On Regge symmetry of Racah-Wigner symbols of $SL(2,\mathbb{C})$ group .

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E. Apresyan, G. Sarkissian and V. P. Spiridonov, "A parafermionic hypergeometric function and supersymmetric 6j-symbols," Nucl. Phys. B **990** (2023), 116170

S. E. Derkachov, G. A. Sarkissian and V. P. Spiridonov, "Elliptic hypergeometric function and 6j-symbols for the $SL(2,\mathbb{C})$ group," Teor. Mat. Fiz. **213** (2022) no.1, 108-128

Racah-Wigner symbols

$$|j_1 j_2, j_{12} m_{12}\rangle = \sum_{m_1, m_2} |j_1 m_1\rangle \otimes |j_2 m_2\rangle \, C^{j_{12} m_{12}}_{j_1 m_1 j_2 m_2}$$

$$|j_{12} j_3, j_4 m_4\rangle = \sum_{m_{12}, m_3} |j_1 j_2, j_{12} m_{12}\rangle \otimes |j_3 m_3\rangle \, C^{j_4 m_4}_{j_{12} m_{12} j_3 m_3}$$

$$|j_{12} j_3, j_4 m_4\rangle = \sum_{m_{12}, m_1, m_2, m_3} |j_1 m_1\rangle \otimes |j_2 m_2\rangle\rangle \otimes |j_3 m_3\rangle \, C^{j_{12} m_{12}}_{j_1 m_1 j_2 m_2} \, C^{j_4 m_4}_{j_{12} m_{12} j_3 m_3}$$

$$|j_{23} j_1, j_4 m_4\rangle = \sum_{m_{23}, m_1, m_2, m_3} |j_1 m_1\rangle \otimes |j_2 m_2\rangle\rangle \otimes |j_3 m_3\rangle \, C^{j_{23} m_{23}}_{j_2 m_2 j_3 m_3} \, C^{j_4 m_4}_{j_{23} m_{23} j_1 m_1}$$

Regge symmetry

$$\left\{ \begin{smallmatrix} j_1 & j_2 \\ j_3 & j_4 \end{smallmatrix} \middle| \begin{smallmatrix} j_{12} \\ j_{34} \end{smallmatrix} \right\} \sim \langle j_{12} j_3, j_4 m_4 | j_{23} j_1, j_4 m_4 \rangle$$
$$= \sum_{m_1, m_2, m_3, m_{12}, m_{23}} C^{j_{12} m_{12}}_{j_1 m_1 j_2 m_2} C^{j_4 m_4}_{j_{12} m_{12} j_3 m_3} C^{j_{23} m_{23}}_{j_2 m_2 j_3 m_3} C^{j_4 m_4}_{j_{23} m_{23} j_1 m_1}$$

The Regge transformation:

$$\left\{ \begin{smallmatrix} j_1 & j_2 \\ j_3 & j_4 \end{smallmatrix} \middle| \begin{smallmatrix} j_{12} \\ j_{23} \end{smallmatrix} \right\} = \left\{ \begin{smallmatrix} S-j_1 & S-j_2 \\ S-j_3 & S-j_4 \end{smallmatrix} \middle| \begin{smallmatrix} j_{12} \\ j_{23} \end{smallmatrix} \right\},$$

where $S = \frac{1}{2}(j_1 + j_2 + j_3 + j_4)$

Unitary principle representation of $SL(2,\mathbb{C})$ group

$$g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL(2,\mathbb{C})$$

$$[T_a(g)\Phi](z,\bar{z}) = (\beta z + \delta)^{a-1}(\bar{\beta}\bar{z} + \bar{\delta})^{\bar{a}-1}\Phi\left(\frac{\alpha z + \gamma}{\beta z + \delta}, \frac{\bar{\alpha}\bar{z} + \bar{\gamma}}{\bar{\beta}\bar{z} + \bar{\delta}}\right)$$

