On the quantum dynamics of states/operators: some expected and unexpected relations

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Problems of the Modern Mathematical Physics

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Extremal principles

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- What can we get from extemal principles?
  - Motion of particle on geodesics
    - in flat space (trivial) and in curves space (nontrivial)
  - Least action principle
    - in (Q)Mechanics and QFT
  - Discrete paths
  - Minimal knowledge to (almost) completely describe a system
    - Paterns and codes , optimization

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  - Minimal knowledge to (almost) completely describe a system
    - Paterns and codes , optimization
- Information about evolving in time complex systems (states/operators)

- example: for Krylov spaces, Lanczos coefficients give information about behavior of the system

$$b_n \sim n^{\delta}, \quad \delta \geq 1 \text{ - chaotic}, \quad 0 < \delta < 1 \text{ - integrable}$$

- for $\delta \geq 1$ one can obtain the Lyapunov exponents

Notion of Complexity

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- Informally, complexity, $C_F(\mathfrak{X})$, quantifies the "information content", the level of "redundancy" and "structures" within a quantity, $\mathfrak{X}$:

$$C_F(\mathfrak{X}) = \min_p \{|p| : F(p) = \mathfrak{X}\},$$

  Here, $p =$ sequence of information/program, $F =$ a computational process or algorithm that generates $\mathfrak{X}$. The complexity measure $=$ the shortest program length $(|p|)$ such that when processed by the algorithm, it yields the desired output, $\mathfrak{X}$.

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- Due to its universality $\Longrightarrow$ many concepts and methods about how to precisely define and measure complexity.
- In our context - the naive notion of complexity C(t): as a correlator for some time dependent operator $A(t)$ (autocorrelation function)

$$C(t) = \langle A(t)|A\rangle, \qquad \langle A|B\rangle = \mathrm{Tr}(A^\dagger \rho_1 B \rho_2) \tag{1}$$

Complexity= Volume conjecture

- *Complexity=Volume* [see for instace: cond-mat/0512165, 0905.1317, 1303.108]

- Complexity of a state $|\psi_i\rangle$ can be obtained from the simplest state by action of a unitary,

$$|\psi_i\rangle = U_i(t)|0\rangle.$$

The complexity of $|\psi_i\rangle$ is the minimum complexity of any unitary that does the job!

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Conjecture: Let A_H is the area of horizon and the rate of change of Complexity is $\dot{\mathcal{C}} \sim \kappa A_H/G$. Thus,

$$\mathcal{C} \sim \frac{(D-3)V}{G r_H}.$$

Complexity=Action conjecture

- *Complexity=Action in holography* [see for instance 1509.07876]

The rate of *quantum complexity for the boundary quantum state* is exactly equal to *the growth rate of the gravitational action on shell* in the bulk region in the WDW patch at the late time approximation. Then the complexity-action duality can be defined by

$$\mathcal{C} = \frac{S}{\pi\hbar}, \tag{2}$$

- $\mathcal{C}$ is the complexity in quantum information theory, whose meaning is that the minimum numbers of quantum gates are required to produce the certain state from the reference state, and S is the total classical gravitational action in the bulk region within the WDW patch.

Complexity Equals Anything

[see for instance 2111.02429, 2210.09647]

Main statements of the conjecture are:
- "new infinite class of gravitational observables in asymptotically Anti-de Sitter space living on codimension-one slices of the geometry, the most famous of which is the volume of the maximal slice and any member of this class of observables is an equally viable candidate as the extremal volume for a gravitational dual of complexity."
- variations of the codimension-zero and codimension-one observables are encoded in the gravitational symplectic form on the semi-classical phase-space, which can then be mapped to the CFT.

• Strong evidence that a wide class of observables are viable candidates for complexity.

Geometric Complexity

Geometric Complexity

- The allowed transformations $U(\sigma)$ - as path ordered exponentials

$$V = i\frac{dU}{d\tau}U^\dagger = T_\alpha V^\alpha \quad \Longrightarrow \quad U(\sigma) = \mathcal{P}e^{-i\int_{s_i}^{\sigma} V(s)ds}$$

- s parametrizes progress along a path, starting at s_i and ending at s_f and $\sigma \in [s_i, s_f]$ is some intermediate value of s. The path-ordering $\mathcal{P}$ is required for non-commuting generators T_α, $V(s) = V^\alpha T_\alpha$.

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- bi-invariant metric

$$ds^2_{bi-inv} = \mathrm{Tr}(V^{\dagger}V)d\tau^2 \tag{3}$$

- The length of a path from s_i to s_f going through $|\Psi(\sigma)\rangle$

$$\ell(|\Psi(\sigma)\rangle) = \int_{s_i}^{s_f} ds(\sigma).$$

- Define the complexity $\mathcal{C}$ as the minimal length/geodesics between states driven by generators $G(s)$

$$\mathcal{C}(|\Psi(s_i)\rangle, |\Psi(s_f)\rangle) = \min_{V(s)} \ell(|\Psi(\sigma)\rangle).$$

Geometric Complexity

• Nielsen's complexity of the evolution operator corresponds to the length of the path with b.c. and velocity that minimizes the length
- penalty factors μ_α $\quad\rightarrow\quad$ the metric (for low cost directions $\mu_\alpha = 1$)

$$C_N(t) = \min_V \int_0^t d\tau \left(\sum_\alpha \mathrm{Tr}(T_\alpha V)^2 + \mu_\alpha \, \mathrm{Tr}(T_\alpha V)^2 \right)^{1/2} ,$$

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• Objective: geodesics connecting the identity to a target unitary $U_{target} = \exp\{-i\mathcal{H}t\}$ at a chosen moment t, with $\mathcal{H}$ being the physical Hamiltonian.
- ambiguity:

$$\mathcal{H} \;\rightarrow\; \mathcal{H} + \frac{2\pi}{t}\kappa, \qquad \kappa \in \mathbb{Z}.$$

- ambiguity in the spectrum

$$E_n \;\rightarrow\; E_n - \frac{2\pi}{t}\kappa_n \equiv 2\pi y_n / t.$$

Geometric Complexity

- accounting for penalties in the metric

$$\sqrt{\sum_\alpha [\mathrm{Tr}(T_\alpha V)^2 + \mu_\alpha \, \mathrm{Tr}(T_\alpha V)^2]} = \sqrt{y_n Q_{nm} y_m}$$

$$\implies \quad Q_{nm} = \sum_\alpha \mu_\alpha \langle n|T_\alpha|n\rangle \langle m|T_\alpha^\dagger|m\rangle, \quad (4)$$

where $\mu_\alpha = 1$ for low cost directions and $\mathrm{Tr}(T_\alpha T_\beta) = \delta_{\alpha\beta}$.

