Black Holes with Electroweak Hair

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No-hair conjecture

Black holes formed by gravitational collapse are characterized by their mass, angular momentum, and electric charge = the only parameters that can survive the collapse $\Rightarrow$ all black holes are described by the Kerr-Newman metrics.

/Ruffini and Wheeler, 1969/

No-hair conjecture

from the archives

From January 1971, pages 30–41

Introducing the black hole

Remo Ruffini and John A. Wheeler

According to present cosmology, certain stars end their careers in a total gravitational collapse that transcends the ordinary laws of physics.

At the time of this article, **Remo Ruffini** and **John Wheeler** were both at Princeton University; Wheeler, on leave from Princeton, was spending a year at the California Institute of Technology and Moscow State University.

The quasi-stellar object, the pulsar, the neutron star have all come onto the scene of physics within the space of a few years. Is the next entrant destined to be the black hole? If so, it is difficult to think of any development that could be of greater significance. A black hole, whether of "ordinary size" (approximately one solar mass, 1 $M_\odot$) or much larger (around $10^6\ M_\odot$ to $10^{10}\ M_\odot$, as proposed in the nuclei of some galaxies), provides our "laboratory model" for the gravitational collapse, predicted by Einstein's theory, of the universe itself.

A black hole is what is left behind after an object has undergone complete gravitational collapse. Spacetime is so strongly curved that no light can come out, no matter can be ejected, and no measuring rod can ever survive being put in. Any kind of object that falls into the black hole loses its separate identity, preserving only its mass, charge, angular momentum, and linear momentum (see figure 1). No one has yet found a way to distinguish between two black holes constructed out of the most different kinds of matter if they have the same mass, charge, and angular momentum. Measurement of these three determinants is permitted by their effect on the Kepler orbits of test objects, charged and uncharged, in revolution about the black hole.

How the physics of a black hole looks depends more upon an act of choice by the observer himself than on anything else. Suppose he decides to follow the collapsing matter through its collapse down into the black hole. Then he will see it crushed to indefinitely high density, and he himself will be torn apart eventually by indefinitely increasing tidal forces. No restraining force whatsoever has the power to hold him away from this catastrophe, once he crossed a certain critical surface known as the "horizon." The final collapse occurs a finite time after the passage of this surface, but it is inevitable. Time and space are interchanged inside a black hole in

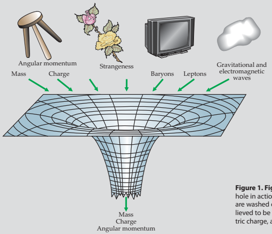


Figure 1. Figurative representation of a black hole in action. All details of the infalling matter are washed out. The final configuration is believed to be uniquely determined by mass, electric charge, and angular momentum.

No-hair theorems and the first hairy black hole

- No-hair theorems /Bekenstein, 1972,.../ confirm the conjecture for a number of special cases. No new black holes holes for gravitating massive scalar, spinor, of vector fields, also for a scalar field with a positive potential, etc.
- First explicit counter-example /M.S.V.+ Gal'tsov, 1989/: static black holes with Yang-Mills hair. Triggered an avalanche of discoveries of other hairy black holes.

Non-Abelian Einstein-Yang–Mills black holes

M. S. Volkov and D. V. Gal'tsov
M. V. Lomonosov Moscow State University

(Submitted 7 September 1989)
Pis'ma Zh. Eksp. Teor. Fiz. **50**, No. 7, 312–315 (10 October 1989)

Solutions of the self-consistent system of Einstein-Yang–Mills equations with the SU(2) group are derived to describe black holes with a non-Abelian structure of gauge fields in the external region.

In the case of the electrovacuum, the most general family of solutions describing spherically symmetric black holes is the two-parameter Reissner–Nordström family, which is characterized by a mass M and an electric charge Q. It was recently shown for the Einstein-Yang–Mills systems of equations with the SU(2) group that a corresponding assertion holds when the hold has a nonvanishing color-magnetic charge. In this case the structure of the Yang–Mills hair is effectively Abelian.[1] In the present letter we numerically construct a family of definitely non-Abelian solutions for Einstein-Yang–Mills black holes in the case of zero magnetic charge. These solutions are characterized by metrics which asymptotically approach the Schwarzschild metric far from the horizon but are otherwise distinct from metrics of the Reissner–Nordström

Zoo of hairy black holes

- Before 2000: Einstein-Yang-Mills black holes and their generalizations – higher gauge groups, additional fields (Higgs, dilaton), non-spherical solutions, stationary generalizations, Skyrme black holes, Gauss-Bonnet, . . .
/M.S.V.+Gal'tsov, Phys.Rep. 319 (1999) 1/

- After 2000: black holes with scalar hair – engineered potential, spinning clouds of massive complex scalar /Herdeiro-Radu/, Horndeski black holes, metric-affine theories, higher dimensions, stringy corrections, hairy black holes with massive gravitons /Gervalle+M.S.V., 2020/, etc, . . .
/M.S.V., 1601.0823/

- Which of these solutions are physical ?

Present status of hairy black holes

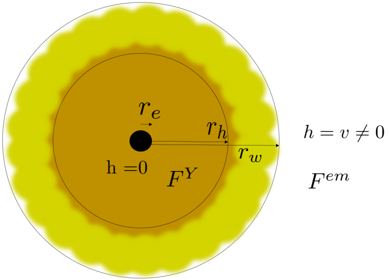
- All known solutions have been obtained within simplified theoretical models. Their physical relevance is not obvious.

