Euclidean path integral and thermal partition function in curved space-time with Killing horizon arXiv:2310.08522

Diakonov Dmitrii

MIPT, ITEP

Problems of the Modern Mathematical Physics

In the Rindler coordinate:

$$ds^2 = r^2 dt^2 - dr^2 - d\vec{z}^2, \tag{1}$$

one can compute the Euclidean path integral:

$$\log Z^E = \tag{2}$$

$$= \int d^4X \sqrt{g} \frac{1}{(\sqrt{g_{00}})^4} \left(\frac{\pi^2}{90} \left[\frac{1}{\beta^4} - \frac{1}{(\beta_c)^4} \right] + \frac{\pi^2}{9} \frac{1}{(\beta_c)^2} \left[\frac{1}{\beta^2} - \frac{1}{(\beta_c)^2} \right] \right).$$

and expectation value of the Hamiltonian:

$$\langle \hat{H} \rangle_\beta = \int d^3x \sqrt{g} \frac{\pi^2}{30} \frac{1}{(\sqrt{g_{00}})^4} \left[\frac{1}{\beta^4} - \frac{1}{\beta_c^4} \right]. \tag{3}$$

The fundamental statistical-mechanical relation does not hold:

$$-\partial_\beta \log Z^E \neq \langle \hat{H} \rangle_\beta = -\partial_\beta \log Tr(e^{-\beta :\hat{H}:}) \tag{4}$$

Statement

Is the Euclidean path integral always equal to the thermal partition function?

$$Z^E = \int d[\varphi] e^{-S[\varphi]} =^? Tr(e^{-\beta :\hat{H}:}) = Z^C \quad (5)$$

$$det^{-\frac{1}{2}} \left[\frac{-\Box_E + m^2 + \xi R}{\mu^2} \right] =^? det^{-1/2} \left[\frac{g_{00} \left(-\Box_E + m^2 + \xi R \right)}{\mu^2} \right] \quad (6)$$

The determinant of the product is equal to the product of the determinants. Hence:

$$det^{-1/2}(g^{00}) = exp \left[-\frac{1}{2} Tr \left[log \left(g^{00} \right) \right] \right] = \quad (7)$$

$$= exp \left[-\frac{1}{2} \int d^4 x \sqrt{g} \sum_i \phi_i^*(x) \phi_i(x) \log \left(g^{00} \right) \right] = exp \left[-\frac{\beta}{2} \int d^3 x \sqrt{g} \log \left(g^{00} \right) \delta^{(4)}(0) \right].$$

In dimensional regularization $\delta^{(4)}(0) = 0$. For space-times with Killing horizons:

$$\delta^{(4)}(0) \log \left(g^{00} \right) \sim 0 \times \infty. \quad (8)$$

YES:

- space-time without Killing horizon

No:

- space-time with Killing horizons

Euclidean path integral

The Euclidean path integral is defined as follows:

$$Z^E = \int d[\varphi] e^{-S[\varphi]}, \qquad \varphi(\tau) = \varphi(\tau + \beta) \tag{9}$$

The functional measure is given by:

$$d[\varphi] = \prod_x \frac{d\varphi(x)}{\sqrt{2\pi}} g^{\frac{1}{4}} = \prod_i \frac{dc_i}{\sqrt{2\pi}}. \tag{10}$$

Here c_i are the Fourier coefficients of the field $\varphi(x) = \sum_i c_i \phi_i(x)$, which are eigenfunction :

$$\left(-\Box_E + m^2 + \xi R\right) \phi_i(x) = \lambda_i \phi_i(x). \tag{11}$$

Path integral in terms of the functional determinant:

$$\int d[\varphi] e^{-S[\varphi]} = det^{-\frac{1}{2}} \left[\frac{-\Box_E + m^2 + \xi R}{\mu^2}\right], \tag{12}$$

where μ is the normalization scale.

