescription.
- Validity of the NSVZ equation with the higher covariant derivative regularization allows to essentially simplify some multiloop calculations.
- The $\mathcal{N}=2$ non-renormalization theorems for $\mathcal{N}=2$ supersymmetric theories formulated in $\mathcal{N}=1$ superspace are valid if a renormalization prescription is compatible with a structure of quantum corrections, do not break the relation between Yukawa and gauge couplings, and is NSVZ.
- The best way to obtain the $\mathcal{N}=2$ non-renormalization theorem is to use the higher covariant derivative regularization in the harmonic superspace.
- The RG invariance of the ratio λ^{ijk}/e in the $P=\frac{1}{3}Q$ theories is equivalent to a certain relation between the anomalous dimensions of the quantum superfields, which should be valid in each order of the perturbation theory.

Thank you for the attention!