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- a pure state in theory with gauge symmetry $\rightarrow$ "generalized length" : curve $\gamma(t)$ on the group manifold (A_i is the gauge connection):

$$C_\gamma = \int_0^1 d\tau ||\dot\gamma(t)|| - \int_0^1 d\tau A_i(\gamma(t))\dot\gamma^i.$$

$\rightarrow$ the state complexity of $|\psi_T\rangle$: the equivalence class of some Gaussian transformation $M \in G$ (group manifold) $\rightarrow$ the length of the geodesic connecting 1 to the point where the equivalence class $[M]$ intersects $\exp(stab_\perp(N))$.

Spread states and Operator growth

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• Unitary evolution mixes the initial state $|\psi\rangle$ with other quantum states as time evolves

$$|\psi(t)\rangle = e^{-i\mathcal{H}t}|\psi(0)\rangle = \sum_{n=0}^{\infty} \frac{(-i\mathcal{H}t)^n}{n!}|\psi\rangle = \sum_{n=0}^{\infty} \frac{(-it)^n}{n!}|\psi_n\rangle. \tag{5}$$

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• The Gram–Schmidt procedure applied to generate an ordered, orthonormal basis $K = \{|K_0\rangle, |K_1\rangle, \dots\}$.

- consider a basis $B = \{|B_i\rangle\; i = 0, 1, \dots\}$ and def *cost finction*

$$C_B(t) = \sum_n c_n |\langle\psi_n|B_n\rangle|^2, \qquad c_n \text{ *positive increasing*},\;\; |B_0\rangle = |\psi(t_0)\rangle$$

- def *Complexity*

$$\boxed{C(t) = \min_B C_B(t)}$$

Spread states and Operator growth

- Operator growth

$$\mathcal{O}(t) = e^{i\mathcal{H}t}\,\mathcal{O}(0)\,e^{-i\mathcal{H}t} = \sum_{n=0}^{\infty} \frac{(it)^n}{n!}\tilde{\mathcal{O}}_n, \tag{6}$$

where

$$\tilde{\mathcal{O}}_0 = \mathcal{O}, \quad \tilde{\mathcal{O}}_1 = [\mathcal{H}, \mathcal{O}], \quad \tilde{\mathcal{O}}_2 = [\mathcal{H}, [\mathcal{H}, \mathcal{O}]] \ldots \tag{7}$$

As time progresses, a simple operator $\mathcal{O}(t)$ "grows" in the space of operators of the theory becoming more "complex".
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- Notion of Liouvillian (superoperator)

$$\mathcal{L} := [\mathcal{H}, *] \quad \Longrightarrow \quad \tilde{\mathcal{O}}_n = \mathcal{L}^n \mathcal{O}(0) \quad \Longrightarrow \quad \mathcal{O}(t) = e^{i\mathcal{L}t}\mathcal{O}(0). \tag{8}$$

- Subtlety: the states $|\mathcal{O}_n(0)) = \mathcal{O}_n|0\rangle$ may not be orthogonal (and the set $\{|\mathcal{O}_n(0))\}$ may not define a basis)

Constructing Krylov spaces

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- The algorithm of orthogonalization (Arnoldi iteration)

1. set $b_0 \equiv 0$ and $|\mathcal{O}_{-1}) \equiv 0$
2. Define $|\mathcal{O})_0 = \frac{1}{\sqrt{(\mathcal{O}|\mathcal{O})}}\mathcal{O})$
3. For $n = 1$:
   - $|A_1) = \mathcal{L}|\mathcal{O}_0)$
   - $b_1 = ||A_1||$
   - If $b_1 \neq 0$ define$|\mathcal{O}_1) = \frac{1}{b_1}|A_1)$
4. For $n > 1$:
   - $|A_n) = \mathcal{L}|\mathcal{O}_{n-1}) - b_{n-1}|\mathcal{O}_{n-2})$
   - $b_n = ||A_n|| \equiv \sqrt{(A_n|A_n)}$
   - If $b_n = 0$ stop the procedure; if not, define $|\mathcal{O}_n) = \frac{1}{b_n}|A_n)$ and go to step 4.

Recurrent relations

- Jacobi operator (Jacobi matrix)

$$L_{mn} = (\mathcal{O}_m|\mathcal{L}|\mathcal{O}_n)$$

- the adjoint action of $H \Longrightarrow$ Jacobi matrix w/ elements Lanczos coefficients $\{b_n, a_n\}$, $n = 0, \ldots$ $[H, \mathcal{O}_n] = \sum_m L_{nm}\mathcal{O}_m$.

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- Explicit form of Jacobi matrix in terms of Lanczos coefficients

$$L = \begin{pmatrix} a_0 & b_0 & 0 & \ddots \\ b_0 & a_1 & b_1 & \ddots \\ 0 & b_1 & a_2 & \ddots \\ \ddots & \ddots & \ddots & \ddots \end{pmatrix}. \tag{9}$$

- the Lanczos coefficients as hopping amplitudes facilitating the traversal of the initial operator along the "Krylov chain"

Recurrent relations

- Define $P_k(x)$ as the determinant made of the first prinicipal $k \times k$ minor w/ $P_1 = a_1 - x$, $P_0 = 1$

$$P_k(\lambda) = \det \begin{pmatrix} a_1 - x & b_1 & 0 & \dots & 0 \\ b_1 & a_2 - x & b_2 & \ddots & \vdots \\ 0 & b_2 & a_3 - x & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & b_{k-1} \\ 0 & \dots & 0 & b_{k-1} & a_k - x \end{pmatrix}.$$

- Expanding the determinant by minors wrt the last row $\Longrightarrow$

$$P_k(x) = (a_k - x) P_{k-1}(x) - b_{k-1}^2 P_{k-2}(x). \tag{10}$$

Krylov complexity

- Decomposition of $\mathcal{O}(t)$ in terms of the Krylov elements:

$$|\mathcal{O}(t)) = \sum_{n=0}^{K-1} \phi_n(t) |\mathcal{O}_n). \tag{11}$$

- The Liouvillian in Krylov basis

$$\mathcal{L} = \sum_{n=0}^{K-1} b_{n+1} \left[\, |\mathcal{O}_n)(\mathcal{O}_{n+1}| + |\mathcal{O}_{n+1})(\mathcal{O}_n| \,\right] \tag{12}$$

