On the entanglement in a two qubit system and the torus action on the Grassmannian Gr(2,4)

L. A. Bordag

joint work with
M. V. Babich, A. Khvedelidze, D. Mladenov
LIT, Dubna
ljudmila@bordag.com

February 2024, Dubna

On the entanglement in a two qubit system and the torus action on the Grassmannian $Gr(2,4)$

Ideas and definitions

- Definition of a state space, one qubit
- The concept of the entanglement
- An n-partite composite system, multi-qubit states
- Grassmannians as an appropriate tool for low dimensional systems

The state of a single spin system, $N = 2$, one qubit

The space $\mathcal{P}(\mathbb{C}^2)$ of a mixed spin state is given by 2x2 positive semi-definite hermitian matrices ρ of trace one: $\rho = \frac{1}{2}(\mathbb{I} + \xi \cdot \sigma)$, $\rho = \rho^+, \rho \geq 0$,
where $Tr\rho = 1$, $\xi^2 \leq 1$, $\sigma = (\sigma_1, \sigma_2, \sigma_3)$, σ_i are Pauli matrices, $\xi = (\xi_1, \xi_2, \xi_3)$ is a Bloch vector, for $\xi^2 = 1, \rho^2 = \rho$ – pure states.

Other formulation: $\rho = U\Lambda U^*,\ U \in \mathcal{U}(2),\ \Lambda = diag(\lambda_1, \lambda_2)$, $\lambda_1 + \lambda_2 = 1,\ \lambda_1, \lambda_2 \geq 0$. If $U' = diag(e^{i\theta_1}, e^{i\theta_2}) \in \mathcal{U}(1) \times \mathcal{U}(1)$ then $U'\Lambda U'^* = \Lambda$ is the simplex of eigenvalues. For $\forall \Lambda$ we obtain the set of ρ that is partitioned into equivalence classes homeomorphic to $\mathcal{U}(2)/\mathcal{U}(1) \times \mathcal{U}(1)$, which is $\mathcal{S}^2$. It is also Riemann sphere, or the projective space $\mathbb{C}P^1$. The state space is the ball $\mathbb{B}^3$, $\dim \mathcal{P}(\mathbb{C}^2) = 3$ and $\lambda_1, \lambda_2 \in [0, 1]$.

The concept of the entanglement. History

- In 1935, A. Einstein, B. Podolsky and N. Rosen formulated EPR paradox.
- In 1935-1936 Schrödinger coined the term *entanglement* to describe this peculiar connection between quantum systems.
- 1964, John S. Bell proved that the principle of locality, as applied by EPR, was mathematically inconsistent, he formulated Bell's inequality.
- The Nobel Prize in Physics 2022 was awarded to A. Aspect, J. Clauser, and A. Zeilinger "for experiments with entangled photons, establishing the violation of Bell inequalities and pioneering quantum information science".

The concept of the entanglement. Definition, applications, entanglement measures

If a pair of entangled particles is generated such that their total spin is known to be zero, and one particle is found to have clockwise spin on a first axis, then the spin of the other particle, measured on the same axis, is found to be anticlockwise.

Def.: An entangled state of a composite system is a state that cannot be written as a product state of the component systems.

Entanglement is a basic property of any interacting quantum state.

Types of entanglement measures: ent. cost, distillable ent., ent. of formation, concurrence, relative entropy of ent., squashed ent., logarithmic negativity. **No single one is standard**, are difficult to compute for mixed states.

A state space of an composite N level system

The state space $\mathcal{P}(\mathbb{C}^N)$ of an N level composite system is given by $N \times N$ positive semi-definite hermitian matrices ρ of trace one acting in the Hilbert space of the composite system $\mathcal{H} = \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n$, where the Hilbert spaces $\mathcal{H}_i$, $i = 1, \ldots, n$, with $\dim \mathcal{H}_i = N_i$, $\dim \mathcal{H} = N = N_i \times \cdots \times N_n$.
The set of density matrices is described in terms of the diagonalizing unitary matrices U, $\rho = U \Lambda U^*$, $U \in \mathcal{U}(N)$, $\Lambda = diag(\lambda_1, \ldots, \lambda_N)$, $\sum_{i=1}^{N} \lambda_i = 1, \quad \lambda_i \geq 0$. For $\forall \Lambda$ we obtain the set of ρ that is partitioned into equivalence classes homeomorphic to $\mathcal{U}(N)/\mathcal{U}(N_1) \times \cdots \times \mathcal{U}(N_n)$ which is in general a flag variety.