Thermal partition function

The thermal partition function of the canonical ensemble is defined as:

$$Z^C = Tr(e^{-\beta :\hat{H}:}). \tag{13}$$

where $:\hat{H}: = \sum_i \omega_i \hat{a}_i^\dagger \hat{a}_i$ is the usual normal ordered Hamiltonian.
The trace is defined as the sum over all distinct (many-particle) states of the system:

$$Tr(e^{-\beta :\hat{H}:}) = \prod_i \left(\sum_{n=0}^{\infty} e^{-\beta n \omega_i} \right) = \prod_i \left(1 - e^{-\beta \omega_i}\right)^{-1}. \tag{14}$$

The energies of single-particle states can be found using Klein-Gordon equation:

$$\left(-\Box + m^2 + \xi R\right) \psi_k(x) = 0. \tag{15}$$

This equation can be rewritten as:

$$g^{00}(\partial_t^2 + H_s^2) e^{-i\omega_k t} f_{\omega_k}(x) = (-\omega_k^2 + H_s^2) f_{\omega_k}(x) = 0, \tag{16}$$

where H_s is the quantum-mechanical single-particle Hamiltonian.

Now using the following factorizations:

$$\sinh\left(\frac{\beta\omega_n}{2}\right) = \frac{\beta\omega_n}{2}\prod_{k=1}^{\infty}\left(1+\frac{\beta^2\omega_n^2}{4\pi^2k^2}\right), \tag{17}$$

One can obtain:

$$\log Z^C = \log\prod_n\left(1-e^{-\beta\omega_n}\right)^{-1} = -\frac{1}{2}\sum_{n,k}\log\left(\frac{\frac{4\pi^2k^2}{\beta^2}+\omega_n^2}{\mu^2}\right). \tag{18}$$

The thermal partition function can be expressed in terms of the functional determinant:

$$Z^C = det^{-\frac{1}{2}}\left(\frac{-\partial_\tau^2+H_s^2}{\mu^2}\right) = det^{-1/2}\left[\frac{g_{00}\left(-\Box_E+m^2+\xi R\right)}{\mu^2}\right], \tag{19}$$

Therefore if determinant $det[g^{00}] \sim e^{\beta\cdots}$ then:

$$Z^E = Z^C = e^{-\beta F[\beta]}. \tag{20}$$

For space-times with Killing horizons, determinant is ill defined, since $g_{00} \to 0$.

Euclidean path integral

Using the Bose-Einstein or Planckian distribution:

$$\langle \hat{a}_i^\dagger \hat{a}_j \rangle_\beta = \delta_{ij} n(\beta\omega_i), \quad \text{where} \quad n(\beta\omega_i) = \frac{1}{e^{\beta\omega_i} - 1}, \tag{21}$$

one can obtain the thermal Green function:

$$G(t, x_1, x_2) = \tag{22}$$

$$= \sum_i \frac{e^{-i\omega_i t}}{2\omega_i} \phi_i(x_1)\phi_i^*(x_2)(1 + n(\beta\omega_i)) + \sum_i \frac{e^{i\omega_i t}}{2\omega_i} \phi_i^*(x_1)\phi_i(x_2) n(\beta\omega_i).$$

Derivative with respect to mass of the Euclidean path integral:

$$\frac{\partial}{\partial m^2} \log \int d[\varphi] e^{-S[\varphi]} = -\frac{1}{2} \int d^4x \sqrt{g}\; G(0, x, x). \tag{23}$$

Then, the Euclidean path integral can be expressed as:

$$\log Z^E = -\beta \int_\infty^{m^2} dm^2 \int d^3x \sqrt{g} \sum_i \frac{1}{2\omega_i} \phi_i(x)\phi_i^*(x) \frac{1}{e^{\beta\omega_i} - 1}. \tag{24}$$