- To be physically relevant, the solution should be obtained within the context of the physical theory = Einstein's gravity +Standard Model of fundamental interactions (QCD+electroweak).

- Classical configurations in the QCD sector are destroyed by large quantum corrections $\Rightarrow$ useless to study. There remains the gravitating electroweak theory = Einstein-Weinberg-Salam. This describes the Kerr-Newman black holes. Does it describe other black holes ?

- Only unphysical limits where $\theta_{\rm W} = 0, \pi/2$ have been analyzed – in the full theory the spherical symmetry is lost.

Magnetic electroweak black hole /Maldacena 2020/



The U(1) hypermagnetic field near the horizon + electroweak "corona" made of Z,W,Higgs fields + radial magnetic field in the far field. No symmetry.

Magnetic monopoles in gauge field theories

Dirac monopole /1930/

$$\vec{\mathcal{B}} = \frac{P\vec{r}}{r^3}, \quad \Rightarrow \quad \vec{\nabla}\cdot\vec{\mathcal{B}} \neq 0, \quad \text{nevertheless} \quad \vec{\mathcal{B}} = \vec{\nabla}\times\vec{\mathcal{A}}_{\pm}$$

where the vector potential contais the Dirac string singularity, but this can be excluded by using two local gauges,

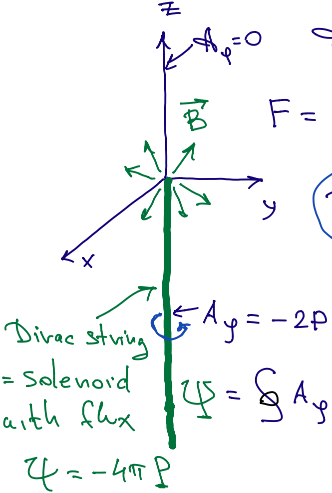
$$\mathcal{A}_- = P(\cos\vartheta - 1)d\varphi \;\; \text{in northern hemisphere} \quad \vartheta \in [0, \pi/2 + \epsilon)$$
$$\mathcal{A}_+ = P(\cos\vartheta + 1)d\varphi \;\; \text{in southern hemisphere} \quad \vartheta \in (\pi/2 - \epsilon, \pi]$$

The two gauges are related in the equatorial region,

$$\mathcal{A}_+ = \mathcal{A}_- + d\,(2P\varphi)\,, \qquad \psi_+ = \exp\left(ie\,2P\varphi\right)\psi_-$$

hence $2eP = n \in \mathbb{Z} \;\Rightarrow \boxed{P = \frac{n}{2e}}$ /n is called "magnetic charge" /

Magnetic field produced by a solenoid



$$A = P(\cos\theta - 1)\, d\varphi$$

$$F = dA = -P \sin\theta \, d\theta \wedge d\varphi$$

$$B = \star F = -\frac{P}{r^2}\, dr$$

radial magnetic field

$$\Psi = \oint A_\varphi \, d\varphi = -4\pi P$$

t'Hooft-Polyakov monopole /1974/

Gauge fiel theory with a triplet Higgs field

$$\mathcal{L} = -\frac{1}{4e^2}\, F^a_{\mu\nu} F^{a\mu\nu} - \frac{1}{2} D_\mu \Phi^a D^\mu \Phi^a - \frac{\lambda}{4} \left(\Phi^a \Phi^a - \Phi_0^2 \right)^2$$

with $D_\mu \Phi^a = \partial_\mu \Phi^a + \epsilon_{abc} A^b_\mu \Phi^c$.

A globally regular solution with a finite energy and magnetic charge $P = 1/e \quad \Rightarrow \quad n = 2$. Enormously popular theoretically, but no observational evidence: does not belong to the Standard Model.

What is known about monopoles in the Standard Model ?

Magnetic monopoles in the electroweak theory

SU(2)×U(1) electroweak theory of Weinberg-Salam

$$\mathcal{L}_{\rm WS} = -\frac{1}{4g^2}\,\mathrm{W}^a_{\mu\nu}\mathrm{W}^{a\mu\nu} - \frac{1}{4g'^2}\,B_{\mu\nu}B^{\mu\nu} - (D_\mu\Phi)^\dagger D^\mu\Phi - \frac{\beta}{8}\left(\Phi^\dagger\Phi - 1\right)^2$$

where Higgs is a complex doublet, $\Phi^{\rm tr} = (\phi_1, \phi_2)$,

$$B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu\,, \qquad \mathrm{W}^a_{\mu\nu} = \partial_\mu \mathrm{W}^a_\nu - \partial_\nu \mathrm{W}^a_\mu + \epsilon_{abc}\mathrm{W}^b_\mu \mathrm{W}^c_\nu\,,$$

$$D_\mu\Phi = \left(\partial_\mu - \frac{i}{2}\,B_\mu - \frac{i}{2}\,\tau^a\mathrm{W}^a_\mu\right)\Phi$$

Couplins $g'^2 = 0.23$, $g^2 = 1 - g'^2$, $\beta = 1.88$. Electron charge $e = gg'$ defines $\boldsymbol{g}_0 = \sqrt{4\pi\alpha/(\boldsymbol{\hbar c}e^2)}$ $\Rightarrow$ length and mass scales

$$\boldsymbol{l}_0 = \frac{1}{\boldsymbol{g}_0\boldsymbol{\Phi}_0} = 1.5\times 10^{-16}\ \mathrm{cm}, \quad \boldsymbol{m}_0 = \frac{\boldsymbol{\hbar}}{\boldsymbol{c}}\,\boldsymbol{g}_0\boldsymbol{\Phi}_0 = 128.6\ \mathrm{GeV}$$

The Z, W, Higgs masses expressed in in unites of $\boldsymbol{m}_0$ are $m_{\rm z} = 1/\sqrt{2}$, $m_{\rm w} = gm_{\rm z}$, $m_{\rm h} = \sqrt{\beta}m_{\rm z}$.

