# Off-diagonal heat kernel expansions for higher-order minimal operators

Władysław Wachowski

Lebedev Physics Institute

Problems of the Modern Mathematical Physics
21 February 2024, Dubna

# What is heat kernel? And why do we need it?

- Let we have a $d$-dimensional Riemannian manifold with $g_{ab}$ and $\nabla_a$. An operator $\hat{F}(\nabla)$ of order $2\nu$ acts on a set of fields $\varphi = \varphi^A$. (We will omit indices of fields and denote matrices by hats, $\hat{1} = \delta^A_B$.)
- The heat kernel:

$$\hat{K}_F(\tau|x,x') = e^{-\tau \hat{F}} \frac{\hat{1}}{\sqrt{g}} \delta(x,x'),$$

$$\left(\partial_\tau + \hat{F}_x\right) \hat{K}_F(\tau|x,x') = 0, \qquad \hat{K}_F(0|x,x') = \frac{\hat{1}}{\sqrt{g}} \delta(x,x').$$

- Coincidence limit: $\hat{K}_F(\tau|x,x) = \tau^{-d/2\nu} \sum_{m=0}^{\infty} \tau^{m/2\nu} \cdot \hat{E}_m(F|x)$.
- One-loop effective action:

$$\Gamma^{(1)}[\varphi] = \tfrac{i}{2} \log\det \hat{F}(\nabla) = \tfrac{1}{2} \int\limits_0^\infty \frac{d\tau}{(4\pi i)^{d/2}} \tau^{-d/2-1} \int d^d x \sqrt{g}\, \mathrm{tr}\, \hat{K}_F(\tau|x,x).$$

# Part I — “Off-diagonal functoriality”

## Non-rigorous statement:

Let $\hat{F}(\nabla)$ be a “good” differential operator, and $f(q)$ be a “good” function. Then

$$\hat{K}(f(F)|x,x') = f(\hat{F})\frac{\hat{1}}{\sqrt{g}}\delta(x,x') = \sum_{\alpha}\mathbb{K}_\alpha(f|\sigma)\cdot\hat{a}_\alpha(F|x,x'),$$

where $\sigma(x,x')$ is the Synge world function.The basis kernels $\mathbb{K}_\alpha(f|\sigma)$ depend only on the type of the operator $\hat{F}(\nabla)$ and the function $f(q)$, and the off-diagonal coefficients $\hat{a}_\alpha(F|x,x')$ depend only on the background fields (i.e., on the geometry of the manifold and the coefficients of the operator).

The Green function: $\hat{G}_F(x,x') = \hat{F}^{-1}\frac{\hat{1}}{\sqrt{g}}\delta(x,x')$.
In spectral theory, a function of the operator is expressed through the integral of its resolvent: $f(\hat{F}) = \frac{1}{2\pi i}\int\limits_C \frac{d\lambda\ f(\lambda)}{\lambda-\hat{F}}$.

# Resolvents and complex powers

## The Mellin transform:

$$\hat{F}^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty d\tau\, \tau^{s-1} e^{-\tau \hat{F}}, \qquad \hat{G}_{F^s}(x,x') = \frac{1}{\Gamma(s)} \int_0^\infty d\tau\, \tau^{s-1} \hat{K}_F(\tau|x,x'),$$

$$e^{-\tau\hat{F}} = \frac{1}{2\pi i} \int_{w-i\infty}^{w+i\infty} \frac{\tau^{-s}\Gamma(s)}{\hat{F}^s} ds, \quad \hat{K}_F(\tau|x,x') = \frac{1}{2\pi i} \int_{w-i\infty}^{w+i\infty} \tau^{-s}\, \Gamma(s)\, \hat{G}_{F^s}(x,x')\, ds.$$

## The Laplace transform:

$$\frac{1}{\hat{F}+\lambda} = \int_0^\infty d\tau\, e^{-\tau(\hat{F}+\lambda)}, \qquad \hat{G}_{F+\lambda}(x,x') = \int_0^\infty d\tau\, e^{-\lambda\tau}\, \hat{K}_F(\tau|x,x'),$$

$$e^{-t\hat{F}} = \frac{1}{2\pi i} \int_C \frac{d\lambda\, e^{-\lambda t}}{\lambda - \hat{F}}, \quad \hat{K}_F(t|x,x') = \frac{1}{2\pi i} \int_C d\lambda\, e^{-\lambda t} \hat{G}_{F-\lambda}(x,x').$$