Thermal partition function

The trace in the definition of the thermal partition function can be rewritten in terms of the trace over all single-particle excitation:

$$\log Tr(e^{-\beta :\hat{H}:}) = -\sum_i \log\left(1 - e^{-\beta\omega_i}\right) = -Tr_s \log\left(1 - e^{-\beta \hat{H}_s}\right), \qquad (25)$$

Using eigen-functions of the single-particle Hamiltonian, one can rewrite the trace as the volume integral:

$$\log Tr(e^{-\beta :\hat{H}:}) = -\int d^3x \sqrt{g} g^{00} \sum_i \phi_i(x)\phi_i^*(x) \log\left(1 - e^{-\beta\omega_i}\right). \qquad (26)$$

For non-compact spaces, one cannot take the volume integral, since it will be proportional to $\delta(0)$.

Is the Euclidean path integral:

$$\log Z^E = -\beta \int_{\infty}^{m^2} dm^2 \int d^3x \sqrt{g} \sum_i \frac{1}{2\omega_i} \phi_i(x) \phi_i^*(x) \frac{1}{e^{\beta\omega_i} - 1} \tag{27}$$

always equal to the thermal partition function:

$$\log Z^C = -\int d^3x \sqrt{g} g^{00} \sum_i \phi_i(x) \phi_i^*(x) \log\left(1 - e^{-\beta\omega_i}\right)? \tag{28}$$

Euclidean path integral:

$$\log Z^E = -\int d^3x\sqrt{g}g^{00}\sum_i \phi_i(x)\phi_i^*(x)\log\left(1-e^{-\beta\omega_i}\right)+ \tag{29}$$
$$+\int d^3x\sqrt{g}g^{00}\sum_i\int_\infty^{m^2} dm^2\partial_{m^2}\left[\phi_i(x)\phi_i^*(x)\right]\log\left(1-e^{-\beta\omega_i}\right)-$$
$$-\beta\int d^3x\sqrt{g}\int_\infty^{m^2} dm^2\sum_i\frac{1}{2\omega_i}\left[\triangle_3\partial_{m^2}\phi_i(x)\phi_i^*(x)-\partial_{m^2}\phi_i(x)\triangle_3\phi_i^*(x)\right]\frac{1}{e^{\beta\omega_i}-1}.$$

Euclidean path integral:

$$\log Z^E = -\int d^3x\sqrt{g}g^{00}\sum_i \phi_i(x)\phi_i^*(x)\log\left(1-e^{-\beta\omega_i}\right) + \tag{29}$$

$$+\int d^3x\sqrt{g}g^{00}\sum_i \int_\infty^{m^2} dm^2\partial_{m^2}\left[\phi_i(x)\phi_i^*(x)\right]\log\left(1-e^{-\beta\omega_i}\right) -$$

$$-\beta\int d^3x\sqrt{g}\int_\infty^{m^2} dm^2\sum_i \frac{1}{2\omega_i}\left[\triangle_3\partial_{m^2}\phi_i(x)\phi_i^*(x) - \partial_{m^2}\phi_i(x)\triangle_3\phi_i^*(x)\right]\frac{1}{e^{\beta\omega_i}-1}.$$

- For compact space 2 and 3 contribution vanish:

$$\int d^3x\sqrt{g}g^{00}\phi_i(x)\phi_i^*(x) = 1 \quad \text{and} \quad \phi_i(x)\big|_{boundary} = 0 \tag{30}$$

- For non-compact space 2 and 3 terms cancel each other.