Dirac monopole embedded into electroweak theory

$$B = W^3 = \frac{n}{2}(\cos\vartheta \pm 1)\,d\varphi, \qquad W^1 = W^2 = 0, \qquad \Phi = \begin{pmatrix}0\\1\end{pmatrix},$$

$$\mathcal{A} = \frac{1}{e}B, \qquad \vec{\mathcal{B}} = \frac{P\vec{r}}{r^3}, \qquad P = \frac{n}{2e}, \qquad n \in \mathbb{Z}$$

Energy is infinite. Remarque:

- Dirac monopole is stable within the U(1) electrodynamics.
- It is unstable within the electroweak theory because the magnetic field $\vec{\mathcal{B}} = P\vec{r}/r^3$ becomes very strong as $r \to 0$ and the electroweak vacuum becomes unstable with respect to condensation. Nobody studied this.

Cho-Maison monopole /1996/

U(1) field $B = (\cos\vartheta - 1)\, d\varphi$ as for the Dirac monopole with $n = 2$ combined with non-Abelian

$$W^a_\mu\, dx^\mu \quad = \quad (1 - f(r))\, \epsilon_{aik} \frac{x^i dx^k}{r^2}\,, \quad \Phi = \phi(r) \begin{pmatrix} \sin\frac{\vartheta}{2}\, e^{-i\varphi} \\ -\cos\frac{\vartheta}{2} \end{pmatrix}$$

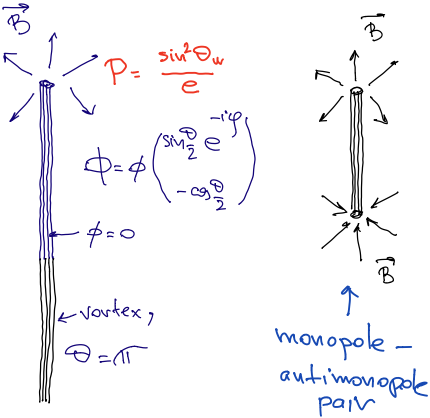
= extended non-Abelian core with a pointlike U(1) hypermagnetic charge in the center. The total magnetic charge

$$P = \frac{1}{e} = \frac{\sin^2\theta_{\rm W}}{e} + \frac{\cos^2\theta_{\rm W}}{e} \equiv P_{\rm U(1)} + P_{\rm SU(2)}$$

where $P_{\rm U(1)}$ is pointlike at the origin and $P_{SU(2)}$ is distributed over the space.
Energy is a sum of a divergent U(1) part and a finite SU(2) part,

$$E \equiv E_{\rm U(1)} + E_{\rm SU(2)} = \frac{2\pi}{g'^2} \int_0^\infty \frac{dr}{r^2} + E_{\rm SU(2)} \qquad /E_{\rm SU(2)} = 15.76/$$

Nambu monopole



time-dependent solutions

Summary

Electroweak theory contains two types of static, spherically symmetric monopole solutions, both with infinite energy:

- Pointlike Dirac monopole for any value of the magnetic charge $n = \pm 1, \pm 2, \ldots$
- Non-Abelian monopole of Cho-Maison for $n = \pm 2 \Rightarrow$ superposition of a pointlike hypermagnetic U(1) monopole and a regular SU(2) field.

New come new results

Part I: Stability of electroweak monopoles

R.Gervalle and M.S.V., Nucl.Phys. B 984 (2022) 115937

Generic perturbations

$$W^a_\mu \to W^a_\mu + \delta W^a_\mu, \qquad B_\mu \to B_\mu + \delta B_\mu, \qquad \Phi \to \Phi + \delta\Phi$$

Linearizing the equations with respect to δW^a_μ, δB_μ, $\delta\Phi$, using the null spacetime tetrad approach and separating the angular variables in terms of the spin-weighted spherical harmonics, assuming the $e^{\pm i\omega t}$ time dependence, the perturbation equations reduce to

$$\left(-\frac{d^2}{dr^2} + \hat{U}\right)\Psi = \omega^2\Psi\,,$$

where Ψ is a 16-component vector and $\hat{U}$ is the symmetric 16×16 matrix. If there are bound states with $\omega^2 < 0$ then the background is unstable.