The unitarity requires

$$a = \frac{N}{2} + i\sigma \quad \bar{a} = -\frac{N}{2} + i\sigma \quad N \in \mathbb{Z}, \sigma \in \mathbb{R}$$

6j-symbols of $SL(2,\mathbb{C})$

$$\boldsymbol{\Gamma}(x,N)=\frac{\Gamma(\frac{N+\mathrm{i}x}{2})}{\Gamma(1+\frac{N-\mathrm{i}x}{2})},\ N\in\mathbb{Z},\ x\in\mathbb{C}$$

$$\left\{\begin{matrix}\sigma_1,N_1 & \sigma_2,N_2 \\ \sigma_3,N_3 & \sigma_4,N_4\end{matrix}\,\middle|\,\begin{matrix}\rho_1,M_1 \\ \rho_2,M_2\end{matrix}\right\}=\frac{\pi^2}{4}\frac{\boldsymbol{\Gamma}\left(\sigma_1-\sigma_2+\rho_2-i,\frac{N_1-N_2+M_2}{2}\right)}{\boldsymbol{\Gamma}\left(-\sigma_3-\sigma_4+\rho_2-i,\frac{-N_3-N_4+M_2}{2}\right)}$$

$$\times\frac{\boldsymbol{\Gamma}\left(\sigma_2-\sigma_3+\rho_1-i,\frac{N_2-N_3+M_1}{2}\right)}{\boldsymbol{\Gamma}\left(\sigma_1+\sigma_4+\rho_1-i,\frac{N_1+N_4+M_1}{2}\right)}$$

$$\times(-)^{(-N_2+M_2+N_4)}\sum_N\int d\nu\prod_{i=1}^{4}\boldsymbol{\Gamma}(R_i-\nu,-N+S_i)\boldsymbol{\Gamma}(U_i+\nu,N+T_i)$$

$$R_1 = -\sigma_1 + \sigma_2 - \rho_2 - i, \quad S_1 = (-N_1 + N_2 - M_2)/2,$$
$$R_2 = \sigma_1 + \sigma_2 - \rho_2 - i, \quad S_2 = (N_1 + N_2 - M_2)/2,$$
$$R_3 = -\sigma_3 - \sigma_4 - \rho_2 - i, \quad S_3 = -(N_3 + N_4 + M_2)/2,$$
$$R_4 = \sigma_3 - \sigma_4 - \rho_2 - i, \quad S_4 = (N_3 - N_4 - M_2)/2$$

$$U_1 = -\rho_1 - \sigma_2 + \sigma_4 + \rho_2, \quad T_1 = (-M_1 - N_2 + N_4 + M_2)/2,$$
$$U_2 = \rho_1 - \sigma_2 + \sigma_4 + \rho_2, \quad T_2 = (M_1 - N_2 + N_4 + M_2)/2,$$
$$U_3 = 0, \quad T_3 = 0,$$
$$U_4 = 2\rho_2, \quad T_4 = M_2.$$

Note that

$$\sum_{a=1}^{4}(R_a + U_a) = -4\mathrm{i} \quad \text{and} \quad \sum_{a=1}^{4}(S_a + T_a) = 0$$

Quantum group $U_q(sl(2,\mathbb{R}))$

Generators: $E,\ F,\ K,\ K^{-1}$
Relations: $KE = qEK,\ KF = q^{-1}FK,\ [E,F] = -\frac{K^2-K^{-2}}{q-q^{-1}}$
Coproduct: $\Delta(K) = K \otimes K,\ \Delta(E) = E \otimes K + K^{-1} \otimes E,$
$\Delta(E) = F \otimes K + K^{-1} \otimes F$
Setting $K = e^{\tau H/2}$ and $q = e^{\tau}$ in the limit $\tau \to 0$ one obtains

$$[H, E^+] = 2E^+, \quad [H, E^-] = 2E^-, \quad [E^+, E^-] = -H$$

commutation relation for $sl(2,\mathbb{R})$. We take $q = e^{i\pi b^2}$.

