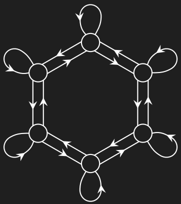

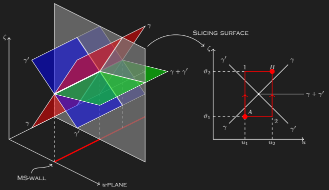

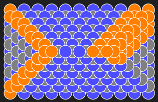

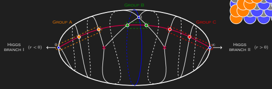

BPS Algebraic Structures Related to Toric Calabi-Yau Manifolds

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Problems of the Modern Mathematical Physics
JINR, Dubna, February 20th, 2024

Motivation

Based on:

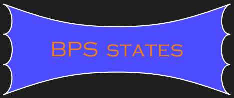
2008.07006	with M.Yamazaki	2311.02751	with W.Li
2106.01230	with W.Li, M.Yamazaki	2307.03150	with A.Morozov, N.Tselousov
2108.10286	with W.Li, M.Yamazaki	2311.00760	with A.Morozov, N.Tselousov
2206.13340	with W.Li, M.Yamazaki	2402.05920	with A.Morozov, N.Tselousov

Enum.Geometry

Categorification

New Algebras & Reps

Modularity



Non-perturbative phenomena

Wall-crossing $\longrightarrow$ phase transition

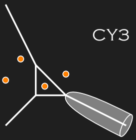
Integrability

Black hole physics

[Nakajima; Kontsevich, Soibelman; Alday, Gaiotto, Tachikawa; Douglass, Moore; Schifman, Vasserot, ...]

UV $\downarrow$ IR

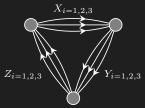
Type II
D6/D4/D2/D0 $\longrightarrow$



4d $\mathcal{N} = 1$ SYM

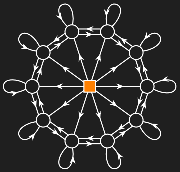
$\downarrow$

1d $\mathcal{N} = 4$ QM



$+ \quad W = \epsilon^{ijk}\,\mathrm{Tr}\, Z_i Y_j X_k$

Quiver BPS Algebras



Q_0 • quiver vertices

Q_1 • quiver arrows

Q_2 • superpotential

$a, b \in Q_0$

$|a| = (|a \to a| + 1) \bmod 2$

$I, J \in Q_1$

$$e^{(a)}(z) = \sum_{n\in\mathbb{Z}_{\geqslant 0}} \frac{e_n^{(a)}}{z^n},$$

$$f^{(a)}(z) = \sum_{n\in\mathbb{Z}_{\geqslant 0}} \frac{f_n^{(a)}}{z^n},$$

$$\psi^{(a)}(z) = \sum_{n\in\mathbb{Z}} \frac{\psi_n^{(a)}}{z^n},$$

$h_I \in \mathbb{C}$ • equiv. weights, flavor charge

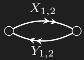
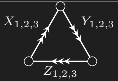
Bond factor: $\varphi^{a\Leftarrow b}(u) \equiv \frac{\prod_{I\in\{a\to b\}} (u+h_I)}{\prod_{J\in\{b\to a\}} (u-h_J)}$

Rational quiver BPS algebra (quiver Yangian) [Li-Yamazaki •20]

$$\begin{aligned}
\psi^{(a)}(z)\,\psi^{(b)}(w) &= \psi^{(b)}(w)\,\psi^{(a)}(z)\,,\\
\psi^{(a)}(z)\,e^{(b)}(w) &\simeq \varphi^{a\Leftarrow b}(z-w)\,e^{(b)}(w)\,\psi^{(a)}(z)\,,\\
e^{(a)}(z)\,e^{(b)}(w) &\simeq (-1)^{|a||b|}\varphi^{a\Leftarrow b}(z-w)\,e^{(b)}(w)\,e^{(a)}(z)\,,\\
\psi^{(a)}(z)\,f^{(b)}(w) &\simeq \varphi^{a\Leftarrow b}(z-w)^{-1}\,f^{(b)}(w)\,\psi^{(a)}(z)\,,\\
f^{(a)}(z)\,f^{(b)}(w) &\simeq (-1)^{|a||b|}\varphi^{a\Leftarrow b}(z-w)^{-1}\,f^{(b)}(w)\,f^{(a)}(z)\,,\\
\left[e^{(a)}(z), f^{(b)}(w)\right\} &\simeq -\delta^{a,b}\,\frac{\psi^{(a)}(z)-\psi^{(a)}(w)}{z-w}\,,
\end{aligned}$$