Stability of Cho-Maison monopole – Jacobi criterion

$$\left(-\frac{d^2}{dr^2}+\hat{U}\right)\Psi=\omega^2\Psi\,,\qquad \Psi^{\rm tr}=(\Psi_1,\ldots,\Psi_{16})\equiv\Psi_k$$

One sets $\omega=0$, finds 16 regular at the origin solutions $\Psi_k^{(a)}(r)$, and computs the determinant

$$\Delta(r)=\left|\Psi_k^{(a)}(r)\right|\qquad a,k=1,\ldots,16$$

If $\Delta(r)>0$ then all eigenvalues $\omega^2>0$. This was checked for the Cho-Maison monopole in sectors with $j=0,1,2,3,4$. For higher j the bound states are unlikely dues to the high centrifugal barrier $\Rightarrow$

The non-Abelian monopole of Cho-Maison is stable with respect to all small perturbations

Stability of Dirac monopole

One perturbative channe decouples

$$\left(-\frac{d^2}{dr^2}+\frac{g^2}{2}-\frac{|n|}{2r^2}\right)\psi=\omega^2\psi \quad \text{if} \quad \boxed{j=\left|\frac{n}{2}\right|-1}, \qquad |n|>1,$$

solution oscillates infinitely many times for $r\to 0$,

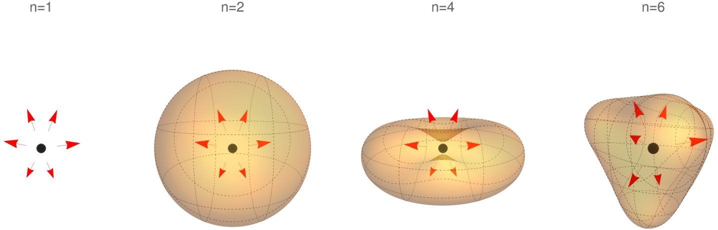
$$\psi=\sqrt{r}\,\cos\left(\frac{\sqrt{2n-1}}{2}\ln\frac{r}{r_0}\right)$$

$\Rightarrow$ all Dirac monopoles with $|n|>1$ are unstable.

The $n=2$ is unstable in the $j=0$ sector $\Rightarrow$ not splitting. The non-Abelian Cho-Maison monopole also has $n=2$ and is stable $\Rightarrow$ it is remnant of Dirac's monopole decay.

Dirac monopoles with $|n|>2$ decay in sectors with $j>0$ and should condense to non spherically-symmetric non-Abelian states.

Conjecture



Dirac monopoles with $|n| \geq 2$ should condense to non-spherically symmetric non-Abelian states. The magnetic charge splits into the pointlike part $n \times \sin^2\theta_W/(2e)$ and part $n \times \cos^2\theta_W/(2e)$ distributed in space. The energy is infinite due to the central singularity.

/R.Gervalle and M.V. Nucl.Phys. B 984 (2022) 115937/

Part II: Non-Abelian multi-monopoles

/R.Gervalle and M.S.V. Nucl.Phys. B 987 (2023) 116112/

Axial symmetry

Rebbi-Rossi ansatz (even parity), with $\mathrm{T}_a = \tau_a/2$,

$$\begin{aligned} W = \mathrm{T}_a \mathrm{W}^a_\mu dx^\mu &= \mathrm{T}_2 \left(F_1\, dr + F_2\, d\vartheta\right) + \frac{n}{2} \left(\mathrm{T}_3\, F_3 - \mathrm{T}_1 F_4\,\right) d\varphi \\ B_\mu dx^\mu &= \frac{n}{2}\, Y\, d\varphi\,, \qquad \Phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}\,, \qquad n \in \mathbb{Z}, \end{aligned}$$

$F_1, F_2, F_3, F_4, Y, \phi_1, \phi_2$ are 7 real-valued functions of r, ϑ.

- System of 7 elliptic PDE's in the domain $r \in [0, \infty)$, $\vartheta \in [0, \pi/2]$, assuming the invariance under $\vartheta \to \pi - \vartheta$.
- For $n = \pm 2$ solution is spherically symmetric = the Cho-Maison monopole. Iteratively increasing n gives axially-symmetric monopoles.

Energy

Splits into an infinite U(1) part and a finite SU(2) part

$$E = \int T_{00} \sqrt{-\mathrm{g}}\, d^3x = \frac{2\pi\nu^2}{g'^2} \int_0^\infty \frac{dr}{r^2} + E_{\mathrm{reg}}.$$

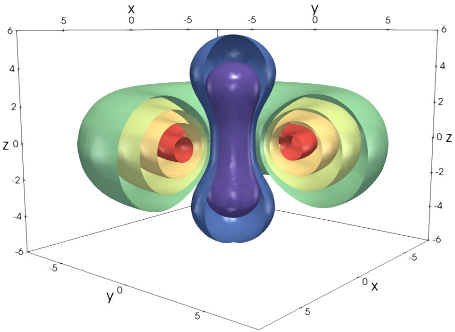
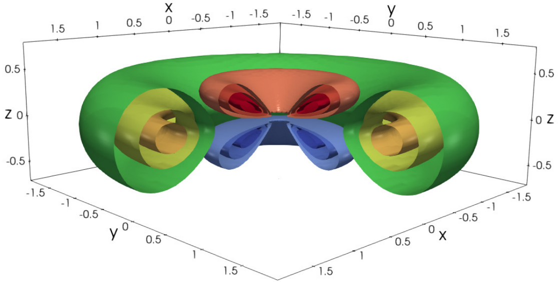


Figure: Surfaces of constant T_{00} for $n = 10$.

Magnetic charge and electric current isosurfaces



Magnetic charge density $\rho_{SU(2)}$ (green) and positive J_φ and negative J_φ densities of the azumuthal electric current for the $n = 4$ monopole. The magnetic charge forms a ring whose magnetic field forces the charged W-bosons to Larmore-orbit, creating two electric currents. These currents create the magnetic field which squeezes the magnetic charge toward equatorial plane.