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- The equation for $\phi_n(t)$

$$-i\dot{\phi}_n = \sum_{m=1}^{K-1} L_{nm}\phi_m(t) = b_{n+1}\phi_{n+1}(t) - b_n\phi_{n-1}(t), \qquad \phi_n(0) = \delta_{n0}.$$

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- *Krylov Complexity and K-entropy (Shannon)*

$$\mathcal{K}(t) = \sum n|\phi_n(t)|^2, \qquad S(t) = \sum |\phi_n(t)|^2 \log |\phi_n(t)|^2 \tag{13}$$

Moments and Hankel determinant

• A key quantity containing equivalent information is the moment matrix $\mathfrak{M}$ defined by

$$\mathfrak{M}_0 = \begin{pmatrix} \int x^0 d\omega & \int x d\omega & \cdots & \int x^n d\omega \\ \int x d\omega & \int x^2 d\omega & \cdots & \int x^{n+1} d\omega \\ . & . & \cdots & \\ \int x^n d\omega & \int x^{n+1} d\omega & \cdots & \int x^{2n} d\omega \end{pmatrix} = \begin{pmatrix} \mu_0 & \mu_1 & \cdots & \mu_n \\ \mu_1 & \mu_2 & \cdots & \mu_{n+1} \\ . & . & \cdots & \\ \mu_n & \mu_{n+1} & \cdots & \mu_{2n} \end{pmatrix}$$

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- Hankel determinant D_n

$$D_n = \det_{1 \le i,j \le N} (\mu_{i+j}) = \begin{vmatrix} \mu_0 & \mu_1 & \cdots & \mu_n \\ \mu_1 & \mu_2 & \cdots & \mu_{n+1} \\ . & . & \dots & \\ \mu_n & \mu_{n+1} & \cdots & \mu_{2n} \end{vmatrix} \tag{14}$$

Orthogonal polynomials

- Moments, Hankel and orthogonal polynomial $D_n(x)$

$$D_n(x) = \begin{vmatrix} \int x^0 d\omega & \int x d\omega & \cdots & \int x^n d\omega \\ \int x d\omega & \int x^2 d\omega & \cdots & \int x^{n+1} d\omega \\ \cdot & \cdot & \cdots & \\ \int x^{n-1} d\omega & \int x^n d\omega & \cdots & \int x^{2n-1} d\omega \\ 1 & x & \cdots & x^n \end{vmatrix}. \tag{15}$$

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- Using D_n and $D_(x)$ $\Longrightarrow$ define an orthogonal polynomial

$$P_n(x) = \frac{D_n(x)}{\sqrt{D_{n-1} D_n}} \tag{16}$$

- Using recurent relations one finds the relations to Lanczos coefficients

$$\boxed{b_n^2 = \frac{D_{n-1} D_{n+1}}{D_n^2}, \qquad a_n = \ln \frac{D_n}{D_{n-1}}.} \tag{17}$$

Schwarz-Christoffel map as a generalized measure

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Schwarz-Christoffel accessory parameters.

- Christoffel-Schwarz mapping

$$\frac{df(w)}{dw} = \gamma \prod_{i=1}^{n} (w - w_i)^{\theta_i - 1}, \tag{18}$$

where w_i are called pre-vertices (on the line), and z_i - the pre-images of the vertices (vertices of the polygon, $z_i = f(w_i)$).

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The Schwarzian differential equation

$$\{f(w), w\} := \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2 = \sum_{i=1}^{n} \left[\frac{1-\theta_i^2}{2(w-w_i)^2} + \frac{2\beta_i}{w-w_i}\right],$$

where n is the number of vertices and $\pi\theta_i$ are the interior angles at each vertex z_i.

Relations to Painleve

• The solutions of Schwarzian equation: given by $z = f(w) = \tilde{y}_1/\tilde{y}_2$, where $\tilde{y}_i$ are the two independent solutions of

$$\tilde{y}''(w) + \sum_{i=1}^{n} \left[\frac{1-\theta_i^2}{4(w-w_i)^2} + \frac{\beta_i}{w-w_i} \right] \tilde{y}(w) = 0. \tag{19}$$

• algebraic constraints on the accessory parameters

$$\sum_i \beta_i = \sum_i (w_i \beta_i + 1 - \theta_i^2) = \sum_i (2 w_i \beta_i^2 + w_i (1 - \theta_i^2)) = 0. \tag{20}$$

For $\tilde{y}(w) = w^{-\theta_0/2}(w-1)^{-\theta_1/2}(w-t)^{-\theta_t/2} y(w) \Longrightarrow$ the Heun equation in *canonical form*

$$y''(w) + \left(\frac{1-\theta_0}{w} + \frac{1-\theta_t}{w-t} + \frac{1-\theta_1}{w-1} \right) y'(w) + \left(\frac{\kappa_- \kappa_+}{w(w-1)} - \frac{t(t-1)K_0}{w(w-1)(w-t)} \right) y(w) = 0. \tag{21}$$

From ODE to Painleve

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$$\frac{1}{f(t)}\Big[P_0(z,t)D^2 + P_1(z,t)D + P_2(z,t)\Big]y(z) = \lambda\, y(z), \quad D = \frac{d}{dz}.$$

Here $P_i(z,t)$ are polynomials in z, $t =$ scaling parameter, λ interpreted as accessory parameter = energy.

$\Longrightarrow$ the Heun class equation as Schrödinger equation

$$H(\hat{q},\hat{p},t)\, y = \lambda\, y, \tag{22}$$

$\Longrightarrow$ "canonically quantize" the system,

$$H(\hat{q},\hat{p},t) = \frac{1}{f(t)}\Big[P_0(\hat{q},t)\hat{p}^2 + P_1(\hat{q},t)\hat{p} + P_2(\hat{q},t)\Big]. \tag{23}$$

Relations to Sasaki-Einstein manifold as string background

Details of the $\mathbb{Y}^{p,q}$ geometry

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- Base manifold

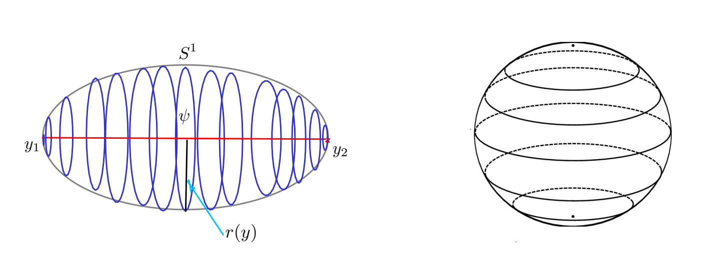

Figure: Squashed sphere as circle fibration parametrized by ψ over the interval $[y_1, y_2]$ and round sphere.