$$\pi_\alpha(K) = e^{i\frac{b}{2}\partial_x}$$

$$\pi_\alpha(E^\pm) = e^{\pm 2\pi bx}\frac{e^{\pm i\pi\alpha b}e^{i\frac{b}{2}\partial_x} - e^{\mp i\pi\alpha b}e^{-i\frac{b}{2}\partial_x}}{q-q^{-1}}$$

This representation will be realized on the space $\mathcal{P}_\alpha$ of entire analytic functions. Unitarity requires $\alpha \in Q/2 + i\mathbb{R}$, $Q = b + b^{-1}$. The coproduct allows us to define tensor product of representations. For any $u \in U_q(sl(2,\mathbb{R})$ let

$$\pi_{1\otimes 2}(u) = (\pi_{\alpha_1} \otimes \pi_{\alpha_2})\Delta(u)$$

This operator generates the representation of $U_q(sl(2,\mathbb{R})$ on $\mathcal{P}_{\alpha_1} \otimes \mathcal{P}_{\alpha_2}$.
In the Lie algebra we have coproduct

$$\Delta(J) = J \otimes 1 + 1 \otimes J$$

for all generators.

6j-symbols of the Faddeev modular quantum double

$6j$-symbols of the Faddeev modular quantum double:
$U_q(sl(2,\mathbb{R}) \otimes U_{\tilde{q}}(sl(2,\mathbb{R}),\ q = e^{\pi i b^2}$ and $\tilde{q} = e^{\pi i b^{-2}}$.

$$\left\{ \begin{smallmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{smallmatrix} \middle| \begin{smallmatrix} \alpha_s \\ \alpha_t \end{smallmatrix} \right\}_b = \frac{S_b(\alpha_s + \alpha_2 - \alpha_1) S_b(\alpha_1 + \alpha_t - \alpha_4)}{S_b(\alpha_t + \alpha_2 - \alpha_3) S_b(\alpha_3 + \alpha_s - \alpha_4)} |S_b(2\alpha_t)|^2 J_h(\beta_a^\circ, \gamma_a^\circ)$$

$$J_h(\beta_a^\circ, \gamma_a^\circ) = \int_{-i\infty}^{i\infty} \prod_{a=1}^{4} S_b(z + \gamma_a^\circ) S_b(-z + \beta_a^\circ) \frac{dz}{i}$$

Ponsot-Teschner parametrization

$$\begin{aligned} \gamma_1^\circ &= \alpha_s + \alpha_1 - \alpha_2, & \beta_1^\circ &= -Q - \alpha_s + \alpha_t + \alpha_4 + \alpha_2, \\ \gamma_2^\circ &= Q + \alpha_s - \alpha_1 - \alpha_2, & \beta_2^\circ &= -\alpha_s - \alpha_t + \alpha_4 + \alpha_2, \\ \gamma_3^\circ &= \alpha_s + \alpha_3 - \alpha_4, & \beta_3^\circ &= Q - 2\alpha_s, \\ \gamma_4^\circ &= Q + \alpha_s - \alpha_3 - \alpha_4, & \beta_4^\circ &= 0. \end{aligned}$$

$$\sum_{a=1}^{4}(\gamma_a^\circ + \beta_a^\circ) = 2(b + b^{-1}) = 2Q\,.$$

The function $S_b(y)$ has the integral representation

$$S_b(y) = \exp\left(-\int_0^\infty \left(\frac{\sinh(2y-Q)x}{2\sinh(bx)\sinh(xb^{-1})} - \frac{2y-Q}{2x}\right)\right)\frac{dx}{x},$$

and obeys the equations:

$$\frac{S_b(y+b)}{S_b(y)} = 2\sin\pi yb, \quad \frac{S_b(y+b^{-1})}{S_b(y)} = 2\sin\frac{\pi y}{b}.$$