Large charge limit

For large n the U(1) field B becomes very strong and drives to zero all other fields in the central monopole region thus restoring the full gauge symmetry. This creates the vacuum bubble in the monopole center.

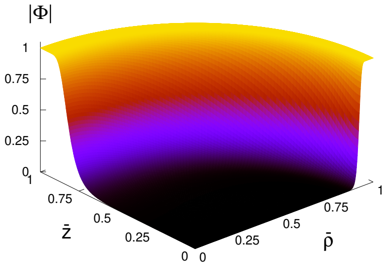
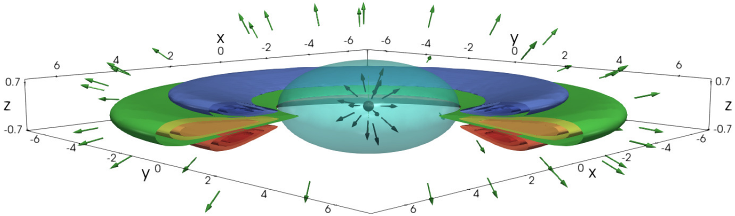


Figure: The norm of the Higgs field $|\Phi|$ for $n = 100$.

Large charge monopole, $n = 80$.



The pointlike hypermagnetic charge $P_{\mathrm{U}(1)} = n \times \sin^2\theta_{\mathrm{W}}/(2e)$ creates a bubble of unbroken phase in the center. Outside the bubble the massive fields condense to a ring of magnetic charge $P_{\mathrm{SU}(2)} = n \times \cos^2\theta_{\mathrm{W}}/(2e)$ squeezed between two superconducting electric currents. Still farther away there remains only the field of the Dirac monopole of charge $P_{\mathrm{U}(1)} + P_{\mathrm{SU}(2)} = n/(2e)$. The energy is infinite due to the central pointlike charge. Perhaps the latter can be shielded by an event horizon ?

Part III. Black holes with electroweak hair

/in preparation/

Einstein-Weinberg-Salam theory

$$\mathcal{L} = \frac{1}{2\kappa}\,R + \mathcal{L}_{\rm WS}$$

with

$$\mathcal{L}_{\rm WS} = -\frac{1}{4g^2}\,{\rm W}^a_{\mu\nu}{\rm W}^{a\mu\nu} - \frac{1}{4g'^2}\,B_{\mu\nu}B^{\mu\nu} - (D_\mu\Phi)^\dagger D^\mu\Phi - \frac{\beta}{8}\left(\Phi^\dagger\Phi - 1\right)^2$$

the length scale and mass scale are electroweak, same as before: $\boldsymbol{l}_0 =1.5\times 10^{-16}$ cm and $\boldsymbol{m}_0 =128.6$ GeV. The couplings

$$g^2 = 0.77,\quad g'^2 = 0.23,\quad \beta = 1.88,\quad \kappa = \frac{8\pi\mathbf{G}\mathbf{\Phi_0^2}}{\boldsymbol{c}^4} = 5.42\times 10^{-33}.$$

Equations to solve

Electroweak:

$$\nabla^\mu B_{\mu\nu} = g'^2 \frac{i}{2} (\Phi^\dagger D_\nu \Phi - (D_\nu \Phi)^\dagger \Phi),$$

$$\mathcal{D}^\mu \mathrm{W}^a_{\mu\nu} = g^2 \frac{i}{2} (\Phi^\dagger \tau^a D_\nu \Phi - (D_\nu \Phi)^\dagger \tau^a \Phi),$$

$$D_\mu D^\mu \Phi - \frac{\beta}{4} (\Phi^\dagger \Phi - 1)\Phi = 0,$$

Einstein:

$$G_{\mu\nu} = \kappa T_{\mu\nu}, \qquad \kappa \sim 10^{-33},$$

and

$$T_{\mu\nu} = \frac{1}{g^2} \mathrm{W}^a{}_{\mu\sigma} \mathrm{W}^a{}_\nu{}^\sigma + \frac{1}{g'^2} B_{\mu\sigma} B_\nu{}^\sigma + 2 D_{(\mu} \Phi^\dagger D_{\nu)} \Phi + \mathrm{g}_{\mu\nu} \mathcal{L}_{\mathrm{WS}}$$

=30 coupled equations. A simple solution:

Magnetically charged Reissner-Nordstrom

Same electroweak fields as for the Dirac monopole,

$$B = W^3 = \frac{n}{2}(\cos\vartheta \pm 1)\, d\varphi, \quad W^1 = W^2 = 0, \quad \Phi = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

and the RN metric,

$$ds^2 = -N(r)\, dt^2 + \frac{dr^2}{N(r)} + r^2 \left(d\vartheta^2 + \sin^2\vartheta\, d\varphi^2\right),$$

$$N(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2}, \qquad Q^2 = \frac{\kappa n^2}{8e^2}, \qquad n \in \mathbb{Z}.$$

The event horizon is at $r_H = M + \sqrt{M^2 - Q^2}$.