The topology of the base is $B \cong S^2 \times S^2$

- S^1 principle bundle over the base $\quad \rightarrow \quad d\alpha + A$
- Isometries are $SU(2) \times U(1) \times U(1)$.

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Toric stuff for $\mathbb{Y}^{p,q}$

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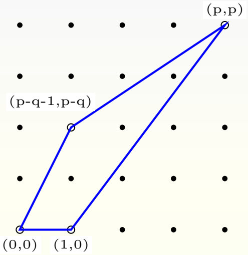

Relations to Sasaki-Einstein manifold as string background

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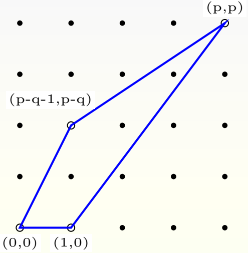

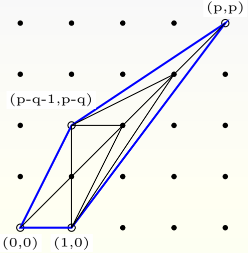

Figure: a) Toric $\mathbb{Y}^{4,2}$; b) The # gauge groups = number $\triangle$, for $\mathbb{Y}^{4,2} = 8$.

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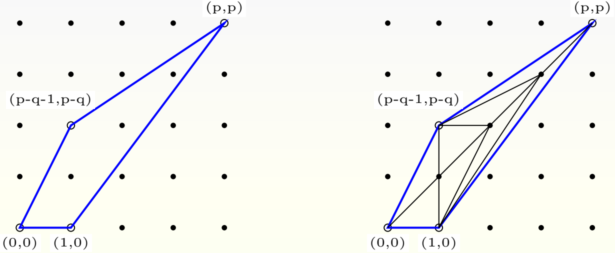

Figure: a) Toric $\mathbb{Y}^{4,2}$; b) The # gauge groups = number $\triangle$, for $\mathbb{Y}^{4,2} = 8$.

- For any toric quiver w/ bifundamentals $X \Longrightarrow$ dibaryonic operator

$$\mathcal{B}[X] = \epsilon^{\alpha_1 \ldots \alpha_N} X_{\alpha_1}^{\beta_1} \ldots X_{\alpha_N}^{\beta_N} \epsilon_{\beta_1 \ldots \beta_N} \tag{24}$$

- interpretation: to each toric divisor Σ_A - a bifundamental X_A whose corresponding d$\mathcal{B}[X]$ is dual to a D3–brane wrapped on Σ_A.

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- the cases $Y^{1,1}, Y^{1,0}$ and $Y^{2,2}$ are also interesting as the only members of $Y^{p,q}$ admitting massive supersymmetric deformations, see for instance (Feng et al, 2000).
- suspended pinch point (SPP) theory and the limits: the superpotential

$$W_{SPP} = X_{12}X_{23}X_{32}X_{21} - X_{23}X_{31}X_{13}X_{32} + X_{13}X_{31}X_{11} - X_{12}X_{21}X_{11}$$

- Higgsing: $\langle X_{23} \rangle \neq 0 \rightarrow$ purely cubic superpotential $\rightarrow \mathcal{N} = 2$; $\langle X_{21} \rangle \neq 0 \rightarrow X_{12}$ and X_{11} - massive $\rightarrow$ conifold.

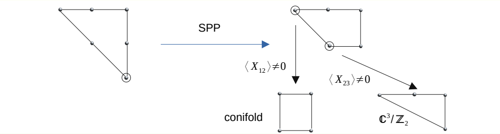

Figure: The (partial) resolution of the $\mathbb{C}^3/(\mathbb{Z}_2 \times \mathbb{Z}_2)$ singularity to the suspended pinch point (SPP) and obtaining toric diagams of $T^{1,1}$ and $\mathbb{C}^3/\mathbb{Z}_2$ by Higgsing.

Further properties

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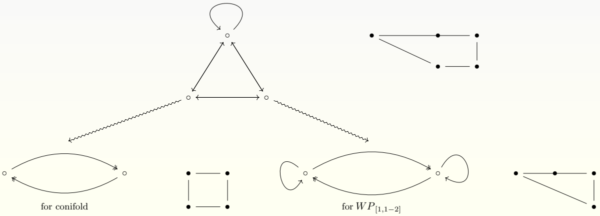

Figure: on the top: quiver for SPP; At the bottom: quiver for conifold ($\mathbb{Y}^{1,0}$) on the left; quiver for $\mathbb{Y}^{1,1}$ on the right.

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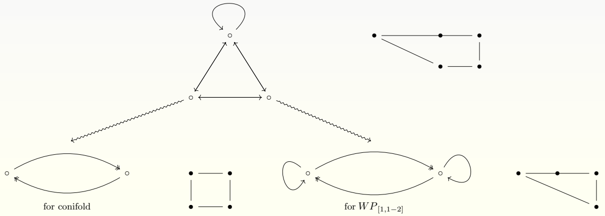

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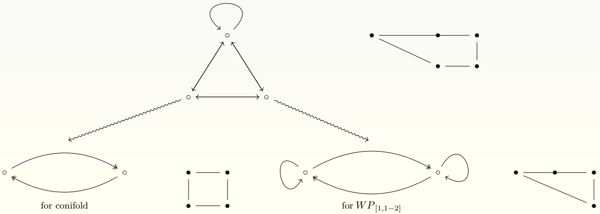

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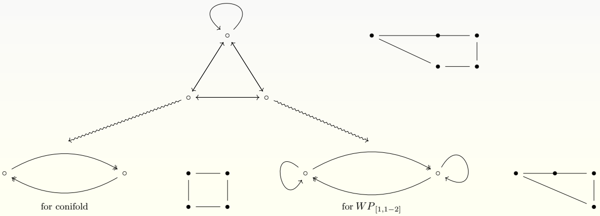

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- The limit ($SPP \to$ conifold) coresponds to $\mathbb{Y}^{2,1} \to$ conifold and the reduction PVI $\to$ PV.
- The limit ($SPP \to \mathbb{WP}_{[1,1,-2]}$) coresponds to $\mathbb{Y}^{2,1} \to \mathbb{Y}^{1,1}$ and the reduction PVI $\to$ PV.
- The limit $\mathbb{Y}^{p,q} \to \mathbb{Y}^{\infty,q}$ coresponds to the reduction PVI $\to$ PV.