A state space of an $N = 4$ level system, two qubit $\mathcal{P}(\mathbb{C}^4)$

Let $N = 4$, then the simplest composite system is bipartite system. A generic state of a pair of qubit A and B is given by $\rho_{AB} = \frac{1}{4}(I_4 + a_i\sigma_{i0} + b_i\sigma_{0i} + c_{ij}\sigma_{ij})$, $\sigma_{i0} = \sigma_i \otimes I_2$, $\sigma_{0i} = I_2 \otimes \sigma_i$, $\sigma_{ij} = \frac{1}{2i}(\sigma_i \otimes \sigma_j)$, $c_{ij} \in C$, C is correlation matrix, **a**, **b** - Bloch vectors of qubits with $\rho_A = \frac{1}{2}(I_4 + \mathbf{a} \cdot \sigma)$, and $\rho_B = \frac{1}{2}(I_4 + \mathbf{b} \cdot \sigma)$. The diagonalizing unitary matrices $U \in \mathcal{U}(4)$ defined by $\rho_{AB} = U\Lambda U^*$, where Λ is the matrix of eigenvalues of ρ. For $\forall\Lambda$ we obtain the set of ρ that is partitioned into equivalence classes homeomorphic to $\mathcal{U}(4)/\mathcal{U}(2) \times \mathcal{U}(2)$ which is a Grassmannian $Gr(2,4)$ in this case.

Invariant transformations of the eigenvalue matrix Λ

The matrix of eigenvalues of the density matrix ρ is equal to $\Lambda = \text{diag}(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$, with $Tr(\Lambda) = \sum_{i=1}^{4} \lambda_i = 1, \quad \lambda_i \geq 0$. In the general case $\lambda_1 > \lambda_2 > \lambda_3 > \lambda_4 > 0$. Because of $Tr(\Lambda) = 1$ this set has 3 independent variables. Correspondingly, the matrix Λ is invariant under the action of a real 3-dimensional torus $\mathbf{T}^3 = \text{diag}(e^{i\phi_1}, e^{i\phi_2}, e^{i\phi_3}, e^{i\phi_4})$, with $\sum_{i=1}^{4} \phi_i = 0$. We have $\mathbf{T}^3 \Lambda (\mathbf{T}^3)^* = \Lambda$. To describe just essential unitary transformation of the density matrix ρ we should exclude such trivial transformations which don't change the physical state of the system at all, and hence the object of study is $\frac{\mathcal{U}(4)}{\mathcal{U}(2) \times \mathcal{U}(2)} / \mathbf{T}^3$.

The Grassmannian $Gr(2,4)$ and affine coordinates

Let V be a 4-dimensional vector space over $\mathbb{C}$, $\dim V = 4$, and $\{e_1, e_2, e_3, e_4\}$ be a basis of V.Then elements of $Gr(2,4)$ are all 2-dimensional sub-spaces in $\mathbb{C}^4$, also planes which containing the origin. Each plane w can be spanned by two vectors $w = \text{span}\{\mathbf{v_1}, \mathbf{v_2}\}$, $\mathbf{v_1}, \mathbf{v_2} \in V = \mathbb{C}^4$ and represented in affine coordinates as 2×4 matrix of rank 2. Here the first and second columns are the vectors $\mathbf{v_1}$, $\mathbf{v_2}$,

$$w = \begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ x_4 & y_4 \end{pmatrix} \to \begin{pmatrix} \tilde{x}_1 & \tilde{y}_1 \\ 1 & 0 \\ \tilde{x}_3 & \tilde{y}_3 \\ 0 & 1 \end{pmatrix}.$$

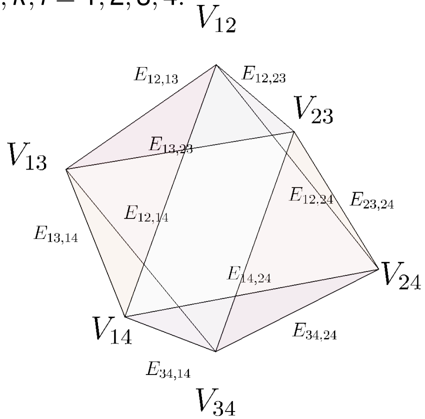
There are six possibilities to arrange rows $(1,0), (0,1)$ on different places, i.e. we obtain a convex body with six vertices (in the case just two rows are different from zero).