This solution is stable at large r_H but becomes unstable at small r_H

Stability of Reissner-Nordstrom

Same instability as for the Dirac monopole: for $j = |n|/2 - 1, \quad |n| > 1)$ one obtains the one-channel problem

$$\left(-\frac{d^2}{dr_\star^2} + N(r)\left[\frac{g^2}{2} - \frac{|n|}{2r^2}\right]\right)\psi(r) = \omega^2\psi(r)$$

with $dr_\star = dr/N(r)$. In flat space $N(r) = 1$ and there are infinitely many bound states with $\omega^2 < 0 \Rightarrow$ Dirac monopoles are unstable.

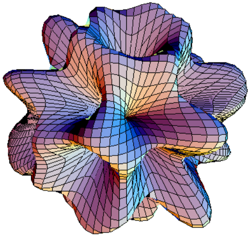
In curved space $N(r) \leq 1 \Rightarrow$ a finite number of bound states if $r_H < r_H^0$ and no bound states if $r_H > r_H^0$. For $r_H = r_H^0$ the first bound state appears as a static zero mode $\psi_0(r)$ which approximates the W-condensate = black hole hair = bifurcation of the RN family with the new family of hairy black holes.

Perturbative black hole hair

Values $r_H^0(n)$ for which the zero mode appears

n	2	4	6	10	20	40	100	200
r_H^0	0.89	1.47	1.93	2.69	4.12	6.19	10.33	15.03

The mode is maximal at the horizon and proportional to $Y_{jm}(\vartheta, \varphi)$ with $j = |n/2| - 1$, describes the W-current tangential to the horizon. This current produce magnetic and Z-fluxes orthogonal to the horizon= vortices of finite length =CORONA. Schematically,



Non-perturbative solutions

Hairy black holes cannot be spherically symmetric for $|n| > 2$ but can be axially symmetric:

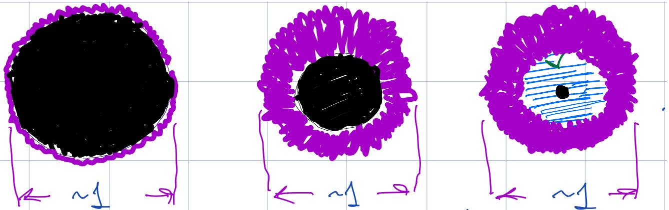
$$ds^2 = -e^{2U} N(r)\, dt^2 + e^{2k-2U} \left(\frac{dr^2}{N(r)} + r^2 d\vartheta^2 + e^{2w} r^2 \sin^2 \vartheta d\varphi^2 \right),$$

$$W = \mathrm{T}_a \mathrm{W}^a_\mu dx^\mu = \mathrm{T}_2 \left(F_1\, dr + F_2\, d\vartheta \right) + \frac{n}{2} \left(\mathrm{T}_3\, F_3 - \mathrm{T}_1 F_4 \right) d\varphi$$

$$B_\mu dx^\mu = \frac{n}{2}\, Y d\varphi\,, \qquad \Phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \qquad \mathrm{T}_a = \tau_a/2\,,$$

where $U, k, w, F_1, F_2, F_3, F_4, Y, \phi_1, \phi_2$ are 10 real functions of r, ϑ and $N(r) = 1 - r_H/r$ where r_H is the black hole "size".

10 coupled PDE's to solve. For $n = \pm 2$ the solution is spherically symmetric, increasing n gives axially-symmetric black holes.



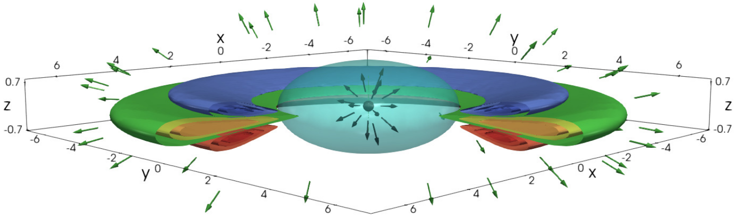
when horizon dicreases, the hair grows, then a bubble of symmetric phase appears

In the extreme limit, inside the hairy region, the metric is ≈ RN for the magnetic charge

$P_{U(1)} = \frac{n\cos^2\theta_W}{2e}$. The hair carries the the rest of the charge, $P_{SU(2)} = \frac{n\sin^2\theta_W}{2e}$,

asymptotiquement the metric is RN with charge $P = n/2e$

Extreme black hole for $n = 80$



At the center – a tiny RN black hole of charge $P_{\mathrm{U}(1)} = n \cos^2 \theta_{\mathrm{W}}/(2e)$ surrounded by the vacuum bubble. Outside the bubble – a condensate of massive fields forming charged rings with charge $P_{\mathrm{SU}(2)} = n \sin^2 \theta_{\mathrm{W}}/(2e)$. Far away – the radial magnetic field of charge $P = n/(2e)$.