# DeWitt's off-diagonal expansion

## Laplace type operator

$$\hat{F}(\nabla) = -\hat{1}\Box + \hat{P}(x) + \frac{\hat{1}}{6}R, \qquad \Box = -g^{ab}\nabla_a\nabla_b.$$

## DeWitt's ansatz

$$\hat{K}_F(\tau|x,x') = \frac{\Delta^{1/2}(x,x')}{(4\pi\tau)^{d/2}} \exp\left(-\frac{\sigma(x,x')}{2\tau}\right) \sum_{m=0}^{\infty} \tau^m \cdot \hat{a}_m(F|x,x')$$

$$= \sum_{m=0}^{\infty} \mathbb{K}_{\frac{d}{2}-m}(\tau,\sigma) \cdot \hat{b}_m(F|x,x'), \qquad \mathbb{K}_\alpha(\tau,\sigma) = \tau^{-\alpha} \exp\left(-\frac{\sigma}{2\tau}\right).$$

## The key result (DeWitt)

$$[\hat{a}_0] = \hat{1}, \qquad [\hat{a}_1] = -\hat{P},$$

$$[\hat{a}_2] = \frac{1}{180}\left(R_{abcd}R^{abcd} - R_{ab}R^{ab} + \Box R\right)\hat{1} + \frac{1}{2}\hat{P}^2 + \frac{1}{12}\hat{\mathcal{R}}_{ab}\hat{\mathcal{R}}^{ab} - \frac{1}{6}\Box\hat{P}.$$

# The direct/inverse Mellin transform trick

## Hadamard's expansion:

$$\hat{G}_{F^s}(x,x') = \sum_{m=0}^{\infty} \mathbb{G}_{\frac{d}{2}-m}(\sigma,s) \cdot \hat{b}_m(F|x,x'),$$

$$\mathbb{G}_\alpha(\sigma,s) = \frac{1}{\Gamma(s)} \int\limits_0^\infty d\tau\, \tau^{s-1} \mathbb{K}_\alpha(\tau,\sigma) = \Gamma(\alpha - s) \left(\frac{\sigma}{2}\right)^{s-\alpha}.$$

## The heat kernel for $\hat{F}^\nu$:

$$\hat{K}_{F^\nu}(\tau|x,x') = \sum_{m=0}^{\infty} \mathbb{K}^{(\nu)}_{\frac{d}{2}-m}(\sigma,\tau) \cdot \hat{b}_m(F|x,x'),$$

$$\mathbb{K}^{(\nu)}_\alpha(\sigma,\tau) = \tfrac{1}{2\pi i} \int\limits_{w-i\infty}^{w+i\infty} \tau^{-s}\, \Gamma(s)\, \mathbb{G}_\alpha(\sigma,\nu s)\, ds = \tau^{-\alpha/\nu} \mathcal{E}_{\nu,\alpha}\left(-\tfrac{\sigma}{2\tau^{1/\nu}}\right).$$

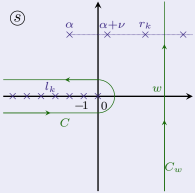
# Properties of generalized exponential functions $\mathcal{E}_{\nu,\alpha}(z)$

A. O. Barvinsky, P. I. Pronin, and W. Wachowski, Phys. Rev. D100, 105004 (2019), arXiv:1908.02161 [hep-th].

## The Mellin transforms:

$$\varepsilon_{\nu,\alpha}(s) = \frac{\Gamma\left(\frac{\alpha-s}{\nu}\right)\Gamma(s)}{\nu\Gamma(\alpha-s)} = \int_0^\infty z^{s-1}\mathcal{E}_{\nu,\alpha}(-z)dz,$$

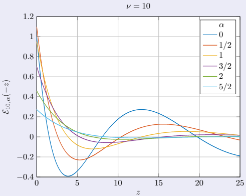
$$\mathcal{E}_{\nu,\alpha}(-z) = \frac{1}{2\pi i}\int_C \varepsilon_{\nu,\alpha}(s)z^{-s}ds.$$




## The Taylor series and differentiation rule

$$\mathcal{E}_{\nu,\alpha}(-z) = \frac{1}{\nu}\sum_{m=0}^{\infty}\frac{\Gamma\left(\frac{\alpha+m}{\nu}\right)}{\Gamma(\alpha+m)}\frac{z^m}{m!},$$

$$\frac{d^\beta}{dz^\beta}\mathcal{E}_{\nu,\alpha}(z) = \mathcal{E}_{\nu,\alpha+\beta}(z), \quad \mathcal{E}_{1,\alpha}(z) = \exp(z).$$