The Grassmannian $Gr(2,4)$ and Plücker embedding

Now we consider w as an alternating decomposable 2-tensor $w = \mathbf{v_1} \wedge \mathbf{v_2}$, called also bivector or binor and as a line spanned on w that $\tilde{w} = Cw$ (as a projective equivalence class).
All bivectors build a linear space $\Lambda^2 V = \Lambda^2 \mathbb{C}^4$,
$\dim \Lambda^2 \mathbb{C}^4 = \binom{4}{2} = 6$ with basis $e_{ij} = e_i \wedge e_j$, $1 \leq i < j \leq 4$.
Each element $w \in \Lambda^2 V \simeq \mathbb{C}^6$ will be expand with respect to this basis $w = \sum_{i,j} p_{ij} e_i \wedge e_j$, $1 \leq i < j \leq 4$ and we obtain new coordinates which we notice as p_{ij}, *where* p_{ij} is equal to the minor with rows i, j of the bivector in affine coordinates.
So we define the Plücker embedding $pl : Gr(2,4) \to \Lambda^2 \mathbb{C}^4$, and each plane in $\mathbf{Gr}(2,4)$ corresponds a point in $\mathbb{CP}^5$
$pl : \tilde{w} \to (p_{12} : p_{13} : p_{14} : p_{23} : p_{24} : p_{34}) \in \mathbb{CP}^5$.

Visualization of the Grassmannian $Gr(2,4)$

We denote six vertices as V_{ij}, twelve edges as $E_{ij,kl}$ and eight faces as $\mathcal{F}_i$ or $\mathcal{F}_{jkl}$, $\;i,j,k,l=1,2,3,4$.



The Grassmannian $Gr(2,4)$ and Plücker quadric

Let us introduce bi-linear symmetric linear form $q(w) = w \wedge w, w \in \Lambda^2\mathbb{C}^4$, because of $q(w) = 0$ in Plücker coordinates we obtain $\Omega : \quad p_{12}p_{34} - p_{13}p_{24} + p_{14}p_{23} = 0.$

Theorem

The Plücker embedding pl *realized bijective mapping between the Grassmannian* $\mathbf{Gr}(2,4)$ *and the Plücker quadric* $\Omega \in \mathbb{CP}^5$.

$$\text{Example: } \mathrm{pl} : V_{12} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \to (1:0:0:0:0:0) \in \Omega,$$

$$\mathrm{pl} : E_{12,13} = \begin{pmatrix} 1 & 0 \\ 0 & 1-t \\ 0 & t \\ 0 & 0 \end{pmatrix} \to (1-t:t:0:0:0:0) \in \Omega, \quad t \in [0,1].$$

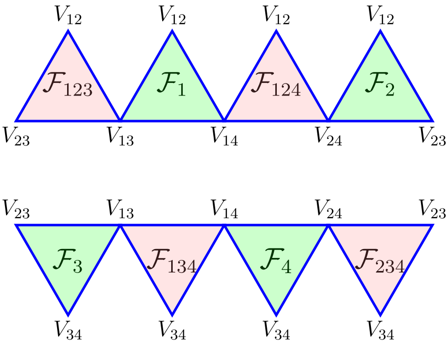


Figure: Eight faces of the octahedron. The green color used for the faces with one common index from $1, 2, 3, 4$ in the limiting edges, correspondingly, i.e. $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, \mathcal{F}_4$. There are so called $\pi-$ planes. The faces of the second type, $\mathcal{F}_{123}, \mathcal{F}_{124}, \mathcal{F}_{134}, \mathcal{F}_{234}$, without any common indexes in the building vertices, so called $\rho-$ planes, colored pink. The $\pi-$ and $\rho-$ planes can have common just one edge.