$\simeq$ • equivalent up to $z^n w^{m\geqslant 0}$, $z^{n\geqslant 0} w^m$

Quiver BPS Algebras II

$\mathbb{C}^3$		 $W = \mathrm{Tr}\ X_1 [X_2, X_3]$	$\mathsf{Y}\left(\widehat{\mathfrak{gl}}_1\right)$
Conifold		 $W = \mathrm{Tr}\ (Y_2 X_2 Y_1 X_1 - Y_2 X_1 Y_1 X_2)$	$\mathsf{Y}\left(\widehat{\mathfrak{gl}}_{1\|1}\right)$
$xy = z^n w^m$	...	...	$\mathsf{Y}\left(\widehat{\mathfrak{gl}}_{n\|m}\right)$
$K_{\mathbb{P}^2}$		 $W = \mathrm{Tr}\ \epsilon^{ijk} Z_i Y_j Z_k$	$\mathsf{Y}\,(K_{\mathbb{P}^2})$???

Representations: MacMahon-like, Fock-like, vector-like and more

$\Big[$ $\mathsf{E}_\tau(Q, W)$  $\xrightarrow{\text{dim. red.}}$ $\mathsf{T}_\beta(Q, W)$  $\xrightarrow{\text{dim. red.}}$ $\mathsf{Y}(Q, W)$  $\Big] \times \left(\text{Toric Calabi-Yau 3-fold}\right)$

D8-D6-D4-D2-D0	D7-D5-D3-D1	D6-D4-D2-D0
$\vartheta_{11}(z\|\tau)$	$2\sinh(\beta z/2)$	z
3d $\mathcal{N} = 2$	2d $\mathcal{N} = (2, 2)$	1d $\mathcal{N} = 4$
Ell. Cohomology	K-theory	Cohomology

Localization

[Denef •02]

[Witten•82, Gaiotto-Moore-Witten• 15,...]

$$\psi_i \rightsquigarrow dx^i, \quad \psi_i^\dagger \rightsquigarrow \iota_{\partial/\partial x^i}, \quad \mathcal{Q}_\alpha, \bar{\mathcal{Q}}_{\dot\alpha} \rightsquigarrow \text{differentials}, \quad \mathcal{H} \rightsquigarrow \text{Laplacian}$$

de Rham — Dolbeault — equivariant — sup. twist

$$\mathcal{Q} = e^{-h}\left(d + \bar\partial + \iota_V + dW\wedge\right)e^{h}$$

Morse height function

BPS states:

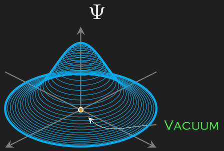
$$\mathscr{H}_{\text{BPS}} = H_G^*\left(\text{Target Space}, \mathcal{Q}\right) \approx \bigoplus_{p\in\mathcal{I}} \mathbb{C}\Psi_p$$

$$\mathcal{I} = \{\text{crit. fixed points}\} = \{\text{classical vacua}\}$$

$$\mathcal{Q}^\dagger \sim \sum_i \left(d\bar{x}^i \partial_{\bar{x}^i} + \omega_i x^i \iota_{\partial/\partial x^i}\right)$$

$$\text{Eul} = \bigwedge_i \left(\omega_i - |\omega_i|\, dx^i \wedge d\bar{x}^i\right) e^{-|\omega_i||x_i|^2} =$$

$$= \prod_i \omega_i \times \exp\left(-\left\{\mathcal{Q}^\dagger, \sum_i \frac{|\omega_i|}{\omega_i}\bar{x}^i\, dx^i\right\}\right)$$