Generalizing the measure

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• Assume without loss of generaylity that the measure $d\omega$ associated with orthogonality of the polynomials is given by

$$d\omega(x) = w_0(x)dx = e^{-v(x)}dx, \qquad \int p_n(x)p_m(x)w_0(x)dx = k_n\delta_{nm},$$

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• conformal function

$$f_2(z) = (z - z_1)^{\theta_1 - 1}(z - z_2)^{\theta_2 - 1},$$

w/ two w_1 & w_2 and interior angles θ_1 & θ_2, and

$$\arg f'(z) = \begin{cases} 0 & \textit{if} \quad z_1 < z_2 < z, \\ (\theta_2 - 1)\pi & \textit{if} \quad z_1 < z < z_2, \\ (\theta_2 - 1)\pi + (\theta_1 - 1)\pi & \textit{if} \quad z_1 < z_2 < z. \end{cases}$$

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• Schwarz-Christoffel map with $n = 2$ defines the measure for the Jacobi polynomials

$$d\omega_0(x) = (1-x)^{\alpha}(1+x)^{\beta}dx \tag{26}$$

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Thus

$$\mathfrak{M} = \left((-1)^{i+j}\partial_\lambda^{i+j}\mathfrak{M}\right)_{ij} = \begin{pmatrix} \mu_0(\lambda) & \mu_1(\lambda) & \cdots & \mu_n(\lambda) \\ \mu_1(\lambda) & \mu_2(\lambda) & \cdots & \mu_{n+1}(\lambda) \\ . & . & \cdots & \\ \mu_n(\lambda) & \mu_{n+1}(\lambda) & \cdots & \mu_{2n}(\lambda) \end{pmatrix}. \tag{27}$$

Generalizing the measure

- Consider Schwarz-Christoffel map of order $n = 3$, w/ $z_1 = 1$, $z_2 = -1$

$$f_3 = (z-1)^\alpha (z+1)^\beta (z-z_3)^{\theta_3 - 1}, \quad \theta_1 - 1 := \alpha,\ \theta_2 - 1 =: \beta, \tag{28}$$

$\implies$ up to irrelevant multiplcative constant

$$d\omega(x) = (1-x)^\alpha (1+x)^\beta e^{v(x)}, \tag{29}$$

where $v(x) = \sum_i \lambda_i x^i$ with $\lambda_i = (1-\theta_3)/i(w_3^i)$.

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- Define another weight

$$w(x,\lambda) \to w(x,\lambda_i) = w_0(x)\, e^{\sum_i \lambda_i x^i} \quad \implies \quad \partial_{\lambda_k}\mu_n = \mu_{n+k} \tag{30}$$

- Expanding in Schur polynomials $\implies$

$$\boxed{\mu_n(\lambda_i) = \int x^n w_0(x)\, e^{\sum_i \lambda_i x^i} dx = \sum_m S_m[\lambda_i]\mu_{n+m}(0).} \tag{31}$$

Relations to Toda

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Flaschka/Manakov-Lax form of the equations

$$a_k = -\frac{y_k}{2} = -\frac{\dot{x}_k}{2}, \qquad b_k = \frac{1}{2} e^{(x_k - x_{k+1})/2}, \tag{32}$$

EoM:

$$\dot{a}_n = b_n^2 - b_{n-1}^2, \qquad \dot{b}_n = \frac{b_n}{2}(a_{n+1} - a_n). \tag{33}$$

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Define

$$T = \begin{pmatrix} a_1 & b_1 & 0 & \dots & 0 \\ b_1 & a_2 & b_2 & \ddots & \vdots \\ 0 & b_2 & a_3 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & b_{n-1} \\ 0 & \dots & 0 & b_{n-1} & a_n \end{pmatrix}; \quad B = \begin{pmatrix} 0 & b_1 & 0 & \dots & 0 \\ -b_1 & 0 & b_2 & \ddots & \vdots \\ 0 & -b_2 & 0 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & b_{n-1} \\ 0 & \dots & 0 & b_{n-1} & 0 \end{pmatrix}$$

$\Longrightarrow$ Toda equations (in Hamiltonian form)

$$\underbrace{\frac{dT}{dt} = [B, T]}_{\text{Lax pair form}} = BT - TB. \tag{34}$$

Derivation from moment matrix

• introduce the following notation for (sub)determinant of the moment matrix $D_m = \det(\mathfrak{M})_{m\times m}$
• denote by $\tau_{n+1} = D$, $\tau_n = D\binom{n+1}{n+1}$, where $D\binom{n}{m}$ denotes the determinant with removed n-th row and m-th column

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• Observe that $\dot{\tau}_n = D\binom{n}{n+1} = D\binom{n+1}{n}$ (the second equality is obvious) and using the symmetries of Hankel determinant one finds $\ddot{\tau} = D\binom{n}{n}$.
• Jacobi identity for determinant $(D \equiv D_{n+1})$ reads

$$D\begin{pmatrix} n \\ n \end{pmatrix} D\begin{pmatrix} n+1 \\ n+1 \end{pmatrix} - D\begin{pmatrix} n \\ n+1 \end{pmatrix} D\begin{pmatrix} n+1 \\ n \end{pmatrix} = D\begin{pmatrix} n\; n+1 \\ n\; n+1 \end{pmatrix} D,$$

In other words, we arrive at the equation

$$\tau_n \ddot{\tau}_n - \dot{\tau}_n^2 = \tau_{n+1}\tau_{n-1}. \tag{35}$$

where the relations between Lanczos coefficients and Hankel determinans are (17), $b_n = \sqrt{D_{n+1}D_{n-1}}/D_n$ and $a_n = \ln(D_n/D_{n-1})$