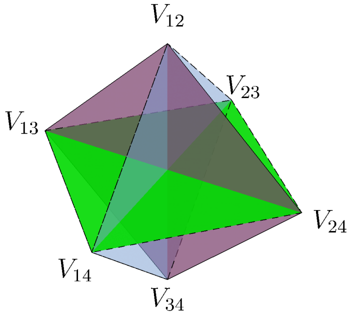


Figure: Three cubic planes. The green one include the vertices $\{V_{13}, V_{14}, V_{24}, V_{23}\}$ (we call it also equatorial plane), the blue one the vertices $\{V_{12}, V_{14}, V_{34}, V_{23}\}$, and the pink one $\{V_{12}, V_{13}, V_{34}, V_{24}\}$. The tetrad axes of symmetry defined by section of these cubic planes, and the vertical one spanned between vertices $\{V_{12}, V_{34}\}$, the other two lie in the equatorial cubic plane between vertices $\{V_{13}, V_{24}\}$ and $\{V_{14}, V_{23}\}$. They connect the antipodal vertices.

Torus action on the points of the Grassmannian $Gr(2,4)$

We take some arbitrary point in Grassmannian in the affine coordinates then we see

$$\mathbf{T}^3 \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} 1e^{i\phi_1} & 0 \\ 0 & 1e^{i\phi_1} \\ \alpha e^{i\phi_3} & \beta e^{i\phi_3} \\ \gamma e^{i\phi_4} & \delta e^{i\phi_4} \end{pmatrix} \to \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \alpha e^{i(\phi_3-\phi_1)} & \beta e^{i(\phi_3-\phi_2)} \\ \gamma e^{i(\phi_4-\phi_1)} & \delta e^{i(\phi_4-\phi_2)} \end{pmatrix}.$$

In the Plücker coordinates we see $p_{kl} = p_{kl}e^{i(\phi_k+\phi_l)}$, it means that the expression $\frac{p_{12}p_{34}}{p_{13}p_{24}}$ left unaltered under this action - it is an invariant. Since we have six homogeneous coordinates $(p_{12}, p_{13}, p_{14}, p_{23}, p_{24}, p_{34}) \in \mathbb{CP}^5$ we have a possibility to use one of the six relations of the type $\frac{p_{ij}p_{kl}}{p_{ik}p_{jl}}$,where $i \neq j \neq k \neq l,\ i,j,k,l = 1,2,3,4.$

Modulo mapping and embedding $Gr(2,4)$ on $\mathbb{CP}^1$

The six cross-ratio relations, also called the double ratio and anharmonic ratio, are modular parameters (invariants of $\mathbf{T}^3$)

$$\frac{p_{13}p_{24}}{p_{14}p_{23}} = \lambda, \qquad \frac{p_{14}p_{23}}{p_{13}p_{24}} = \frac{1}{\lambda}, \qquad \frac{p_{12}p_{34}}{p_{14}p_{23}} = 1-\lambda,$$
$$\frac{p_{14}p_{23}}{p_{12}p_{34}} = \frac{1}{1-\lambda}, \quad \frac{p_{12}p_{34}}{p_{13}p_{24}} = \frac{\lambda-1}{\lambda}, \quad \frac{p_{13}p_{24}}{p_{12}p_{34}} = \frac{\lambda}{\lambda-1},$$

where λ is a parameter. It associated with the six-element anharmonic group (generated by $\lambda \to \frac{1}{\lambda}$ and $\lambda \to 1-\lambda$), which maps 4 points z_1, z_2, z_3, z_4 on a projective line $\mathbb{CP}^1$ in another ones, and corresponds for instance to $\frac{(z_2-z_1)(z_4-z_3)}{(z_3-z_1)(z_4-z_2)}$. The set of fixed points is $\{0, 1, \infty\}$. Its action on $\{0, 1, \infty\}$ gives an isomorphism of S_3.