Let us consider a model metric of the following form:

$$ds^2 = (1 + f(x))(-dt^2 + dx^2) + d\vec{z}^2. \tag{31}$$

where $\lim_{x\to\pm\infty} f(x) = 0$.
Field operator has the following form:

$$\hat{\varphi} = \int_0^\infty \frac{dp}{\sqrt{2\pi}} \int \frac{d^2k}{(\sqrt{2\pi})^2} \frac{e^{-i\omega t}}{\sqrt{2\omega}} e^{i\vec{k}\cdot\vec{z}} \left[\overrightarrow{\phi}_p(x)\hat{a}_{p,\vec{k}} + \overleftarrow{\phi}_p(x)\hat{b}_{p,\vec{k}} \right] + h.c., \tag{32}$$

where $\omega = \sqrt{m^2 + p^2 + \vec{k}^2}$ and $\overrightarrow{\phi}_p(x)$, $\overleftarrow{\phi}_p(x)$ are the scattering eigen-functions of the effective Schrodinger equation:

$$\left[-\partial_x^2 + \mathbf{V} \right] \overleftarrow{\overrightarrow{\phi}}_p(x) = p^2 \overleftarrow{\overrightarrow{\phi}}_p(x), \tag{33}$$

with the effective potential $\mathbf{V} = f(x)(k^2 + m^2) + (1 + f(x))\xi R$.

The asymptotics of the right moving waves are:

$$\overrightarrow{\phi}_p(x) \approx \theta(-x)\left(e^{ipx} + \overrightarrow{R}_p e^{-ipx}\right) + \theta(x)\overrightarrow{T}_p e^{ipx}, \tag{34}$$

while of the left moving waves are:

$$\overleftarrow{\phi}_p(x) \approx \theta(-x)\overleftarrow{T}_p e^{-ipx} + \theta(x)\left(e^{-ipx} + \overleftarrow{R}_p e^{ipx}\right), \tag{35}$$

Third term can be written in the form:

$$\begin{aligned} \log Z_3^E = -\beta \int_\infty^{m^2} dm^2 \int d^3x\sqrt{g} \int_{-\infty}^{\infty} \frac{d^2k}{(2\pi)^2} \int_0^\infty \frac{dp}{2\pi}\frac{1}{2\omega} n(\beta\omega)\times \\ \times\left(\left[\left(\triangle_3 \partial_{m^2}\overrightarrow{\phi}_p(x)\right)\overrightarrow{\phi}^*_p(x) - \left(\partial_{m^2}\overrightarrow{\phi}_p(x)\right)\triangle_3\overrightarrow{\phi}^*_p(x)\right] + \left[\overrightarrow{\phi} \to \overleftarrow{\phi}\right]\right) = \\ = \beta A \int_\infty^{m^2} dm^2 \int_{-\infty}^{\infty} \frac{d^2k}{(2\pi)^2} \int_0^\infty \frac{dp}{2\pi}\frac{2p}{2\omega} n(\beta\omega)\partial_{m^2}\left[\theta_{\overrightarrow{R}_p} + \theta_{\overleftarrow{R}_p}\right], \end{aligned} \tag{36}$$

where $\theta_{\overrightarrow{R}_p}, \theta_{\overleftarrow{R}_p}$ are the phases of the reflection amplitudes.

Friedel formula connects the integrated density of states and the energy derivative of scattering phaseshifts:

$$\int_{-\infty}^{\infty} dx \left(\left[\overrightarrow{\phi}_p(x) \overrightarrow{\phi}^*_p(x) + \overleftarrow{\phi}_p(x) \overleftarrow{\phi}^*_p(x) \right] - \left[\overrightarrow{\phi}_{0p}(x) \overrightarrow{\phi}^*_{0p}(x) + \overleftarrow{\phi}_{0p}(x) \overleftarrow{\phi}^*_{0p}(x) \right] \right) = \tag{37}$$

$$= \frac{d}{dp} \left[\theta_{\overrightarrow{R}_p} + \theta_{\overleftarrow{R}_p} \right],$$

where $\overrightarrow{\phi}_{0p}(x) = e^{ipx}$ and $\overleftarrow{\phi}_{0p}(x) = e^{-ipx}$ are the modes for the case of the absence of the scattering potential.
The second and third terms cancel each other:

$$\log Z_2^E + \log Z_3^E = 0. \tag{38}$$

Euclidean path integral for space-time with Killing horizon is defined only by a third term of:

$$\log Z^E = -\int d^3x\sqrt{g}g^{00}\sum_i \phi_i(x)\phi_i^*(x)\log\left(1-e^{-\beta\omega_i}\right) + \tag{39}$$

$$+\int d^3x\sqrt{g}g^{00}\sum_i \int_\infty^{m^2} dm^2\partial_{m^2}\left[\phi_i(x)\phi_i^*(x)\right]\log\left(1-e^{-\beta\omega_i}\right) -$$

$$-\beta\int d^3x\sqrt{g}\int_\infty^{m^2} dm^2\sum_i \frac{1}{2\omega_i}\left[\triangle_3\partial_{m^2}\phi_i(x)\phi_i^*(x) - \partial_{m^2}\phi_i(x)\triangle_3\phi_i^*(x)\right]\frac{1}{e^{\beta\omega_i}-1}.$$

Sine the spectrum of the theory does not depent on the mass $\omega \neq \omega(m)$:

$$\sqrt{-g}m^2\varphi^2 \to 0, \quad \text{at the horizon} \tag{40}$$

Therefore 1 and 2 terms cancel each other.

The Euclidian path integral for a massless scalar field in the Rindler space-time has the following form:

$$\log Z^E = \tag{41}$$

$$= -\beta \int d^3x\sqrt{g}\int_{\infty}^{m^2} dm^2 \sum_i \frac{1}{2\omega_i}\left[\triangle_3\partial_{m^2}\phi_i(x)\phi_i^*(x) - \partial_{m^2}\phi_i(x)\triangle_3\phi_i^*(x)\right]\frac{1}{e^{\beta\omega_i}-1} =$$

$$= \int d^4X\sqrt{g}\frac{1}{(\sqrt{g_{00}})^4}\left(\frac{\pi^2}{90}\left[\frac{1}{\beta^4} - \frac{1}{(2\pi)^4}\right] + \frac{\pi^2}{9}\frac{1}{(2\pi)^2}\left[\frac{1}{\beta^2} - \frac{1}{(2\pi)^2}\right]\right).$$

If one first takes the volume integral and then the momentum integral:

$$\log Z^E = \beta A \int d^2k d\omega \frac{1}{e^{\beta\omega}-1}\log\det S(k,\omega) = A\frac{\pi^2}{3}\left[\frac{1}{\beta^2} - \frac{1}{\beta_c^2}\right]\int_{\delta^2}^{\infty}\frac{ds}{(4\pi s)^{\frac{d}{2}}}, \tag{42}$$

where δ is an ultraviolet cutoff.
The thermal partition function:

$$\log Z^C = -\int d^3x\sqrt{g}g^{00}\sum_i \phi_i(x)\phi_i^*(x)\log\left(1 - e^{-\beta\omega_i}\right) = \tag{43}$$

$$= \int d^4x\sqrt{g}\frac{\pi^2}{90}\frac{1}{(\sqrt{g_{00}})^4}\left[\frac{1}{\beta^4} - \frac{1}{(2\pi)^4}\right].$$

Three definitions of the energy

Stress energy tensor is defined as:

$$T_{\mu\nu} = \partial_\mu\varphi(x)\partial_\nu\varphi(x) - \frac{1}{2}g_{\mu\nu}\left(\partial_\rho\varphi(x)\partial^\rho\varphi(x) + m^2\varphi^2(x)\right) + \\ +\xi\left[\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right)\varphi^2(x) + g_{\mu\nu}\Box\varphi^2(x) - \nabla_\mu\nabla_\nu\varphi^2(x)\right]. \qquad (44)$$

The energy operator:

$$\hat{E} = -\int d^3x\sqrt{g}\,\hat{T}_0^0. \qquad (45)$$

The canonical Hamiltonian:

$$\hat{H}_c = \int d^3x\sqrt{g}\,\hat{\mathbf{H}}, \qquad (46)$$

where:

$$\mathbf{H} = \frac{1}{2}\left[-g^{00}\partial_0\varphi(x)\partial_0\varphi(x) + g^{ij}\partial_i\varphi(x)\partial_j\varphi(x) + (m^2 + \xi R)\varphi^2(x)\right]. \qquad (47)$$