The function $S_b(y)$ has the asymptotics:

$$\lim_{y\to\infty} S_b(y) = e^{\mp\frac{i\pi}{2}\left(B_{2,2}(y,b,b^{-1})\right)}, \quad \text{for} \quad \mathfrak{Im}(y) \to \pm\infty,$$

where $B_{2,2}(y,b,b^{-1})$ is the second order Bernoulli polynomial

$$B_{2,2}(y;b,b^{-1}) = \left(y-\frac{Q}{2}\right)^2 - \frac{b^2+b^{-2}}{12}.$$

Hyperbolic hypergeometric function $J_h(\underline{\beta}, \underline{\gamma})$

Define

$$J_h(\underline{\beta}, \underline{\gamma}) = \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \prod_{a=1}^{4} S_b(\beta_a - z) S_b(\gamma_a + z) \frac{dz}{\mathrm{i}}$$

with the parameters β_a, γ_a satisfying the balancing condition

$$\sum_{a=1}^{4} (\gamma_a + \beta_a) = 2Q\,.$$

$$J_h(\underline{\beta},\underline{\gamma}) = \prod_{j,k=1}^{2} S_b(\gamma_j+\beta_{k+2})S_b(\gamma_{j+2}+\beta_k)J_h(G-\gamma_1, G-\gamma_2,$$

$$Q-G-\gamma_3, Q-G-\gamma_4; G-\beta_1, G-\beta_2, Q-G-\beta_3, Q-G-\beta_4),$$

where $G = \frac{1}{2}(\gamma_1+\gamma_2+\beta_1+\beta_2)$.

$$J_h(\underline{\beta}^\circ, \underline{\gamma}^\circ) = \Omega(\underline{\alpha}) J_h(\underline{\beta}^\diamond, \underline{\gamma}^\diamond)$$

where

$$\gamma_1^\diamond = \alpha_{1234}\,, \qquad \gamma_3^\diamond = 2Q\,, \qquad \beta_1^\diamond = -\alpha_{23t}\,, \qquad \beta_3^\diamond = -\alpha_{12s}\,,$$
$$\gamma_2^\diamond = \alpha_{13st}\,, \qquad \gamma_4^\diamond = \alpha_{24st}\,, \qquad \beta_2^\diamond = -\alpha_{14t}\,, \qquad \beta_4^\diamond = -\alpha_{34s}\,,$$

$$\alpha_{ijk} \equiv \alpha_i + \alpha_j + \alpha_k\,, \quad \alpha_{ijkl} \equiv \alpha_i + \alpha_j + \alpha_k + \alpha_l$$

and

and

$$\Omega(\underline{\alpha}) = S_b(Q+\alpha_s-\alpha_3-\alpha_4)S_b(Q+\alpha_s-\alpha_1-\alpha_2)$$
$$\times S_b(Q-\alpha_t+\alpha_2-\alpha_3)S_b(Q-\alpha_t+\alpha_4-\alpha_1)$$
$$\times S_b(-Q+\alpha_t+\alpha_2+\alpha_3)S_b(Q-\alpha_s+\alpha_3-\alpha_4)$$
$$\times S_b(-Q+\alpha_t+\alpha_4+\alpha_1)S_b(Q-\alpha_s+\alpha_1-\alpha_2).$$

Using this identity we obtain for the Ponsot-Teschner $6j$-symbols:

$$\left\{ \begin{smallmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{smallmatrix} \middle| \begin{smallmatrix} \alpha_s \\ \alpha_t \end{smallmatrix} \right\}_b = S_b(Q+\alpha_s-\alpha_3-\alpha_4)S_b(Q+\alpha_s-\alpha_1-\alpha_2)$$
$$\times S_b(Q-\alpha_t+\alpha_2-\alpha_3)S_b(Q-\alpha_3+\alpha_4-\alpha_s)$$
$$\times S_b(-Q+\alpha_t+\alpha_2+\alpha_3)S_b(Q-\alpha_s+\alpha_3-\alpha_4)$$
$$\times S_b(-Q+\alpha_t+\alpha_4+\alpha_1)S_b(Q-\alpha_t+\alpha_3-\alpha_2)J_h(\underline{\beta}^{\diamond},\underline{\gamma}^{\diamond}) \ .$$

