Reconstruction of instantonic systems from exactly solvable Schroedinger operators

based on arXiv:2402.07165

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Instanton is ...

- Partition function

$$Z_\beta = \int\limits_{\text{periodic}} \mathcal{D}x \; e^{-S[x]}, \qquad S[x] = \int d\tau \left(\frac{1}{2}\dot{x}^2 + V(x) \right),$$

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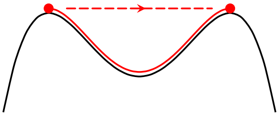
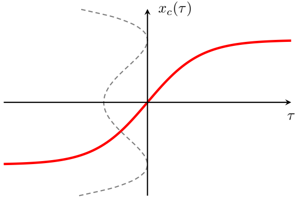
$$\ddot{x}_c - V'(x_c) = 0 \qquad \text{or} \qquad \dot{x}_c \pm \sqrt{2V(x_c)} = 0$$





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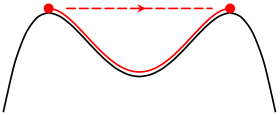

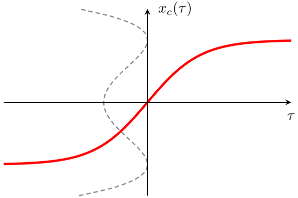

- Responsible for
  - Tunneling amplitudes
  - False vacuum decay rate
  - Energy level splitting

How to calculate?

- Perturbative loop expansion comes from

$$Z = Z_{1-\text{loop}}\, e^{\frac{1}{2}\int d\tau\, d\tau'\, \frac{\delta}{\delta\eta(\tau)} G(\tau,\tau') \frac{\delta}{\delta\eta(\tau')}}$$
$$\times \left[1 + \frac{1}{\|\dot{x}_c\|}\int_0^\beta d\tau\, \eta_0(\tau)\dot{\eta}(\tau)\right] e^{-S^{\text{int}}[x_c,\eta]}\Big|_{\eta=0}$$
$$Z_{\text{1-loop}} = \frac{\beta\|\dot{x}_c\|}{\sqrt{2\pi}} \frac{1}{\sqrt{\text{Det}'(-\partial_\tau^2 + V''(x_c(\tau)))}} e^{-S[x_c]}$$

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  - **(0-loop)** Instanton solution $x_c(\tau)$ and classical action $S[x_c]$

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  - **(0-loop)** Instanton solution $x_c(\tau)$ and classical action $S[x_c]$
  - **(1-loop)** Functional determinant $\text{Det}'(-\partial_\tau^2 + V''(x_c(\tau)))$
  - **(higher loops)** Green's function

$$\left[-\partial_\tau^2 + V''(x_c(\tau))\right] G(\tau,\tau') = \delta(\tau-\tau') - \eta_0(\tau)\eta_0(\tau')$$

Calculations are hard

- Why? Green's functions are cumbersome

$$G_{\mathrm{sG}}(\tau, \tau') = \frac{1}{4}\big(-\partial_\tau + \tanh\tau\big)\big(-\partial_{\tau'} + \tanh\tau'\big)\big(1 - |\tau - \tau'|\big)e^{-|\tau-\tau'|}$$

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- But doable
  - **(2 loops)** sine-Gordon **[Lowe, Stone 1978]**
  - **(3 loops)** sine-Gordon, double-well **[Escobar-Ruiz, Shuryak, Turbiner 2015]**
  - **(3 loops)** non-symmetric double-well **[Bezuglov, Onishchenko 2017]**

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- Uniform WKB result **[Dunne, Ünsal, 2014]**

$$Z \sim \exp\Big[-\frac{8}{\hbar}\Big]\Big[1 - \frac{7}{8}\Big(\frac{\hbar}{8}\Big) - \frac{59}{128}\Big(\frac{\hbar}{8}\Big)^2 - \dots\Big]$$

Brute force loop expansion result **[Escobar-Ruiz, Shuryak, Turbiner 2015]**

$$Z \sim \exp\Big[-\frac{8}{\hbar}\Big]\Big[1 - \frac{7}{8}\Big(\frac{\hbar}{8}\Big) - 0.460937498\Big(\frac{\hbar}{8}\Big)^2 - \dots\Big]$$

Inverse problem

- Direct problem is hard. Maybe set an inverse one?

Can one reconstruct the full action functional $S[x]$, from the known operator $K = -\partial_\tau^2 + W(\tau)$?

where $W(\tau) = V''(x_c(\tau))$ with unknown $V(x)$.