The structure of a base for the fiber bundle $Gr(2,4)/\mathbf{T}^3$

We have seen that the value of modular mappings unaltered under action of $\mathbf{T}^3$ and from six possible equivalent invariant mappings $\frac{p_{ij}p_{kl}}{p_{ik}p_{jl}} = \mathcal{D}_1(i,j,k,l)$ we can chose one - the modulo map $\mathcal{D}_1 = \mathcal{D}_1(1,2,3,4) = \frac{p_{13}p_{24}}{p_{14}p_{23}}$. We see that the image of the open octahedron (without all vertices, edges, faces and cubic planes) surjective cover $\mathbb{CP}^1$. It is evident quite for all points, may be except three points $(0,1,\infty)$. Let

$$w_0 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{pmatrix} = (-1,0,1,1,1,1) \in \Omega, \text{ then } \mathcal{D}_1 w_0 = P_0 \in \mathbb{CP}^1.$$

The points P_1, P_∞ we obtain with permutations. We build the mapping $\mathcal{D}_1 : Gr(2,4) \to \mathbb{CP}^1$, the Riemann sphere is the base space.

The structure of a fiber

We try now to reduce the number of variables in description of the octahedron and make it's projection real.
Using the modular parameter $\mathcal{D}_1 = \frac{p_{13}p_{24}}{p_{14}p_{23}}$ we transform the Plücker quadric $p_{12}p_{34} - p_{13}p_{24} + p_{14}p_{23} = 0$ to the expression $\frac{p_{12}p_{34}}{p_{13}p_{24}} = 1 - \frac{1}{\mathcal{D}_1} = 1 - \mathcal{D}_{-1} = \mathcal{D}_3$. Both modular parameters are constant on the fiber. Then we obtain the Plücker coordinates in the form $(p_{12} : p_{13} : p_{14} : \mathcal{D}_{-1}\frac{p_{13}p_{24}}{p_{14}} : p_{24} : \mathcal{D}_3\frac{p_{13}p_{24}}{p_{12}})$. To get the affine coordinates we divide all components, by $p_{24} \neq 0$ and obtain $\left(x : y : z : \mathcal{D}_{-1}\frac{y}{z} : 1 : \mathcal{D}_3\frac{y}{x}\right)$, where $x = \frac{p_{12}}{p_{24}}$, $y = \frac{p_{13}}{p_{24}}$, $z = \frac{p_{14}}{p_{24}}$. The torus $\mathbf{T}^3$ act on these coordinates in the following way $x \to xe^{i(\phi_1-\phi_4)}$, $y \to ye^{i(\phi_1+\phi_3-\phi_2-\phi_4)}$, $z \to ze^{i(\phi_1-\phi_2)}$. If we choose $\phi_1 = -\arg x$, $\phi_2 = \arg z - \arg x$, $\phi_3 = \arg z - \arg y$, $\phi_4 = 0$, then the images x, y, z after torus action are real and positive $x, y, z \in R^3_+$.

Projective mapping of $\mathbf{Gr}(2,4)/\mathbf{T}^3$ from $\mathbb{CP}^5$ into $\mathbb{R}^3$

We introduce now in the space $\mathbb{CP}^6$ the new frame of references and provide projection from the last two vertices to $\mathbb{R}^3$, then the six vertices of the octahedron are placed in

$V_{12}, V_{34} \to (1,0,0), (-1,0,0);$
$V_{13}, V_{24} \to (0,1,0), (0,-1,0);$
$V_{14}, V_{23} \to (0,0,1), (0,0,-1).$

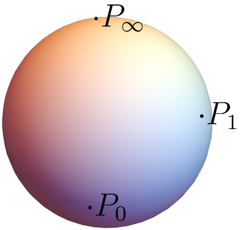
Then the tetrad axes are $\nu_{12,34} = \frac{1}{p_s}(x - \frac{y}{x})$, $\nu_{13,24} = \frac{1}{p_s}(y-1)$, $\nu_{14,23} = \frac{1}{p_s}(z - \frac{y}{z})$, where $p_s = x + y + z + \frac{y}{x} + \frac{y}{z} + 1$. All six vertices of the real octahedron

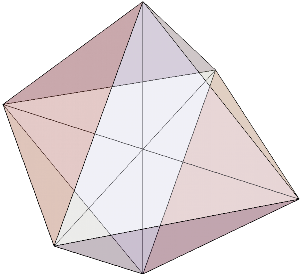
$\frac{1}{p_s}\left(x(1,0,0) + \frac{y}{x}(-1,0,0) + y(0,1,0) + (0,-1,0)\right)$
$\frac{1}{p_s}\left(+z(0,0,1) + \frac{y}{z}(0,0,-1)\right)$

have positive *weights*, because of all values $x, y, z, p_s > 0$ are positive

Visualization of a base and a fiber

The fiber bundle $Gr(2,4)/\mathbf{T}^3$ with a base which is a Riemann sphere and a fiber which is an octahedron





