Open Toda chain: Q-operator, orthogonality and completeness of eigenfunctions

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Motivation

Monodromy matrix $T(u)$

$$T(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}$$

Yang-Baxter relation

$$R_{12}(u-v)\,\overset{1}{T}(u)\,\overset{2}{T}(v) = \overset{2}{T}(v)\,\overset{1}{T}(u)\,R_{12}(u-v)$$

$$\overset{1}{T}(u) = T(u) \otimes \mathbb{1} \;\; ; \;\; \overset{2}{T}(v) = \mathbb{1} \otimes T(v)$$

4×4 R-matrix $R_{12}(u) = u + P_{12}$ $\leftrightarrow$ Spin chain models XXX_s, Toda chain
Yang-Baxter relation $\longrightarrow$

$$[A(u)\,, A(v)] = 0 \;\;,\;\; [A(u)+D(u)\,, A(v)+D(v)] = 0$$

- Diagonalization of $A(u)+D(u)$ $\longleftrightarrow$ Spectrum is discrete. Bethe equations, Baxter equation.
- Diagonalization of $A(u)$ $\leftrightarrow$ Spectrum is continuous and simple. Iterative construction of eigenfunctions. Closed integral representations.
- New type of useful integral transformations. Separation of variables.

Plan

- Quantum Toda open chain. Hamiltonian and family of commuting operators.
- Lax operator, monodromy matrix and Yang-Baxter commutation relations.
- Main Problem – construction of common eigenfunctions of the family of commuting operators. Spectrum is continuous and simple. Orthogonality and completeness.
- Q-operator. Definition, kernel identity.Motivation from classical Baecklund transformation.
- Λ-operators. Iterative construction of eigenfunction in x-space – Gauss-Givental representation for eigenfunctions.
- Feynman diagrams. Proof of the main relations.
- Calculation of scalar product and hidden role of Q-operator.
- $\hat{\Lambda}$-operators. Iterative construction of eigenfunction in λ-space – Mellin-Barnes representation for eigenfunctions.
- Gustafson integrals and completeness.
  Sklyanin, Semenov-Tian-Shansky, Kharchev, Lebedev, Silantiev, Gerasimov, Oblezin, Kozlowski, ...

Hamiltonian and commuting operators

Open Toda chain – the system of n one-dimensional particles with coordinates $x_k \in \mathbb{R}$. The Hamiltonian of the system has the form

$$H = \frac{1}{2}\sum_{k=1}^{n} p_k^2 + \sum_{k=1}^{n-1} e^{x_k - x_{k+1}} \quad ; \quad p_k = \frac{1}{i}\frac{\partial}{\partial x_k}$$

The Hilbert space of the system is $L_2\left(\mathbb{R}^n\right)$.
Momentum operator

$$P = \sum_{k=1}^{n} p_k \quad ; \quad HP = PH$$

Integrability – existence of n commuting operators

$$P\,,H \to P\,,H\,,H_3\,,\ldots\,,H_n$$

Generating function $A(u)$

$$A_n(u) = u^n - u^{n-1}P + u^{n-2}\left(\frac{1}{2}P^2 - H\right) - u^{n-3}H_3 + \ldots + H_n$$

$$A_n(u)A_n(v) = A_n(v)A_n(u)$$

Construction of generating function $A(u)$

Lax operator

$$L_k(u) = \begin{pmatrix} u - p_k & e^{-x_k} \\ -e^{x_k} & 0 \end{pmatrix}$$

Yang-Baxter relation $R_{12}(u) = u + P_{12}$

$$\overset{1}{L}_k(u) = L_k(u) \otimes \mathbb{1} \;\; ; \;\; \overset{2}{L}_k(v) = \mathbb{1} \otimes L_k(v)$$

$$R_{12}(u-v)\,\overset{1}{L}_k(u)\,\overset{2}{L}_k(v) = \overset{2}{L}_k(v)\,\overset{1}{L}_k(u)\,R_{12}(u-v)$$

Monodromy matrix $T(u) = L_n(u) \ldots L_2(u)\, L_1(u)$

$$T(u) = \begin{pmatrix} u - p_n & e^{-x_n} \\ -e^{x_n} & 0 \end{pmatrix} \cdots \begin{pmatrix} u - p_1 & e^{-x_1} \\ -e^{x_1} & 0 \end{pmatrix} = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}$$

$$R_{12}(u-v)\,\overset{1}{T}(u)\,\overset{2}{T}(v) = \overset{2}{T}(v)\,\overset{1}{T}(u)\,R_{12}(u-v)$$

Vusualisation

Basis in $\mathbb{V}_1 \otimes \mathbb{V}_2 = \mathbb{C}^2 \otimes \mathbb{C}^2$:

$$e_1 = |\uparrow\uparrow\rangle \ , \ e_2 = |\uparrow\downarrow\rangle \ , \ e_3 = |\downarrow\uparrow\rangle \ , \ e_4 = |\downarrow\downarrow\rangle$$