Subsectors of Virasoro - SL case

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- The most general $SU(2, R)$ coherence preserving Hamiltonian

$$H = \alpha L_+ + \alpha^* L_- + \gamma L_0. \tag{36}$$

- consider highest weight state h, $|h\rangle = \mathcal{O}_{-h}|0\rangle$

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- For primary operators

$$[L_m, \mathcal{O}_{-h}] = [h(m-1) - m]\, \mathcal{O}_{-h+m}, \qquad m = \{-1, 0, 1\}, \tag{38}$$

and

$$[\mathcal{O}_m^{(i)}, \mathcal{O}_n^{(j)}] = \binom{m+h-1}{2h-1} \delta^{ij} \delta_{m+n,0} + \sum_k C_k^{ij} p_k^{ij}(m,n) \mathcal{O}_{m+n}^{(k)}. \tag{39}$$

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- The action of $SL(2,R)$ on a Fock state

$$L_0|h,n\rangle = (h+n)|h,n\rangle, \quad L_{-1}|h,n\rangle = \sqrt{(n+1)2h+n}|h,n+1\rangle \tag{40}$$

$$L_1|h,n\rangle = \sqrt{n(2h+n-1)}|h,n-1\rangle \tag{41}$$

Successfive applications of L_{-1} on the ground state $|h\rangle$

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- The explicit form of a state

$$\boxed{|z,h\rangle = (1-|z|^2)^h\sum_{n=0}^{\infty}z^n\sqrt{\frac{\Gamma(2h+n)}{n!\Gamma(2h)}}\,|h,n\rangle.} \tag{44}$$

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- The state generated by Liouvillian

$$|\mathcal{O}(t)) = e^{i\alpha(L_{-1}+L_1)t}|h\rangle = |z = i\tanh(\alpha t); h = \eta/2\rangle \tag{45}$$

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$$b_n = \alpha\sqrt{n(2h+n-1)}. \tag{46}$$

$\Longrightarrow$ the wavefunctions are just coefficients of the coherent state.

- Krylov Complexity for $SL(2,R)$

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- *Comment*: IHO $\quad K_{\mathcal{O}} \sim \sinh^2\left(\alpha t\sqrt{1-\frac{\gamma^2}{4\alpha^2}}\right)$

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- Virasoro algebra

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}n(n^2-1)\delta_{m+n,0}, \tag{48}$$

- construct $SL(2,\mathbb{R})$ from L_0 and $L_k = L^{\dagger}_{-k}$ using

$$[L_k, L_{-k}] = 2kL_0 + \frac{c}{12}k(k^2-1), \qquad [L_0, L_{\pm k}] = \mp L_{\pm k}. \tag{49}$$

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$$\tilde{L}_\pm = \frac{1}{k}L_{\pm k}, \qquad \tilde{L}_0 = \frac{1}{k}\left(L_0 + \frac{c}{12}k(k^2-1)\right). \tag{50}$$

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- Autocorrelation function for SL case

$$C(t) = (1|\psi_{\mathcal{O}}(t)) = \frac{1}{\cosh^{2h}(\alpha t)}$$

- heavier primaries decay faster.

Subsectors of Virasoro

In the above def basis the coherent state can be written as

$$|z, h, k\rangle = \sum_{n=0}^{\infty} e^{in\phi} \frac{\tanh^n(kr)}{\cosh^{2h_k}(kr)} \sqrt{\frac{\Gamma(2h_k)}{n!\Gamma(2h_k + n)}} |h, nk\rangle. \tag{52}$$

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$$|z,h,k\rangle = \sum_{n=0}^{\infty} e^{in\phi}\frac{\tanh^n(kr)}{\cosh^{2h_k}(kr)}\sqrt{\frac{\Gamma(2h_k)}{n!\Gamma(2h_k+n)}}|h,nk\rangle. \tag{52}$$

The action of Virasoro operators on the states $|\mathcal{O}) = |h,nk\rangle$

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The Liouvillian and K-basis

$$\mathcal{L}_k = \alpha(L_{-k} + L_k), \tag{54}$$

$$|\mathcal{O}(t)) = e^{it\mathcal{L}_k}|h\rangle = \sum_n \frac{\tanh^n(k\alpha t)}{\cosh^{2h_k}(k\alpha t)} \sqrt{\frac{\Gamma(2h_k+n)}{n!\Gamma(2h_k)}} |K_n\rangle = \sum_n \phi_n(t)|K_n\rangle$$

Subsectors of Virasoro

- The Lanczos coeffcients: $\boxed{b_n = \alpha k \sqrt{n(2h_k + n - 1)}}$
- relations between Lanczos koefficients

$$b_{n+1}^2 - b_n^2 = 2k^2\alpha^2(h_k + n).$$

- large n asymptotics of Lanczos coefficients

$$b_n \simeq k\alpha n + \frac{2h_k - 1}{2} + O(1/n), \qquad \lambda_L = 2\alpha.$$

- Krylov Complexity

$$K_{\mathcal{O}} = \sum_n n|\phi_n(t)|^2 = 2h_k \sinh^2(k\alpha t). \tag{55}$$

- asymptotics: $K_{\mathcal{O}}(t \to \infty) \simeq \frac{h}{2} e^{2\alpha t}$

$\Longrightarrow$ the total K-complexity is not sensitive enough to distinguish between the $SL(2,R)$ and Virasoro cases for simple primary operators.

Subsectors of Virasoro - subtleties

• consider subset of descendants & compute $K_{\mathcal{O}}$ for particular Young diagram - fine graining

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• *Example*: typical states, (in oscillator basis, $N = \sum i m_i$) :

$l_{-1}\Phi_{\{m_k\}} = \sum_{n=1}^{N} \sqrt{n(n+1)m_n(m_{n+1}+1)}\Phi_{m_1,m_2,\dots,m_n-1,m_n+1,\dots} + *\sqrt{2(m_n+1)}\Phi_{m_1+1,m_2\dots}$

- states with (c,h) dependence, $n \ll N$

$$b_{\{m_i\}\to\{\dots,m_n-1,m_n+1,\dots\}} \quad \Longrightarrow \quad b_n \sim \sqrt{N}$$

- states without (c,h) dependence, $n \ll N$

$$b_{\{m_i\}\to\{m_1+1,m_2,\dots\}} \quad \Longrightarrow \quad b_n \sim \sqrt[4]{n}$$

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• fluctuations - K-variance

$$\delta_{\mathcal{O}}(t) = \frac{\sum_n n^2|\phi(t)|^2 - (\sum_n n|\phi_n(t)|^2)^2}{(\sum_n n|\phi_n(t)|^2)^2} \overset{t\to\infty}{\approx} \frac{1}{\sqrt{2h}}$$

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$$(A|B)^g_\beta = \int_0^\beta g(\lambda)\langle e^{\lambda H} A^\dagger e^{-\lambda H} B\rangle_\beta d\lambda, \quad \langle A\rangle = \frac{1}{Z}\,\mathrm{Tr}(e^{-\beta H} A)$$

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- Conclusion: the correct Lyapunov exponent arises precisely after minimizing over all possible choices of the ambiguous inner product, and over all choices of basis.