For quiver QFT target space ≈ Quiver Moduli Space

Fixed Points

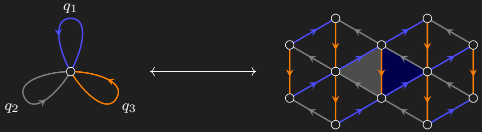
D-term + F-term (ADHM instantons in 6D):

$$\sum_{x\in Q_0}\sum_{I\in\{a\to x\}} q_I {q_I}^\dagger - \sum_{y\in Q_0}\sum_{J\in\{y\to a\}} q_J^\dagger q_J = \zeta_a \mathrm{Id}_{d_a\times d_a}, \quad \forall\ a\in Q_0;$$

$$\Phi_b q_I - q_I \Phi_a - h_I q_I = 0, \quad \forall\ a,b\in Q_0,\ I\in\{a\to b\};$$

$$\partial_{q_I} W = 0, \quad \forall\ I\in Q_1.$$

Periodic quiver:



$$W = \sum_{\text{faces}} (-1)^{\text{ori}}\, \mathrm{Tr} \prod_{\text{loop}} q = \triangle - \triangle = \mathrm{Tr}\ q_1\,[q_2, q_3]$$

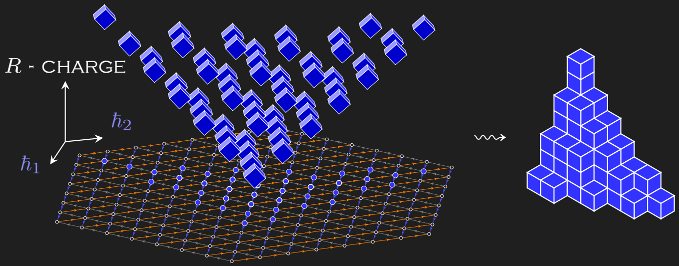
Constraints on flavor charges (masses):

$$\left.\begin{array}{lll} \text{Loop:} & \sum_{\text{loop}} h_I = 0, & \forall \text{faces}; \\ \text{Vertex:} & h_I \sim h_I - \epsilon_a + \epsilon_b, & \forall I\in\{a\to b\}. \end{array}\right\} \ h_I = x_I \hbar_1 + y_I \hbar_2.$$

Equivariant toric action on CY3: $(z_1, z_2, z_3) \longmapsto \left(e^{\hbar_1} z_1, e^{\hbar_2} z_2, e^{-\hbar_1-\hbar_2} z_3\right)$

Crystals

Quiver path algebra: $\mathbb{C}Q/\langle dW\rangle \rightsquigarrow \prod q$ - Baryons
Crystal = Possible baryons:



$\square$ • atom of a crystal

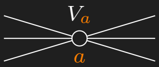
Color of $\square$ denoted $\hat{\square} \in Q_0$ is a color of atom projection to $(\hbar_1, \hbar_2)$

Melting rule: K • molten crystal

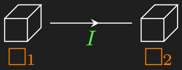
> For any atom $\square$ such that $I \cdot \square \in \mathrm{K}$ for some arrow I, then $\square$ is also contained in K

[Szendroi; Mozgovoy, Reyneke; Nagao, Nakajima; Ooguri-Yamazaki; Jafferis, Chuang, Moore; Sulkowski; Aganagic, Schaeffer; Aganagic, Vafa; ...]

Euler classes



$$\mathcal{V} = \bigoplus_{a \in Q_0} V_a$$



Quiver representation in Crystal basis:

$$V_a = \bigoplus_{\square \in \mathrm{K},\, \hat{\square} = a} \mathbb{C}|\square\rangle, \quad a \in Q_0,$$

$$q_I \in \mathrm{Hom}\left(V_{\mathrm{beg}\,I}, V_{\mathrm{beg}\,J}\right)$$
$$q_I = \langle q_I \rangle + \delta q_I$$
$$\langle \square_2 | q_I | \square_1 \rangle = \begin{cases} 1, & \text{link present} \\ 0, & \text{otherwise} \end{cases}$$

G-action:

$$\delta q_{I \in \{a \to b\}} \mapsto \delta q_{I \in \{a \to b\}} + g_a \langle q_I \rangle - \langle q_I \rangle g_b, \quad g_a \in \mathfrak{gl}(d_a, \mathbb{C}), \; g_b \in \mathfrak{gl}(d_b, \mathbb{C})$$