The difference between the energy and the Hamiltonian is:

$$\hat{E} = \hat{H}_c - \xi\int dA^i\sqrt{|g_{00}|}\left(\partial_i(\hat{\phi}^2(x)) - \hat{\phi}^2(x)\frac{1}{2}\partial_i\log|g_{00}|\right). \qquad (48)$$

The Hamiltonian operator is defined as follows:

$$\hat{H} = \sum_i \omega_i \hat{a}_i^\dagger \hat{a}_i, \tag{49}$$

and the following Heisenberg equation are valid:

$$i\partial_t \varphi\hat{(x)} = [\hat{\varphi}(x), \hat{H}] \quad , \quad i\partial_t \hat{\pi}(x) = [\hat{\pi}(x), \hat{H}]. \tag{50}$$

The canonical Hamiltonian:

$$\hat{H}_c = \sum_i \ \omega_i \ \hat{a}_i^\dagger \hat{a}_i + \frac{1}{4} \int d^3x \partial_i [g^{ij} \sqrt{g} \partial_j \hat{\varphi}^2(x)]. \tag{51}$$

Therefore, we have the following relations:

$$\hat{H}_c = \hat{H} + \hat{B}_H \quad \text{and} \quad \hat{E} = \hat{H} + \hat{B}_H + \hat{Q}_\xi, \tag{52}$$

The expectation values of the charge and boundary terms:

$$\langle \hat{Q}_\xi \rangle = -\frac{\xi}{2} \int d^3X \sqrt{g} (\sqrt{g_{00}})^4 \left[\frac{1}{\beta^4} - \frac{1}{(2\pi)^4} \right] \sim \langle \hat{B}_H \rangle \tag{53}$$

Fundamental statistical-mechanical relation

The derivatives with respect to the inverse temperature of the Euclidean path integral:

$$-\partial_\beta \log Z^E \neq \langle \hat{E} \rangle, \quad \langle \hat{H} \rangle, \quad \langle \hat{H}_c \rangle \tag{54}$$

and thermal partition function:

$$-\partial_\beta \log Z^C = \langle \hat{H} \rangle. \tag{55}$$

Conformal anomaly

Conformal anomaly in the de Sitter space-time

$$\int d^3x\sqrt{g}T^{\mu}_{\mu} = -\frac{1}{\beta}\frac{\partial}{\partial\alpha}\log Z^E(g_{\mu\nu},\alpha\mu)\big|_{\alpha=1} = \frac{1}{720\pi}\left[3+\left(\frac{2\pi}{\beta}\right)^4\right] \tag{56}$$

For $\beta = 2\pi$ one can restore the "classic"result:

$$\langle T_{\mu\nu}\rangle = \frac{1}{960\pi^2}g_{\mu\nu}. \tag{57}$$

From the relation $-\partial_\beta \log Z = \langle\hat{E}\rangle_\beta$ one can obtain:

$$\log Z = -\int^{\beta} d\beta E_\beta = ... - \frac{\beta}{720\pi}\left[3+\left(\frac{2\pi}{\beta}\right)^4\right]\log(\epsilon/2), \tag{58}$$

We define the thermal partition function as:

$$Z^C = Tr(e^{-\beta :\hat{H}:}). \tag{59}$$

But one can construct a thermal partition function using the stress energy tensor:

$$Z^C_T = Tr\left[exp\left(\int d\Sigma^\mu \hat{T}_{\mu\nu}\beta^\nu\right)\right] = Tr\left[e^{-\beta \hat{E}}\right] \tag{60}$$

or canonical Hamiltonian:

$$Z^C_{H_c} = Tr\left[e^{-\beta \hat{H}_c}\right]. \tag{61}$$