# Can we do more? Operator $\hat{F}^{-\mu}\exp(-\tau\hat{F}^{\nu})$

Using an integral transform, we can compute basis kernels in this case:

$$\hat{K}_F^{(\mu,\nu)}(\tau|x,x') = \frac{1}{\hat{F}^{\mu}}\exp(-\tau\hat{F}^{\nu})\frac{\hat{1}}{\sqrt{g}}\delta(x,x') = \sum_{m=0}^{\infty}\mathbb{K}_{\frac{d}{2}-m}^{(\mu,\nu)}(\tau,\sigma)\cdot\hat{b}_m(F|x,x'),$$

$$\mathbb{K}_{\alpha}^{(\mu,\nu)}(\tau,\sigma) = \frac{1}{\Gamma\left(\frac{\mu}{\nu}\right)}\int_t^{\infty} ds\,(s-t)^{\frac{\mu}{\nu}-1}\mathbb{K}_{\alpha}^{(0,\nu)}(s,\sigma) = t^{\frac{\mu-\alpha}{\nu}}\mathcal{K}_{\alpha}^{(\mu,\nu)}\left(-\frac{\sigma}{2t^{1/\nu}}\right),$$

where $\mathcal{K}_{\alpha}^{(\mu,\nu)}(-z)$ are new hypergeometric-type functions. Their properties can be found from the Mellin-Barnes integral:

$$\mathcal{K}_{\alpha}^{(\mu,\nu)}(-z) = \frac{1}{\Gamma\left(\frac{\mu}{\nu}\right)}\int_1^{\infty} d\tau\,(\tau-1)^{\frac{\mu}{\nu}-1}\tau^{-\frac{\alpha}{\nu}}\mathcal{E}_{\nu,\alpha}\left(-\frac{z}{\tau^{1/\nu}}\right) = \frac{1}{2\pi i}\int_C \kappa_{\alpha}^{(\mu,\nu)}(s)z^{-s}ds,$$

$$\kappa_{\alpha}^{(\mu,\nu)}(s) = \frac{\Gamma(s)\Gamma\left(\frac{\alpha-\mu-s}{\nu}\right)}{\nu\Gamma(\alpha-s)} = \int_0^{\infty}\mathcal{K}_{\alpha}^{(\mu,\nu)}(-z)z^{s-1}dz.$$

# And even more: operator $(\hat{F}^\mu + \lambda^\mu)^{-1}$

Using an integral transform, we can compute basis kernels in this case:

$$\hat{G}_F^{(\mu)}(\lambda|x,x') = \frac{1}{\hat{F}^\mu + \lambda^\mu}\frac{\hat{1}}{\sqrt{g}}\delta(x,x') = \sum_{m=0}^{\infty} \mathbb{G}^{(\mu)}_{\frac{d}{2}-m}(\lambda,\sigma)\cdot \hat{b}_m(F|x,x'),$$

$$\mathbb{G}_\alpha^{(\mu)}(\lambda,\sigma) = \int_0^\infty d\tau\, e^{-\lambda^\mu \tau}\mathbb{K}_\alpha^{(\mu)}(\tau,\sigma) = \lambda^{\alpha-\mu}\mathcal{G}_\alpha^{(\mu)}\left(\frac{1}{2}\lambda\sigma\right),$$

where $\mathcal{K}_\alpha^{(\mu,\nu)}(-z)$ are new hypergeometric-type functions. Their properties can be found from the Mellin-Barnes integral:

$$\mathcal{G}_\alpha^{(\mu)}(z) = \int_0^\infty dt\, t^{-\alpha/\mu}e^{-t}\mathcal{E}_{\mu,\alpha}\left(-\frac{z}{t^{1/\mu}}\right) = \frac{1}{2\pi i}\int_C \gamma_\alpha^{(\mu)}(s)z^{-s}ds,$$

$$\gamma_\alpha^{(\mu)}(s) = \frac{\Gamma\left(\frac{\alpha-s}{\mu}\right)}{\mu\Gamma(\alpha-s)}\Gamma(s)\Gamma\left(\frac{s-\alpha}{\mu}+1\right) = \int_0^\infty \mathcal{G}_\alpha^{(\mu)}(z)z^{s-1}dz.$$

# Part II — Perturbation theory

Generalized Fourier method: A. O. Barvinsky and W. Wachowski, Phys. Rev. D105, 065013 (2022), arXiv:2112.03062 [hep-th].