Fixed point structure:

$$\underbrace{\text{Fixed Point K}}_{\text{Baryons, } \langle q_I \rangle} + \underbrace{(\text{Tangent Space}/G)}_{\text{Mesons, } \delta q_I}$$

IR meson flavor charges:

$$h_{\mathrm{eff}}\left(\langle \square_2 | \delta q_I | \square_1 \rangle\right) = h_{\square_2} - h_{\square_1} - h_I$$

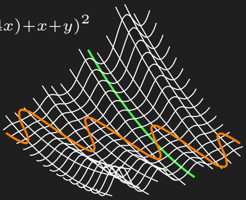
Meson space:

$$\mathcal{M}_{\mathrm{meson}} = \mathrm{Span}\left\{ q_\alpha, h_\alpha \right\}_{\alpha=1}^{N_{\mathrm{meson}}}, \quad \mathrm{Eul}(\mathcal{M}) \sim \prod_\alpha h_\alpha$$

However it is singular!

(Regularized) Euler classes

$U(x,y)=(\sin(4x)+x+y)^2$



$$\Psi_{\mathrm{UV}} \longrightarrow \Psi_{\mathrm{IR}} \sim \left[\begin{array}{c}\text{FREE}\\ \text{PARTICLE ON CLASS.}\\ \text{VACUUM LOCUS}\end{array}\right] \times \left[\ \text{FLUCTUATIONS}\ \right]$$

$$m\,(\text{FLUCTUATION}) \sim |h|$$

NEED TO ADD HIGHER LOOPS!

$$\mathcal{Q} = \quad e^{-sh}\left(d+\bar{\partial}+\iota_{sV}+s\,dW\wedge\right)e^{sh} \quad \Rightarrow \quad e^{-s_1 h}\left(d+\bar{\partial}+\iota_{s_1 V}+s_2\,dW\wedge\right)e^{s_1 h}$$

$$\text{IR:} \qquad s\to\infty \quad \Rightarrow \quad s_1\to\infty \text{ THEN } s_2\to\infty$$

SUPERPOTENTIAL FOR MASSLESS MODES:

$$W \sim A\phi_1\phi_2 \quad \Rightarrow \quad \Psi_{\mathrm{IR}}(\phi_1\phi_2) \sim (-A)(A) \sim (-1)$$

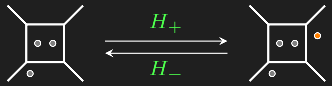
CONJECTURE: [G.-YAMAZAKI '20]

$$\mathcal{N} = \mathrm{Span}\{q_\alpha, h_\alpha\}_{\alpha=1}^{N}$$

$$\widetilde{\mathrm{Eul}}(\mathcal{N}) = (-1)^{\left\lfloor \sum_{\alpha:\ h_\alpha=0} \frac{1}{2} \right\rfloor} \prod_{\alpha:\ h_\alpha\neq 0} h_\alpha$$

Hecke modification

Adding/deleting branes → Hecke modifications:



[Nakajima • 99; Kontsevich-Soibelman • 11; ...]

Fourier-Mukai transform:

$$\mathrm{Rep}(Q,\vec{d}) \underset{f}{\overset{e}{\rightleftarrows}} \mathrm{Rep}(Q,\vec{d}')$$

$$\mathrm{Rep}(Q,\vec{d}) \leftarrow \mathrm{Rep}(Q,\vec{d}) \times \mathrm{Rep}(Q,\vec{d}') \rightarrow \mathrm{Rep}(Q,\vec{d}')$$

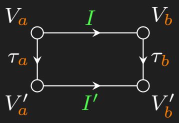
$$\vec{d}' = \vec{d} + \vec{1}_{a\in Q_0}$$

$$\vec{1}_a := \left(0,\ldots,0,\overset{a^{\mathrm{th}}\ \mathrm{place}}{1},0,\ldots,0\right)$$

with a kernel given by $\mathcal{O}_{\mathcal{I}}$ where $\mathcal{I}$ is an incidence locus:

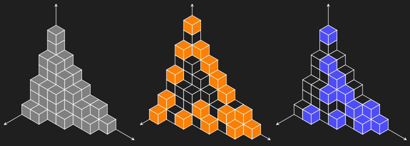
$$\mathcal{I} = \left\{ \mathrm{Rep}(Q,\vec{d}) \xhookrightarrow[\text{HOMO}]{} \mathrm{Rep}(Q,\vec{d}') \right\}$$

Homomorphism of quiver reps: $\{\tau_a\}_{a\in Q_0}$, $\tau_a : V_a \to V_a{}'$

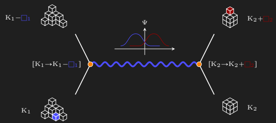
so that  commutes $\forall I \in Q_1$

Matrix elements

Fixed point: $\mathcal{I} = \{\mathrm{K} \subset \mathrm{K}'\}$,
denote corresponding Euler class as $\widetilde{\mathrm{Eul}}(\mathrm{K}, \mathrm{K}')$
Vacant positions:



a) K   b) Add (K)   c) Rem (K)



$$e|\mathrm{K}\rangle = \sum_{\square\in \mathrm{Add}(\mathrm{K})} [\mathrm{K}\to\mathrm{K}+\square]|\mathrm{K}+\square\rangle \qquad f|\mathrm{K}\rangle = \sum_{\square\in \mathrm{Rem}(\mathrm{K})} [\mathrm{K}\to\mathrm{K}-\square]|\mathrm{K}-\square\rangle$$

$$[\mathrm{K}\to\mathrm{K}+\square] = \frac{\widetilde{\mathrm{Eul}}(\mathrm{K})}{\widetilde{\mathrm{Eul}}(\mathrm{K},\mathrm{K}+\square)} \qquad [\mathrm{K}\to\mathrm{K}-\square] = \frac{\widetilde{\mathrm{Eul}}(\mathrm{K})}{\widetilde{\mathrm{Eul}}(\mathrm{K}-\square,\mathrm{K})}$$

Numerical results ($[a\to b\to c] := [a\to b]\cdot[b\to c]$):

$$[\mathrm{K}+\boxed{a}_1 \to \mathrm{K}+\boxed{a}_1+\boxed{b}_2 \to \mathrm{K}+\boxed{b}_2] = [\mathrm{K}+\boxed{a}_1\to\mathrm{K}\to\mathrm{K}+\boxed{b}_2]\;,$$

$$\frac{[\mathrm{K}\to\mathrm{K}+\boxed{b}_2\to\mathrm{K}+\boxed{a}_1+\boxed{b}_2]}{[\mathrm{K}\to\mathrm{K}+\boxed{a}_1\to\mathrm{K}+\boxed{a}_1+\boxed{b}_2]} = \varphi^{a\Leftarrow b}\left(h_{\boxed{a}_1}-h_{\boxed{b}_2}\right)\;,$$

$$\frac{[\mathrm{K}+\boxed{a}_1+\boxed{b}_2\to\mathrm{K}+\boxed{b}_2\to\mathrm{K}]}{[\mathrm{K}+\boxed{a}_1+\boxed{b}_2\to\mathrm{K}+\boxed{a}_1\to\mathrm{K}]} = \varphi^{a\Leftarrow b}\left(h_{\boxed{a}_1}-h_{\boxed{b}_2}\right)\;,$$

$$[\mathrm{K}\to\mathrm{K}+\boxed{a}\to\mathrm{K}] = \operatorname*{res}_{t=h_{\boxed{a}}} \Psi_{\mathrm{K}}^{(a)}(t)$$

,

$$\Psi_{\mathrm{K}}^{(a)}(z) = \left(\prod_{I\in\{a\to a\}} \frac{1}{-h_I}\right)\times$$
$$\times \prod_{\boxed{b}\in\mathrm{K}} \varphi^{a\Leftarrow b}\left(z-h_{\boxed{b}}\right)$$

Spectral parameters

$$\mathcal{N}=4 \text{ SQM} \xleftarrow{\text{DIM.RED.}} \mathcal{N}=1 \text{ 4D SYM}$$