This expression is explicitly invariant under the Regge transformation:

$$\left\{ \begin{smallmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{smallmatrix} \middle| \begin{smallmatrix} \alpha_s \\ \alpha_t \end{smallmatrix} \right\}_b = \left\{ \begin{smallmatrix} S-\alpha_1 & S-\alpha_2 \\ S-\alpha_3 & S-\alpha_4 \end{smallmatrix} \middle| \begin{smallmatrix} \alpha_s \\ \alpha_t \end{smallmatrix} \right\}_b ,$$

where $S = \frac{1}{2}(\alpha_1+\alpha_2+\alpha_3+\alpha_4)$, since it just permutes with each other the prefactor terms, as well as the parameters $\underline{\beta}^{\diamond}$ and $\underline{\gamma}^{\diamond}$.

$$S_{\mathrm{i}+\delta}(\mathrm{i}(n+x\delta)) \underset{\delta\to 0}{=} e^{\frac{\pi \mathrm{i} n^2}{2}}(4\pi\delta)^{\mathrm{i}x-1}\mathbf{\Gamma}(x,n).$$

The integral

$$\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty}\Delta(z)\frac{dz}{\mathrm{i}}$$

where $\Delta(z)$ is a product of $S_b(z)$, in this limit becomes

$$\sum_{N\in\mathbb{Z}}\int_{-\infty}^{\infty}\left[\lim_{\delta\to 0}\delta\Delta(\mathrm{i}\sqrt{\omega_1\omega_2}(N+y\delta))\right]dy.$$

Setting

$$\alpha_1 = Q/2 - \mathrm{i}(N_1/2 + \sigma_1\delta), \qquad \alpha_3 = Q/2 - \mathrm{i}(N_3/2 + \sigma_3\delta),$$
$$\alpha_2 = Q/2 - \mathrm{i}(N_2/2 + \sigma_2\delta), \qquad \alpha_4 = Q/2 + \mathrm{i}(N_4/2 + \sigma_4\delta),$$

$$\alpha_t = Q/2 - \mathrm{i}(M_1/2 + \rho_1\delta), \quad \alpha_s = Q/2 + \mathrm{i}(M_2/2 + \rho_2\delta),$$

$$\lim_{\delta\to 0}\left\{ \begin{smallmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{smallmatrix} \Big| \begin{smallmatrix} \alpha_s \\ \alpha_t \end{smallmatrix} \right\}_{i+\delta} = e^{\frac{\pi \mathrm{i}}{2}F} \frac{M_1^2 + 4\rho_1^2}{16\pi^4 \mathrm{i}\delta} \left\{ \begin{smallmatrix} \sigma_1, N_1 & \sigma_2, N_2 \\ \sigma_3, N_3 & \sigma_4, N_4 \end{smallmatrix} \Big| \begin{smallmatrix} \rho_1, M_1 \\ \rho_2, M_2 \end{smallmatrix} \right\}$$

where on the rhs we have $6j$-symbols for the group $SL(2,\mathbb{C})$ and F is a sign factor.

$$\mathcal{J}_{cr}(\underline{s},\underline{n};\underline{t},\underline{m}) = \frac{1}{4\pi}\sum_{N\in\mathbb{Z}}\int_{-\infty}^{\infty}\prod_{a=1}^{4}\boldsymbol{\Gamma}(s_a-y,n_a-N)\boldsymbol{\Gamma}(t_a+y,m_a+N)dy.$$

$$\sum_{a=1}^{4}(n_a+m_a)=0, \qquad \sum_{a=1}^{4}(s_a+t_a)=-4\mathrm{i}.$$

$$\mathcal{J}_{cr}(\underline{s}, \underline{n}; \underline{t}, \underline{m}) =$$