Solution of inverse problem

- Fluctuation operator on time-dependent background always have zero mode $\eta_0(\tau) = \dot{x}_c(\tau)/\|\dot{x}_c(\tau)\|$

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- Having given zero mode $\eta_0(\tau)$, we obtain instanton solution

$$\dot{x}_c(\tau) = \nu\, \eta_0(\tau) \quad \to \quad x_c(\tau) = \nu \int^\tau d\tau\, \eta_0(\tau)$$

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- Exploiting Euclidean energy conservation, we reconstruct the potential

$$V(x_c) = \frac{\dot{x}_c^2(\tau)}{2} \quad \rightarrow \quad \boxed{\left(x,\, V(x)\right) = \left(\nu \int^\tau d\tau'\, \eta_0(\tau'),\; \frac{\nu^2}{2}(\eta_0(\tau))^2\right)}$$

Simplest example

- QM textbook potential

$$K = -\partial_\tau^2 - 2\,\delta(\tau) + 1, \qquad \eta_0(\tau) = e^{-|\tau|}$$

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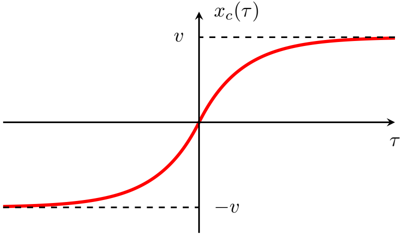
$$x_c(\tau) = \nu\,(1 - e^{-|\tau|})\,\mathrm{sign}\,\tau$$



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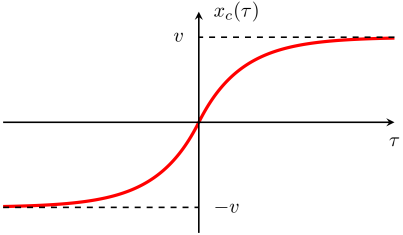
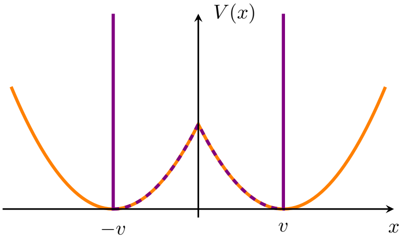


- Reconstructed potential

$$V(x) = \frac{1}{2}\nu^2\,e^{-2|\tau|} = \frac{1}{2}\big(|x| - \nu\big)^2$$



Prototypical tunneling potential

- Most known tunneling QM potentials

$$V_{DW}(x) = \frac{1}{2}(1-x^2)^2, \quad V_{IDW}(x) = \frac{1}{2}x^2(1-x^2),$$
$$V_{SG}(x) = 1+\cos(x), \qquad V_{CW}(x) = 2\,x^2(1-x).$$

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- Corresponding instanton solutions

Potential:	Double Well	Inverted DW	Sine-Gordon	Cubic Well
$x_c(\tau)$:	$\tanh\tau$	$\operatorname{sech}\tau$	$2\arcsin\tanh\tau$	$\operatorname{sech}^2\tau$

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- Fluctuation operators are reflectionless Pöschl-Teller operators

$$K_{\ell,m} = -\partial_\tau^2 + V''(x_c(\tau)) = -\partial_\tau^2 + m^2 - \frac{\ell(\ell+1)}{\cosh^2\tau}$$

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(ℓ, m):	$(2, 2)$	$(2, 1)$	$(1, 1)$	$(3, 2)$

General reflectionless Pöschl-Teller case I

- What are the potentials $V_{\ell,m}(x)$ for general ℓ, m?

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$$x_c(\tau) = \int^\tau d\tau\, P_\ell^m(\tanh\tau) = \int^{\tanh\tau} \frac{dy}{1-y^2} P_\ell^m(y),$$

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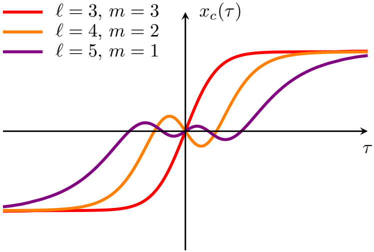
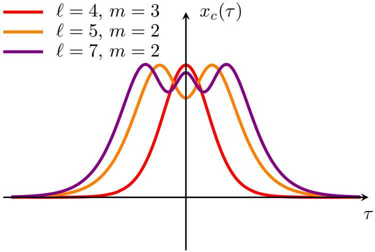
- Plots





General reflectionless Pöschl-Teller case II

- Parametric form

$$\left(x,\, V_{\ell,m}(x)\right) = \left(\int^{y} \frac{dy'}{1-y'^2} P_\ell^m(y'),\ \frac{1}{2}\left(P_\ell^m(y)\right)^2\right)$$