Asymptotic analysis – multipole moments

For $r \gg 1$ the theory reduces to Einstein-Maxwell,

$$\mathcal{L} = \frac{1}{2\kappa} R - \frac{1}{4e^2} B_{\mu\nu} B^{\mu\nu} \,.$$

The static spacetime metric being

$$ds^2 = -e^{2U} dt^2 + e^{-2U} dl^2, \qquad dl^2 = h_{ik} dx^i dx^k \,,$$

the Ernst potentials Ψ and ξ are defined by

$$\frac{1-\xi}{1+\xi} = e^{2U} - \Psi^2, \qquad B_{ik} = \sqrt{\frac{2e^2}{\kappa}} \sqrt{h}\, e^{-2U} \epsilon_{iks}\, \partial^s \Psi$$

Choosing the Weyl coordinates, $dl^2 = e^{2K}(d\rho^2 + dz^2) + \rho^2 d\varphi^2$,
the multipoles m_k, s_k are defined by (equivalent to Geroch-Hanson)

$$\xi(\rho = 0, z \gg 1) = \sum_{k \geq 0} \frac{m_k}{z^{k+1}}, \quad \Psi(\rho = 0, z \gg 1) = \sum_{k \geq 0} \frac{s_k}{z^{k+1}},$$

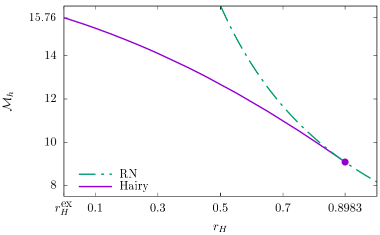
this gives the mass, charge, (dipole=0), quadrupole moments, etc.

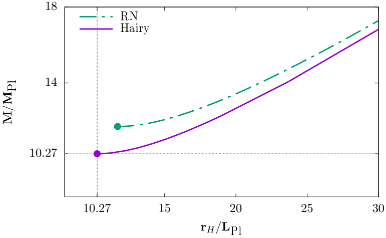
ADM mass

Obtained from the asymptotics or from the integral formula

$$g_{00} = 1 - \frac{2M}{r} + \dots,$$
$$M = \frac{\mathrm{k_H A_H}}{4\pi} + \frac{\kappa}{4\pi}\int_{r>r_\mathrm{H}} (2T_{\hat{0}\hat{0}} + T)\sqrt{-g}\, d^3x$$

Hairy solutions are less energetic than the RN of the same size.





For small r_H the mass is below the mass of the extreme RN.

Horizon mass $M_{\rm H}$ and hair mass $M_{\rm hair}$

$$P = \frac{n}{2e} = P_{\rm H} + P_{\rm hair}, \qquad M = M_{\rm H} + M_{\rm hair}$$

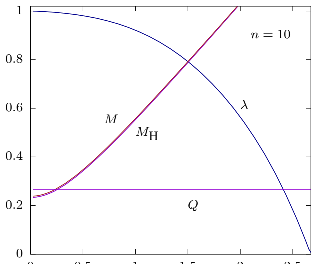
where $P_{\rm H}$, $M_{\rm H}$ are the charge and mass inside the horizon and $P_{\rm hair}$, $M_{\rm hair}$ are the charge and mass contained in the hair

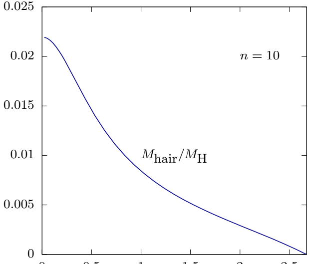
$$P_{\rm hair} = \lambda \times \frac{n \sin^2\theta_{\rm W}}{2e}, \qquad P_{\rm H} = P - P_{\rm hair},$$

whereas, with $4\pi r_H^2 = A_{\rm H}$,

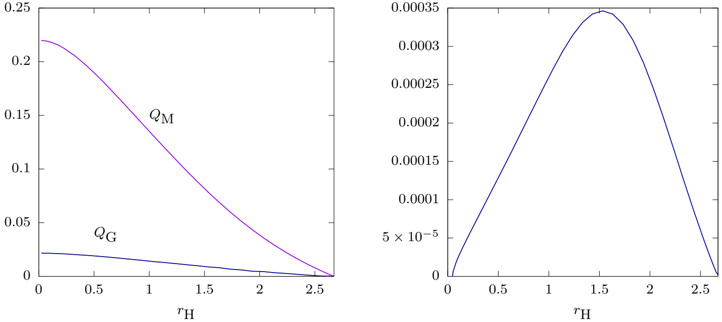
$$M_{\rm H} = \frac{r_H}{2} + \frac{\kappa}{4 r_H} P_{\rm H}^2, \qquad M_{\rm hair} = M - M_{\rm H}$$

is the mass of RN black hole of same area and charge $P_{\rm H}$.





Quadrupole moments and horizon oblateness



The quadrupole moments Q_{G} and Q_{M} become maximal in the extreme limit and vanish in the RN limit. The horizon oblateness

$$(L_{\mathrm{equator}} - L_{\mathrm{polar}})/L_{\mathrm{equator}}$$

vanishes in both limits and assumes a maximal value in between.

Extreme limit

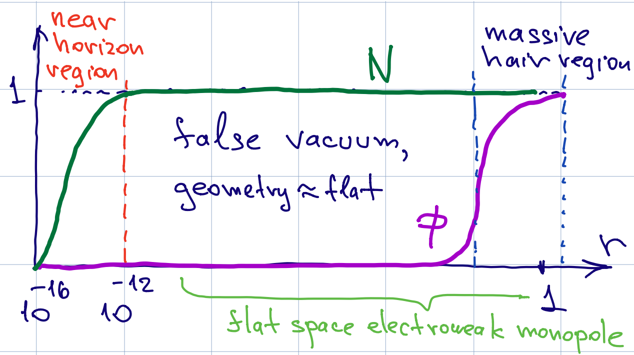
- As r_H approaches the lower bound $r_H \to r_H^{\min} \sim n$, the horizon becomes degenerate, surface gravity and the temperature vanish. The extreme horizon is exactly spherical, while for non-extreme solutions it is squashed.