$$e^{\pi \mathrm{i} A} \prod_{j,k=1}^{2} \Gamma(s_j + t_{k+2}, n_j + m_{k+2}) \prod_{j,k=1}^{2} \Gamma(s_{j+2} + t_k, n_{j+2} + m_k)$$

$$\times \mathcal{J}_{cr}(Y - t_1, K - m_1, Y - t_2, K - m_2,$$

$$-2i - t_3, -m_3, -2i - t_4, -m_4; -s_1, -n_1, -s_2, -n_2,$$

$$-2i - Y - s_3, -K - n_3, -2i - Y - s_4, -K - n_4),$$

$$K = n_1 + n_2 + m_1 + m_2, \qquad Y = s_1 + s_2 + t_1 + t_2,$$

$$A = (n_1 + n_2)(m_3 + m_4) + (n_3 + n_4)(m_1 + m_2)$$

$$+ \frac{1}{2}(1 - (-1)^K)\left(1 + \sum_{a=1}^{4} m_a\right).$$

Inserting this into Ismagilov formula we obtain

$$\left\{ \begin{smallmatrix} \sigma_1, N_1 & \sigma_2, N_2 \\ \sigma_3, N_3 & \sigma_4, N_4 \end{smallmatrix} \middle| \begin{smallmatrix} \rho_1, M_1 \\ \rho_2, M_2 \end{smallmatrix} \right\} =$$

$$e^{\pi i B} \frac{\pi^2}{4} \Omega(\underline{\sigma}, \underline{\rho}, \underline{N}, \underline{M}) \mathcal{J}_{cr}(\underline{\tilde{R}}, \underline{\tilde{S}}; \underline{\tilde{U}}, \underline{\tilde{T}})$$

where

$$\begin{aligned}
&\Omega(\underline{\sigma}, \underline{\rho}, \underline{N}, \underline{M}) = \mathbf{\Gamma}(\sigma_1 - \sigma_2 + \rho_2 - i, (N_1 - N_2 + M_2)/2) \\
&\times \mathbf{\Gamma}(-\sigma_1 + \sigma_2 + \rho_2 - i, (-N_1 + N_2 + M_2)/2) \\
&\times \mathbf{\Gamma}(-\rho_1 - \sigma_2 - \sigma_3 - i, -(M_1 - N_2 - N_3)/2) \\
&\times \mathbf{\Gamma}(-\rho_1 + \sigma_4 - \sigma_1 - i, (-M_1 + N_4 - N_1)/2) \\
&\times \mathbf{\Gamma}(\sigma_3 - \sigma_2 + \rho_1 - i, (N_3 - N_2 + M_1)/2) \\
&\times \mathbf{\Gamma}(\sigma_3 - \sigma_4 - \rho_2 - i, (N_3 - N_4 - M_2)/2) \\
&\times \mathbf{\Gamma}(\sigma_2 - \sigma_3 + \rho_1 - i, (N_2 + N_3 + M_1)/2) \\
&\times \mathbf{\Gamma}(\sigma_2 + \sigma_1 - \rho_2 - i, (N_2 + N_1 - M_2)/2)
\end{aligned}$$

and

$$\tilde{R}_1 = \sigma_4 - \sigma_2 - \sigma_1 - \sigma_3, \quad \tilde{S}_1 = (N_4 - N_2 - N_1 - N_3)/2,$$
$$\tilde{R}_2 = -\rho_2 - \sigma_1 - \sigma_3 - \rho_1, \quad \tilde{S}_2 = (-M_2 - N_1 - N_3 - M_1)/2,$$
$$\tilde{R}_3 = 0, \quad \tilde{S}_3 = 0,$$
$$\tilde{R}_4 = \sigma_4 - \rho_2 - \rho_1 - \sigma_2, \quad \tilde{S}_4 = (N_4 - M_2 - M_1 - N_2)/2,$$