The strata on the Grassmannian $Gr(2,4)$

Def. The equivalence classes of subspaces in $Gr(2,4)$ are called *strata*.
The strata on $Gr(2,4)$ can be defined as the union of all the orbits of $\mathbf{T}^3$ whose images under the momentum map give the same polytope (convex hypersimplex). Let $w \in Gr(2,4)$ we denote $\mathbf{T}^3 \cdot w$ the orbit of w under the action of $\mathbf{T}^3$.
Def. The map $\mu_{\mathbf{T}^3}(w)$:Gr(2,4)$\rightarrow \mathbb{R}^3 \subset \mathbb{R}^4$ into convex polytope $\Delta_{2,4}$ defined by $\frac{\sum_J |p^J|^2 \delta_J}{\sum_J |p^J|^2}$, $J \subset \{1,2,3,4\}$,
$(\delta_J)_i = \left\{ \begin{array}{ll} 1, & i \in J, \\ 0 & i \neq J. \end{array} \right\}$ is called the momentum map. The vertices of the hypersimplex $\Delta_{2,4}$ are given by
$Vert(\mu(\mathbf{T}^3 \cdot w)) = \{\delta_J | p^J(w) \neq 0\}$.
Two orbits $\mathbf{T}^3 \cdot w_1$ and $\mathbf{T}^3 \cdot w_2$ are called equivalent if
$\mu(\mathbf{T}^3 \cdot w_1) = \mu(\mathbf{T}^3 \cdot w_2)$.

An example of the three parameter $\rho_{\alpha\beta\gamma}$ matrix

The Schlienz-Mahler matrix $M = \rho - \rho_A \otimes \rho_B$ provide a tool to measure the correlations between subsystems in the entangled states. One study the invariants of M in the case composed bipartite system with ρ_A, ρ_B. The mixed entangled states are complementary to "separable states", consequently to such with the representation $\rho_{sep} = \sum_i w_i \frac{1}{2}(I_2 + a_i\sigma_i) \otimes \frac{1}{2}(I_2 + b_i\sigma_i)$ where $\sum_i w_i = 1, w_i \geq 0$.

The example of the three parameter matrix

$$\rho_{\alpha\beta\gamma} = \frac{1}{4}\begin{pmatrix} 1+\alpha & 0 & 0 & 0 \\ 0 & 1-\beta & i\gamma & 0 \\ 0 & -i\gamma & 1+\beta & 0 \\ 0 & 0 & 0 & 1-\alpha \end{pmatrix},$$ represents both types of states: separable state spaces and entangled ones.

To do list

- to determine parametrization of all strata $Gr(2,4)/\mathbf{T}^3$
- to prove known kriteria of entanglement and formulate one which is convenient for the geometrical approach
- formulate the geometrical setting for *n*-partite systems of qubits

Literature I

- I. M. Gel'fand and V/V/ Serganova (1987). "Combinatorial geometries and torus strata on homogeneous compact manifolds". Uspekhi Mat. Nauk 42(2):107-134
- Reinhard F. Werner (1989). "Quantum states with Einstein-Podolsky-Rosen correlations admitting a hidden-variable model". Physical Review A. 40 (8): 4277-4281. Bibcode:1989PhRvA..40.4277W. doi:10.1103/PhysRevA.40.4277. PMID 9902666.
- V. Gerd, A. Khvedelidze, and Yu. Palii (2011). "Constraints on $SU(2) \otimes SU(2)$ invariant Polynomials for a pair of entagled qubits". Physics of Atomic NNuclei, v. 74 (6): 893-900
- E. Ghasemian and M. K. Tavassoly (2021)"Generation of Werner-like states via a two-qubit system plunged in a thermal reservoir and their application in solving binary classification problems". Scientific Reports | (2021) 11:3554 | https://doi.org/10.1038/s41598-021-82880-3, www.nature.com/scientificreports