- The extreme black hole supports the maximal amount of hair. It contains inside only a part $P_{\mathrm{U}(1)} = \cos^2\theta_{\mathrm{W}} P$ of the total magnetic charge $P = n/(2e)$ and the rest $P_{\mathrm{SU}(2)} = \sin^2\theta_{\mathrm{W}} P$ is in the hair. It is smaller than the extreme RN black hole of charge P and has a smaller mass $\Rightarrow$ asymptotically hairy solutions are overcharged RN (/stable !?/) with

$$M < Q = \sqrt{\kappa/2}\,P \equiv M_{\mathrm{BPS}}$$

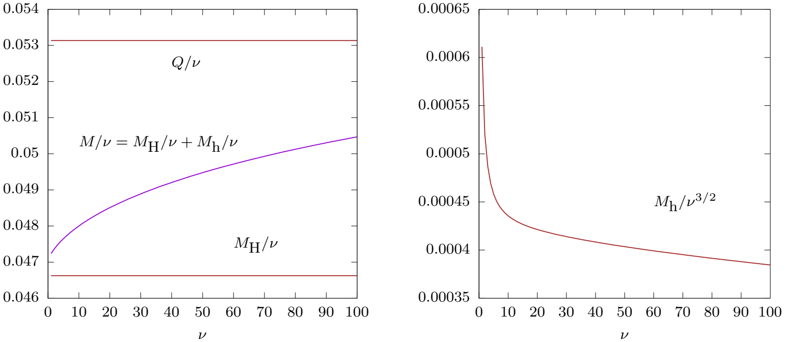
- Close to the horizon the hypermagnetic field $B \propto |n|/r^2$ is very strong and drives to zero the SU(2) and Higgs fields, creating a bubble of symmetric phase. The geometry in the bubble is very close to RN of charge $P_{\mathrm{U}(1)} = \cos^2\theta_{\mathrm{W}} P$.

Horizon size vs hair size



horizon region size is parametrically small as compared to the hair region size.
The hair decoples from the horizon and lives in flat geometry and reduces to the flat space monopole solution.

Mass for large magnetic charge $n = \nu/2$

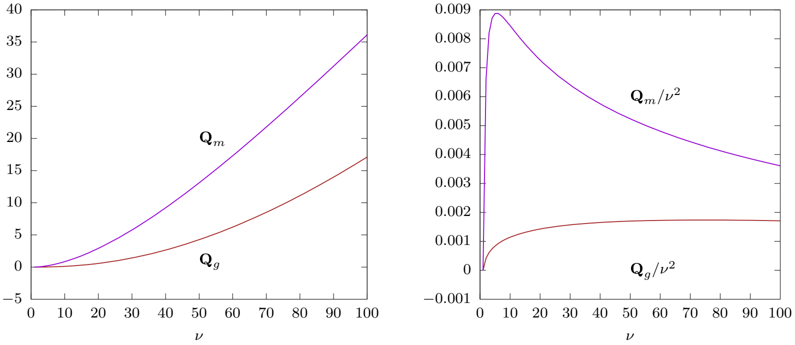


One has for extreme solutions

$$M = M_{\rm H} + M_{\rm hair} < Q \sim n, \quad M_{\rm H} \sim n, \quad M_{\rm reg} \sim n^{3/2}$$

$\Rightarrow$ there is an upper bound for n.

Quadrupole moments for large magnetic charge



One has for extreme solutions $Q_{\rm G} \sim Q_{\rm M} \sim n^2$ and $R \sim \sqrt{n} =$ radius of the charged rings, therefore

$$Q_{\rm G} \sim Q_{\rm M} \sim \text{charge} \times R^2$$

corresponding to the quadrupole moment of a charged torus.

Limiting solution

- Upper limit for n. The minimal value of the event horizon $r_H^{\mathrm{min}} \propto \sqrt{\kappa}\,|n|$ increases with n faster than the maximal value $r_H^{\mathrm{max}} \propto \sqrt{n}$. The two merge for $\boxed{n \sim 1/\kappa \approx 10^{32}}$, exactly,

$$n_{\mathrm{max}} = \frac{\sin\theta_{\mathrm{W}}}{\sqrt{\beta}\kappa} = 0.64 \times 10^{32}$$

The limiting solution develops in this limit an infinitely long tube and becomes geodesically complete.

- The size and mass of the limiting solution

$$r_H \approx 1 \text{ cm}, \qquad M \approx 10^{25} \text{ kg},$$

typical for planetary size black holes $\Rightarrow$ extremely large values, since hairy black holes in other models are typically very small.

Non-axially symmetric black holes

- The perturbative zero mode corresponding to the bifurcation of the RN and hairy branches has the structure

$$\psi(r) \times Y_{jm}(\vartheta, \varphi), \qquad j = \frac{|n|}{2} - 1$$

For $m = 0$ it is axially symmetric but for $m \neq 0$ depends on φ.

- Therefore, for a given n there should be also $|n|$ different non-axially symmetric hairy black holes $\Rightarrow$ CORONA. Their number is $|n| \sim r_H^2 \sim A_{\mathrm{H}}$, as for the horizon entropy. However, their energy is not the same.

Conclusion

Solutions describing black hole with electroweak hair are constructed. They can be large and perhaps astrophysically relevant