Vector multiplet: $A_0,\ X_{i=1,2,3},\ \psi_\alpha,\ D$
Notice for $\Phi_a = A_{1,a} - \mathrm{i}A_{2,a}$, $a \in Q_0$:

$$[\mathcal{Q}, \Phi_a] = 0$$

therefore $\mathrm{Tr}\ \Phi_a{}^k$, $k \in \mathbb{Z}$ is a BPS operator.
For example, for a resolvent:

$$\mathrm{Tr}\ (z-\Phi_a)^{-1}\,|\mathrm{K}\rangle = \left(\sum_{\boxed{a}\in\mathrm{K}} \frac{1}{z-h_{\boxed{a}}}\right)|\mathrm{K}\rangle$$

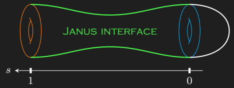
Eventually, we define:

$$\begin{aligned} e^{(a)}(z) &= \left[\mathrm{Tr}\ (z-\Phi_a)^{-1}, e\right] \\ f^{(a)}(z) &= -\left[\mathrm{Tr}\ (z-\Phi_a)^{-1}, f\right] \\ \psi^{(a)}(z) &= \exp\left(\sum_{b\in Q_0} \mathrm{Tr}\ \log \varphi^{a\Leftarrow b}(z-\Phi_a)\right) \end{aligned}$$

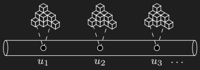
$e^{(a)}(z),\ f^{(a)}(z),\ \psi^{(a)}(z)$ are a basis of a quiver Yangian

INTEGRABILITY

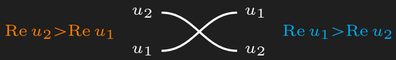
[WORK IN PROGRESS...]



$$Z_i(s=1) = \sum_j M_{i,j}(1,0) Z_j(s=0)$$

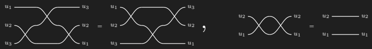


[BULLIMORE-KIM-LUKOWSKI • 17]



$$R_{12}(u_{12}) = \frac{u_{12}}{u_{12}+\hbar}\mathrm{Id} + \frac{\hbar}{u_{12}+\hbar}P_{12}$$

YBE+UNITARITY:

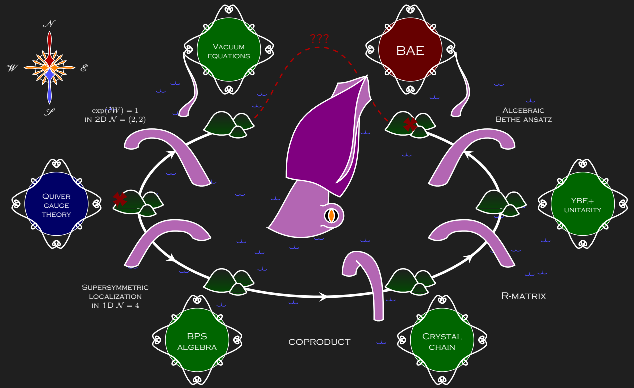


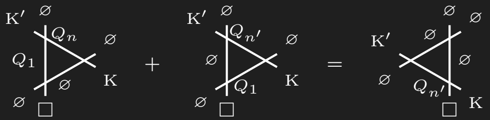
TRANSFER MATRIX: $T(z) := \mathrm{Tr}' R_{0n}(z-u_n)\ldots R_{01}(z-u_1)$, $T(z) : \mathcal{F}^{\otimes n} \to \mathcal{F}^{\otimes n}$

$$[T(u), T(v)] = 0$$

BAEs FOR OFF-SHELL BETHE VECTORS FOLLOWS FROM THE BETHE/GAUGE CORRESPONDENCE

Is there Gauge/Bethe correspondence?





$$\sum_{k\geqslant 1}\Psi_{\varnothing,-k}\frac{\left(h_{\square'}+u_3-u_1\right)^k-\left(h_{\square}+u_2-u_1\right)^k}{\left(h_{\square}+u_2\right)-\left(h_{\square'}+u_3\right)}=0 \Longrightarrow$$ NO SHIFTS!

Open Problems

- Non-toric Calabi-Yau manifolds
- Wall-crossing
- Stable envelopes
- Categorification
- Vortices $\to$ 4d (solid) partitions $\to$ CY_4 $\to$ Mama-algebra
- ...

Thank you for your attention