$$\tilde{U}_1 = \sigma_3 + \sigma_2 + \rho_1 - i, \quad \tilde{T}_1 = (N_3 + N_2 + M_1)/2,$$
$$\tilde{U}_2 = \sigma_1 - \sigma_4 + \rho_1 - i, \quad \tilde{T}_2 = (N_1 - N_4 + M_1)/2,$$
$$\tilde{U}_3 = \sigma_1 + \sigma_2 + \rho_2 - i, \quad \tilde{T}_3 = (N_1 + N_2 + M_2)/2,$$
$$\tilde{U}_4 = \sigma_3 - \sigma_4 + \rho_2 - i, \quad \tilde{T}_4 = (N_3 - N_4 + M_2)/2.$$

$$\mathcal{S} = \frac{1}{2}(\sigma_1 + \sigma_2 + \sigma_3 - \sigma_4)$$
$$\mathcal{N} = \frac{1}{2}(N_1 + N_2 + N_3 - N_4)$$

$$\sigma_1 \to \mathcal{S} - \sigma_1, \quad \sigma_2 \to \mathcal{S} - \sigma_2, \quad \sigma_3 \to \mathcal{S} - \sigma_3, \quad \sigma_4 \to -(\mathcal{S} + \sigma_4)$$
$$N_1 \to \mathcal{N} - N_1, \; N_2 \to \mathcal{N} - N_2, \; N_3 \to \mathcal{N} - N_3, \; N_4 \to -(\mathcal{N} + N_4)$$

$$B=\left[\frac{1}{2}(N_2-N_1-N_3-N_4)-M_2\right]\left[\frac{1}{2}(N_4-N_2)+\frac{3}{2}M_2-\frac{1}{2}M_1\right]$$
$$+\left[\frac{1}{2}(N_1+N_2+N_3-N_4)-M_2\right]\left[\frac{1}{2}(N_4-N_2)+\frac{1}{2}M_2+\frac{1}{2}M_1\right]$$
$$+\frac{1}{2}\left(1-(-1)^{\frac{1}{2}(-N_3-N_1+M_1-M_2)}\right)(1+2M_2+N_4-N_2)$$
$$+M_2+N_4-N_2$$

$$
\begin{aligned}
B' = \\
&\left[-\frac{1}{2}(N_2 - N_1 - N_3 - N_4) - M_2\right]\left[-\frac{1}{2}(N_1 + N_3) + \frac{3}{2}M_2 - \frac{1}{2}M_1\right] \\
&+ \left[\frac{1}{2}(N_1 + N_2 + N_3 - N_4) - M_2\right]\left[-\frac{1}{2}(N_1 + N_3) + \frac{1}{2}M_2 + \frac{1}{2}M_1\right] \\
&+\frac{1}{2}\left(1 - (-1)^{\frac{1}{2}(N_4 - N_2 + M_1 - M_2)}\right)(1 + 2M_2 - N_1 - N_3) \\
&+M_2 - N_1 - N_3
\end{aligned}
$$

Regge symmetry of Racah-Wigner symbols of $SL(2,\mathbb{C})$

$$\left\{ \begin{matrix} \mathcal{S}-\sigma_1,\mathcal{N}-N_1 & \mathcal{S}-\sigma_2,\mathcal{N}-N_2 \\ \mathcal{S}-\sigma_3,\mathcal{N}-N_3 & -(\mathcal{S}+\sigma_4),-(\mathcal{N}+N_4) \end{matrix} \middle| \begin{matrix} \rho_1,M_1 \\ \rho_2,M_2 \end{matrix} \right\} = (-1)^{B-B'} \left\{ \begin{matrix} \sigma_1,N_1 & \sigma_2,N_2 \\ \sigma_3,N_3 & \sigma_4,N_4 \end{matrix} \middle| \begin{matrix} \rho_1,M_1 \\ \rho_2,M_2 \end{matrix} \right\}$$