## The main idea of the method

We already know the expansion for the degree of the Laplace type operator $\hat{H}^\nu(\nabla)$. The minimal operator $\hat{F}(\nabla)$ of higher order $2\nu$ generally cannot be represented in this form, but in any case it can be represented in the form

$$\hat{F}(\nabla) = \hat{1}(-\Box)^\nu + \hat{P}(\nabla) = \hat{H}^\nu(\nabla) + \hat{W}(\nabla).$$

Then it is natural to consider $\hat{H}^\nu$ as the "unperturbed part", and $\hat{W}$ as the "perturbation".

## Perturbation theory

$$\hat{W}_\tau = e^{\tau \hat{H}^M}\, \hat{W}(\nabla)\, e^{-\tau \hat{H}^M},$$

$$\hat{U}_\tau = e^{\tau \hat{H}^M}\, e^{-\tau \hat{F}} = \overline{T} \exp\Big(-\int_0^\tau dt\, \hat{W}_t\Big).$$

# Heat kernel deformation operator

## Deformation operator

$$e^{-\tau \hat{F}} = e^{-\tau \hat{H}^M}\, \hat{U}_\tau = \hat{U}'_\tau\, e^{-\tau \hat{H}^M},$$

$$\hat{U}'_\tau = T \exp\Big(-\int\limits_0^\tau dt\, \hat{W}_{-t}\Big).$$

## Expansion in powers $\tau$

$$\hat{W}_{-t} = \sum_{k=0}^{\infty} \frac{t^k}{k!}\, \hat{V}_k(\nabla), \quad \text{where} \quad \hat{V}_k(\nabla) = [\![\hat{W}(\nabla), \hat{H}^M]\!]_k,$$

$$\int\limits_{\{t\}} dt^n\; t^{\boldsymbol{\alpha}} = \frac{\tau^{n+|\boldsymbol{\alpha}|}}{c(\boldsymbol{\alpha})}, \qquad c(\boldsymbol{\alpha}) = (\alpha_1+1)(\alpha_1+\alpha_2+2)\cdots(|\boldsymbol{\alpha}|+n),$$

$$\hat{U}'_\tau = \sum_{k=0}^{\infty} \tau^k\, \hat{U}_k(\nabla), \quad \hat{U}_k(\nabla) = \sum_{n+|\boldsymbol{\alpha}|=k} \frac{(-1)^n}{\boldsymbol{\alpha}!\, c(\boldsymbol{\alpha})} \hat{V}_{\alpha_1}(\nabla)\cdots\hat{V}_{\alpha_n}(\nabla).$$

# The result of perturbation theory

## Double functional series

$$\hat{K}_F(\tau|x,x') = (4\pi\tau^{1/\nu})^{-d/2} \sum_{m=-\infty}^{\infty} \sum_{k\geq K_m}^{\infty} \tau^{\frac{m}{\nu}} \mathcal{E}_{\nu,\frac{d}{2}+\nu k-m}\left(-\frac{\sigma}{2\tau^{1/\nu}}\right) \hat{a}_{m,k}(F|x,x'),$$

$$\hat{a}_{m,k}(F|x,x') = \sum_{l\geq L_{m,k}}^{m+(\nu-1)k} \left\langle \hat{U}_k(\nabla) \right\rangle_n \hat{a}_l(H|x,x'),$$

$$K_m = \max\{0, -m/(\nu-1)\}, \qquad L_{m,k} = \max\{0, m-\nu k\}.$$

## Three step algorithm:

1. Compute operators $\hat{V}_k(\nabla)$ and $\hat{U}_k(\nabla)$
2. Compute operators $\langle \hat{U}_k(\nabla) \rangle_n$
3. Compute the coefficients $\hat{a}_{m,k}$

# Conclusion

- The use of off-diagonal expansions allows us to significantly strengthen the "functional property". Roughly speaking, if we know the heat kernel expansion for a "good" operator, then we can automatically find the kernel expansion for the "good" function of this operator.
- We obtained off-diagonal heat kernel expansions for HOMO in the form of a double functional series in GEF. A feature of these expansions is the presence of nontrivial coefficients at negative powers of $\tau$.
- We developed two different algorithms for calculating the coefficients of these expansions: based on the "generalized Fourier transform" and on perturbation theory. These algorithms were implemented in Wolfram Mathematica.
- We have constructed a generalization of both algorithms to an even wider class of so-called *causal operators*. In this case, the heat kernel expansion outside of its diagonal is given not by a double, but by a triple functional series.
- Both considered algorithms are not "smart" in the sense that they generate a lot of "extra" coefficients. Therefore they need improvement. In some cases we know how to do this, in others we don't.