Supersymmetric case, $osp(2|1)$

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• $osp(2|1)$ spanned by bosonic generators (B_+, B_0, B_-) and farmionic ones (F_+, F_-) w/ commutation relations

$$[B_-, B_+] = \{F_+, F_-\} = 2B_0, \tag{56}$$

$$[B_0, B_\pm] = \{F_\pm, F_\pm\} = \pm B_\pm, \tag{57}$$

$$[F_\pm, B_\mp] = 2[B_0, F_\pm] = \pm F_\pm. \tag{58}$$

• general element

$$U(g) = \exp\left[aB_0 + bB_+ - \bar{b}B_- + dF_+ + \bar{d}F_-\right]. \tag{59}$$

Using a BCH relation we want $U(g)$ to be rewritten as

$$U(g) = e^{\alpha B_0} e^{\beta B_+} e^{\gamma B_-} e^{\xi F_+} e^{\eta F_-}. \tag{60}$$

Supersymmetric case, $osp(2|1)$

- the standard procedure

$$e^{t\left[aB_0+bB_+-\bar{b}B_-+dF_++\bar{d}F_-\right]} = e^{\alpha(t)B_0}e^{\beta(t)B_+}e^{\gamma(t)B_-}e^{\xi(t)F_+}e^{\eta(t)F_-}.$$

- equations for (α,β,γ) and (ξ,η)

$$\begin{aligned}
\dot{\alpha} &= a + 2\beta e^{\alpha} + \xi\bar{d}e^{\frac{\alpha}{2}} \\
\dot{\beta} &= be^{-\alpha} - \beta^2\bar{b}e^{\alpha} + \frac{\xi d}{2}e^{-\alpha/2} - \frac{1}{2}\beta\xi e^{\alpha/2} \\
\dot{\gamma} &= \bar{b}e^{\alpha} - \frac{\gamma(\gamma\xi+\eta d)}{2}e^{-\alpha/2} + [\gamma\xi(1-\frac{\beta\gamma}{2}) + \frac{(1-\beta\gamma)\eta}{2}]\bar{d}e^{\alpha/2} \\
\dot{\xi} &= de^{-\alpha/2} + \beta\bar{d}e^{\alpha/2} \\
\dot{\eta} &= (1-\beta\gamma)(1-\frac{\xi\eta}{2})\bar{d}e^{\alpha/2} - \gamma(1-\frac{\xi\eta}{2})de^{-\alpha/2}
\end{aligned} \tag{61}$$

Supersymmetric case, $osp(2|1)$

- notations

$$\kappa = \frac{1}{4}a^2 + b\bar{b}, \quad \sigma = \cosh\kappa - \frac{a}{2\kappa}\sinh\kappa, \tag{62}$$

$$\tilde{\sigma} = \kappa\sinh\kappa - \frac{a}{2}\cosh\kappa = \kappa\frac{d}{d\kappa}\sigma, \tag{63}$$

$$\zeta = \frac{1}{\kappa^2}S^{-2}\left[\sigma^{-1}\left[1 + \frac{2b\bar{b}}{a}\cosh\kappa\right] - \frac{2\kappa^2}{a} - 1\right]. \tag{64}$$

- the parameters (α, β, ξ) in terms of $(a, b, \bar{b}, d, \bar{d})$

$$\alpha = -2\ln\sigma + \sigma^2\zeta d\bar{d}, \tag{65}$$

$$\beta = \frac{b}{\kappa}\sigma^{-1}\sinh\kappa\left[\frac{b}{2\kappa^3}\sigma^{-3}(\sinh\kappa - \kappa)\right]d\bar{d}, \tag{66}$$

$$\xi = \frac{1}{\sigma}\left[\frac{1}{\kappa^2}\left[\sigma + \frac{a}{2}\right]d + \frac{b}{\kappa^2}(\cosh\kappa - 1)\bar{d}\right]. \tag{67}$$

Supersymmetric case, $osp(2|1)$

- the action of the generators on the states

$$
\begin{aligned}
B_0|j,m;\eta,n\rangle &= (\epsilon_j^\eta + n)\,|j,m;\eta,n\rangle \\
B_\pm|j,m;\eta,n\rangle &= \sqrt{(\epsilon_j^\eta + n)(\epsilon_j^\eta + n \pm 1) - \epsilon_j^\eta(\epsilon_j^\eta - 1)}\,|j,m;\eta,n\rangle \\
F_\pm|j,m;\eta,n\rangle &= \frac{1}{2}\sqrt{2(\epsilon_j^\eta + n) \pm \frac{1}{2} \pm 2\eta\epsilon_j^\eta}\,|j,m;\eta,n\rangle
\end{aligned}
\tag{68}
$$

where $\epsilon_j^\eta = \frac{\sqrt{j(j+1)-j_0(j_0+1)}+1}{2} - \frac{\eta}{4}$ and $\eta = \pm 1$ is the fermionic number operator.

Supersymmetric case, $osp(2|1)$ - coherent states

- the states

$$|\alpha,\theta\rangle = N e^{\alpha K_+ + \theta F_+}|n,h\rangle = N e^{\alpha K_+} e^{\theta F_+}|n,h\rangle,$$

where α is a complex number, θ is a Grassmann number.