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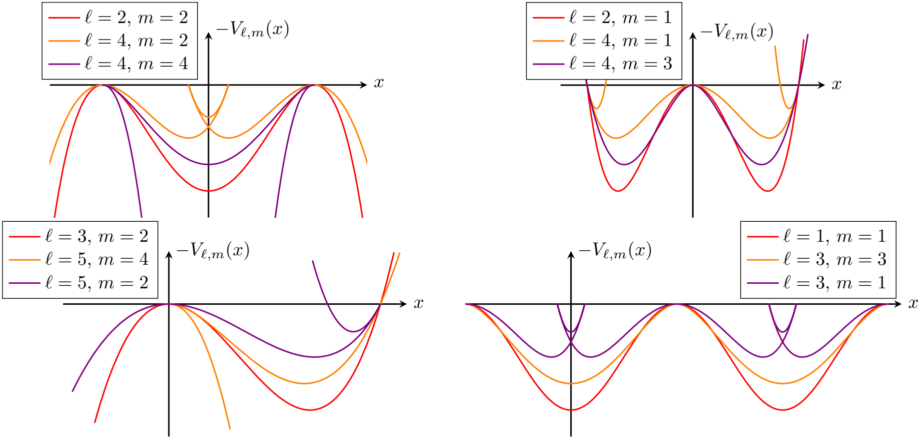

General reflectionless Pöschl-Teller case III

- Potentials are non-analytic for > 1 negative mode!
  $(\ell - m > 1,\ m > 2)$

$$V_{\ell,m}(x) \sim (x - x_{\text{itp}})\left(1 + c_0\sqrt{x - x_{\text{itp}}} + \ldots\right),$$
$$V_{\ell,m}(x) \sim (x - x_{\text{btp}})^2\left(1 + c_1(x - x_{\text{btp}})^{2/m} + \ldots\right)$$

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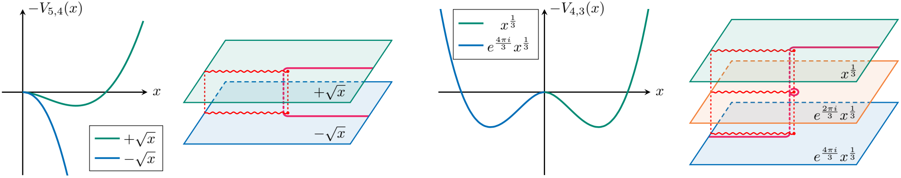
- No new analytic potentials :(

These potentials can be defined on appropriate Riemann surfaces. Examples read:

- $(\ell, m) = (\ell, \ell - 1)$

$$V_{\ell,\ell-1}(x) = \frac{1}{2}(\ell-1)^2\,x^2\Big(1 - 4\,x^{\frac{2}{\ell-1}}\Big).$$



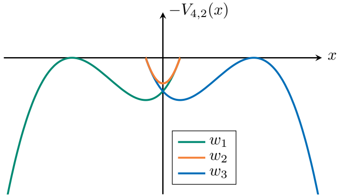
General reflectionless Pöschl-Teller case V

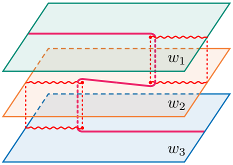
- $\ell = 4,\ m = 4, 2, 1$

$$V_{4,4}(x) = \frac{9}{8}\left(1 - y^2\right)^4, \qquad y = -\left(w + w^{-1}\right),$$

$$V_{4,2}(x) = \frac{63}{8}(7y^4 - 8y^2 + 1)^2, \qquad y = w + w^{-1},$$

$$V_{4,1}(x) = \frac{63}{512}u^2(1 - u^2)(4 - 7u^2)^2, \qquad u = -\frac{2}{\sqrt{7}}\left(w + w^{-1}\right).$$





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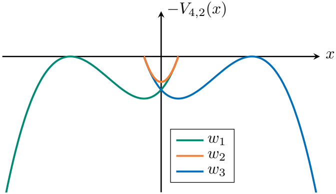

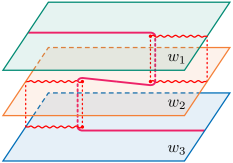

- **However, these are in strong coupling regime!**

Natanzon operators: overview

- Natanzon operator

$$W_N(\tau) = \frac{fz(z-1) + h_0(1-z) + h_1 z + 1}{R(z)} + \left[a + \frac{a + (c_1 - c_0)(2z-1)}{z(z-1)} - \frac{5}{4}\frac{\Delta}{R(z)}\right]\frac{z^2(1-z)^2}{R^2(z)}$$

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- Parameterization

$$\frac{(\dot{z}(\tau))^2 R(z(\tau))}{4z(\tau)^2(1-z(\tau))^2} = 1, \qquad R(z) = az(z-1) + c_0(1-z) + c_1 z$$

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- Eigenfunctions

$$\eta(\tau) = (\dot{z}(\tau))^{-1/2}(z(\tau))^{(\lambda_0+1)/2}(1-z(\tau))^{(\lambda_1+1)/2}{}_2F_1(\ldots; z(\tau))$$