- normalization

$$N = \frac{1}{\mathrm{Sdet}\, M}, \qquad M = \begin{pmatrix} 1 & \frac{\theta'}{\sqrt{2}} \\ \frac{\bar{\theta}}{\sqrt{2}} & 1 - \alpha\alpha^* \end{pmatrix}$$

- expansion

$$|\theta,\alpha\rangle = \sum_{n=0}^{\infty} \frac{(2n)!}{2^n} \frac{\alpha^n}{n!}|2n\rangle + \frac{\theta}{2} \sum_{n=1}^{\infty} \sqrt{\frac{n(2n)!}{2^n}} \frac{\alpha^n}{n!}|2n-1\rangle$$

- The wave functions

$$\langle \zeta, x|\alpha,\theta\rangle = N \frac{\sqrt{\zeta}}{\sqrt[4]{\pi}} \frac{e^{-\zeta^2 x^2/2}}{\sqrt{1+\alpha}} \exp \frac{\zeta^2 \alpha x^2}{1+\alpha} \left(1 + \frac{1}{2} \frac{\theta \zeta x}{1+\alpha}\right)$$

where $\zeta = \sqrt{\frac{mw}{\hbar}}$ in oscillator basis.

Supersymmetric case, $osp(2|1)$ - coherent states

- Explicit form of the states expanded into $Osp(1|2)$ basis

$$|h,\alpha,\theta\rangle = \sum_{n=0}^{\infty} \frac{\tanh^n(\alpha t)}{\cosh^{2h}(\alpha t)} \sqrt{\frac{\Gamma(2h+n)}{n!\Gamma(2h)}} |K_{2n}\rangle + \theta \sum_{n=1}^{\infty} \frac{\tanh^{n-1}(\alpha t)}{\cosh^{2h-1}(\alpha t)} \sqrt{\frac{\Gamma(2h+n+\frac{1}{2})}{(n-1)!\Gamma(2h)}} |K_{2n-1}\rangle\rangle$$

- Estimation of Complexity and autocorrelation functions rates

- autocorrelation functions rate

$$C^0(t) \sim \frac{1}{\sinh^{2h}(\alpha t)}, \qquad C^1(t) \sim \frac{\theta}{\sinh^{2h-1}(\alpha t)}$$

- Krylov Complexity: has the same rate

$$K_{\mathcal{O}}(t) \overset{t\to\infty}{\sim} \cosh(2\alpha t), \qquad L = 2\alpha$$

Subsectors of $N = 1$ Super-Virasoro algebra

- $N = 1$ Super-Virasoro algebra.

$$
\begin{aligned}
[L_n, L_m] &= (n-m)L_{n+m} + \frac{c}{12}n(n^2-1)\delta_{m+n,0}, \\
[L_k, L_{-k}] &= 2kL_0 + \frac{c}{12}k(k^2-1), \\
[L_0, L_{\pm k}] &= \mp L_{\pm k} \\
[L_m, G_r] &= \left(\frac{m}{2} - r\right) G_{m+r}, \\
\{G_r, G_s\} &= 2L_{r+s} + \frac{c}{3}(r^2 - \frac{1}{4})\delta_{r+s}.
\end{aligned}
\tag{69}
$$

Subsectors of $N = 1$ Super-Virasoro algebra

- the subalgebra (for NS case $k =$ odd; for Ramond case $k =$ even)

$$\begin{aligned}
&[L_k, L_{-k}] = 2kL_0 + \frac{c}{12}k(k^2-1), \\
&[L_{\pm k}, G_{\pm k/2}] = 0, \quad [L_{\pm k}, G_{\mp k/2}] = \pm k G_{\pm k/2}, \\
&[L_0, G_{\pm k/2}] = \mp G_{\pm k/2}, \\
&\{G_{k/2}, G_{-k/2}\} = 2L_0 + \frac{c}{12}(k^2-1), \\
&\{G_{\pm k}, G_{\pm k}\} = 2L_{\pm k}.
\end{aligned} \tag{70}$$

- to obtain $osp(2|1)$ for each $k \rightarrow$ rescaling

$$\begin{aligned}
&L_0 \;\rightarrow\; \tilde{L}_0(k) = \frac{1}{k}\left(L_0 + \frac{c}{24}(k^2-1)\right), \qquad L_{\pm k} \;\rightarrow\; \tilde{L}_{\pm k} = \frac{1}{k}L_{\pm k} \\
&G_{\pm k/2} \;\rightarrow\; \tilde{G}_{\pm k/2} = \frac{1}{\sqrt{k}}G_{\pm k/2}
\end{aligned} \tag{71}$$

- super-Liouvillian: $\mathcal{L}(k) = \alpha(L_k + L_{-k}) + \xi(G_{k/2} + G_{-k/2})$.

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Nielsen for Jacobian states

- *Comments*

- the lowest weight reference state

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-*Remarks*:

∗ For the bosonic case → the metric of the hyperbolic disk

$$ds^2 = 2h\, dz d\bar{z} \Big/ (1 - z\bar{z})^2$$

∗ for $r = \alpha t \implies \quad Vol(t) = 2\pi h \sinh^2(\alpha t) = \pi K_{\mathcal{O}}(t)$

Nielsen for Jacobian states

$$ds^2 = 2h\left(1 + \frac{(1+|z|^2)}{2(1-|z|^2)}\bar{\theta}\theta\right)\frac{dz \wedge d\bar{z}}{(1-|z|^2)^2} - h\frac{d\theta d\bar{\theta}}{1-|z|^2}$$
$$- h\theta\bar{z}\frac{dz d\bar{\theta}}{(1-|z|^2)^2} + hz\bar{\theta}\frac{d\theta d\bar{z}}{(1-|z|^2)^2}$$

- The Nielsen Complexity $\rightarrow$ compute geodesic on $z_1 = 0, \theta_1 = 0$ submanifold

$*$ geodesic solution in the bosonic case

$$z(\sigma) = e^{i\phi}\tanh(c\sigma) \quad \xrightarrow{plugin} \quad \ell = \int_{\sigma_i}^{\sigma_f} \sqrt{\left| g_{a\bar{b}}\partial_\sigma z(\sigma)\partial_\sigma \bar{z}(\sigma)\right|}\, d\sigma$$

$*$ geodesic solution in the supersymmetric case

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Future directions:

- Analysis of Complexity for vertain CFT and QFT models
- Complexity for various (black hole, D-brane) backgrounds
- Uses of integrable systems
- Information geometry approach to Complexity
- Complexity and Seiberg-Witten curves?
- ...

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• Information spaces and Geometry (Entropies, Entaglement etc)
- emerging phenomena, ...

END

THANK YOU!