Natanzon operators: application

- Formal solution

$$x(z) = \int dz\,\big(R(z)\big)^{3/4}\, z^{\bar{\lambda}_0/2-1}(1-z)^{\bar{\lambda}_1/2-1},$$

$$V_N(x(z)) = \sqrt{R(z)}\, z^{\bar{\lambda}_0}(1-z)^{\bar{\lambda}_1},$$

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- Shape-invariant operators

$$R(z) = z^{q_0}(1-z)^{q_1}, \qquad q_0, q_1 = 0, 1, 2, \quad q_0 + q_1 \leq 2$$

	$z(\tau)$	$\dot{z}$	$R(z)$
Maning-Rosen	$(1+e^{2\tau})^{-1}$	$2z(z-1)$	1
hyperbolic P-T	$\tanh^2\tau$	$2z^{1/2}(1-z)$	z
Eckart	$1-e^{-2\tau}$	$2(1-z)$	z^2
trigonometric PT	$\sin^2\tau$	$2z^{1/2}(1-z)^{1/2}$	$z(1-z)$
Scarf	$\frac{1}{2}(1-i\sinh\tau)$	$(-z)^{1/2}(1-z)^{1/2}$	$-4z(1-z)$
Rosen-Morse	$\frac{1}{2}(1+i\cot\tau)$	$-2iz(1-z)$	-1

Manning-Rosen

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$$W_{\mathsf{MR}}(\tau) = -\frac{\ell_+(\ell_+ + 1)}{\cosh^2 \tau} + 2\,\ell_+\ell_- \tanh\tau + \ell_+^2 + \ell_-^2, \quad \ell_\pm = \frac{1}{N_0} \pm \frac{1}{N_1}$$

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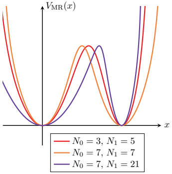
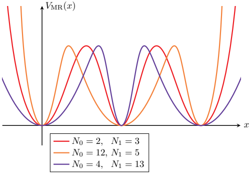
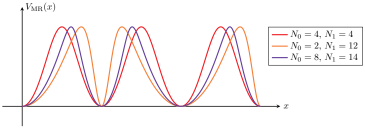
- Plots







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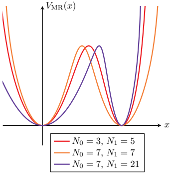

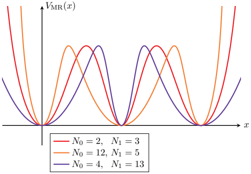

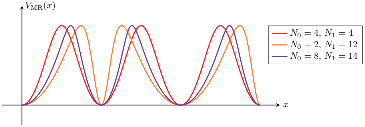

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Hyperbolic Pöschl-Teller

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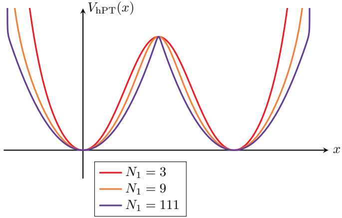

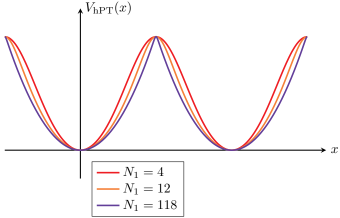

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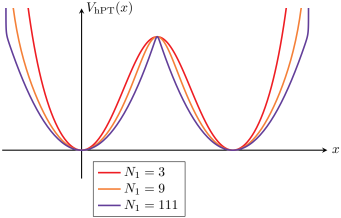

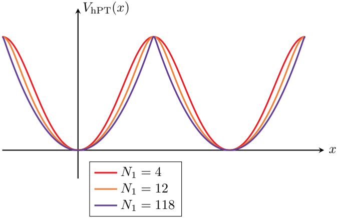

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Scarf

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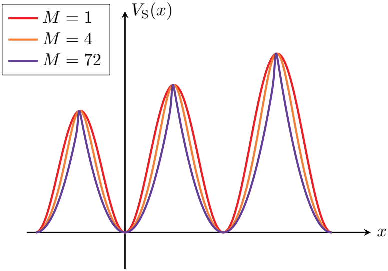

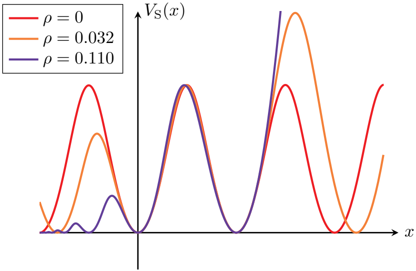

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- Unusual quasi-periodicity property

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- Classification of tunneling potentials, having solvable fluctuation operators!

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Thank you for your attention!