Matrices of operators $R_{12}(u-v)\,,\overset{1}{T}(u)\,,\overset{2}{T}(v)$ in this basis

$$R_{12}(u-v) = u - v + P_{12} = \begin{pmatrix} u-v+1 & 0 & 0 & 0 \\ 0 & u-v & 1 & 0 \\ 0 & 1 & u-v & 0 \\ 0 & 0 & 0 & u-v+1 \end{pmatrix}$$

$$\overset{1}{T}(v) = \begin{pmatrix} A(v) & 0 & B(v) & 0 \\ 0 & A(v) & 0 & B(v) \\ C(v) & 0 & D(v) & 0 \\ 0 & C(v) & 0 & D(v) \end{pmatrix}$$

$$\overset{2}{T}(u) = \begin{pmatrix} A(u) & B(u) & 0 & 0 \\ C(u) & D(u) & 0 & 0 \\ 0 & 0 & A(u) & B(u) \\ 0 & 0 & C(u) & D(u) \end{pmatrix}$$

Matrix RTT-relation $\leftrightarrow$ 16 commutation relations for A, B, C, D

$$R_{12}(u-v)\,\overset{1}{T}(u)\,\overset{2}{T}(v) = \overset{2}{T}(v)\,\overset{1}{T}(u)\,R_{12}(u-v) \to A(u)\,A(v) = A(v)\,A(u)$$

Eigenfunctions of $A(u)$

Common eigenfunction of n commuting operators $P\,, H\,, H_3\,, \ldots\,, H_n$ is eigenfunction of $A_n(u)$ with polynomial eigenvalue

$$A_n(u)\,\Psi_{\lambda_1\ldots\lambda_n}(x_1\,,\ldots\,,x_n) = (u-\lambda_1)\cdots(u-\lambda_n)\,\Psi_{\lambda_1\ldots\lambda_n}(x_1\,,\ldots\,,x_n)$$

Spectrum is continuous and simple. Eigenfunctions are labelled by n zeroes $\lambda_k \in \mathbb{R}\ \ ; k = 1\,,\ldots\,,n$ of eigenvalue of operator $A_n(u)$

Iterative construction of eigenfunctions $\Psi_{\lambda_1\ldots\lambda_n}(x)$

$$\Psi_{\lambda_1\ldots\lambda_n} = \Lambda_{\lambda_n}\,\Psi_{\lambda_1\ldots\lambda_{n-1}}$$

$$\Psi_{\lambda_1\ldots\lambda_n}(x_1\,,\ldots\,,x_n) = \int_{\mathbb{R}^{n-1}} dy_1\cdots dy_{n-1}\,\Lambda_{\lambda_n}(x_1\,,\ldots\,,x_n|y_1\,,\ldots\,,y_{n-1})\,\Psi_{\lambda_1\ldots\lambda_{n-1}}(y_1\,,\ldots\,,y_{n-1})$$

$$\Psi_{\lambda_1}(x_1) = e^{i\lambda_1 x_1} \longrightarrow \Psi_{\lambda_1\lambda_2}(x_1\,,x_2) = \Lambda_2(\lambda_2)\, e^{i\lambda_1 x_1} \longrightarrow \dots$$

$$\Psi_{\lambda_1\lambda_2}(x_1\,,x_2) \;=\; \int_{\mathbb{R}} dy_1\; \exp\left(i\lambda_2\left(x_1+x_2-y_1\right) - e^{x_1-y_1} - e^{y_1-x_2}\right) e^{i\lambda_1 y_1}$$

Orthogonality relation

$$\int_{\mathbb{R}^n} \overline{\Psi_{\lambda_1\dots\lambda_n}(x)}\,\Psi_{\lambda'_1\dots\lambda'_n}(x)\,\prod_{j=1}^{n}\frac{dx_j}{2\pi} = \mu_n^{-1}(\lambda)\,\delta_{sym}(\lambda-\lambda')$$

Completeness relation

$$\int_{\mathbb{R}^n} \overline{\Psi_{\lambda_1\dots\lambda_n}(x)}\,\Psi_{\lambda_1\dots\lambda_n}(y)\,\mu_n(\lambda)\,\prod_{j=1}^{n}\frac{d\lambda_j}{2\pi} = \prod_{k=1}^{n}\delta(x_k-y_k)$$

$$\mu_n(\lambda) = \frac{1}{n!}\,\frac{1}{\prod_{j<k}^{n}\Gamma(i\lambda_k-i\lambda_j)\Gamma(i\lambda_j-i\lambda_k)}$$

$$\delta_{sym}\left(\lambda-\lambda'\right) = \frac{1}{n!}\sum_{\sigma\in S_n}\prod_{k=1}^{n}\delta\left(\lambda_k-\lambda'_{\sigma(k)}\right)\;.$$

Q-operator

$$Q(u)\,Q(v) = Q(v)\,Q(u) \quad ; \quad Q(v)\,A(u) = A(u)\,Q(v)$$

$$[Q(\lambda)\Psi]\,(x_1\,,\ldots\,,x_n) = \int_{\mathbb{R}^n} dy_1 \cdots dy_n\, Q_\lambda(x_1\,,\ldots\,,x_n|y_1\,,\ldots\,,y_n)\,\Psi(y_1\,,\ldots\,,y_n)$$

Integral kernel

$$Q_\lambda(x|y) = \exp\left(i\lambda \sum_{k=1}^{n}(x_k - y_k) - \sum_{k=1}^{n} e^{x_k - y_k} - \sum_{k=1}^{n-1} e^{y_k - x_{k+1}} \right)$$

Integral operators

$$Q_1(\lambda) \leftrightarrow \exp\left(i\lambda(x_1 - y_1)\right)$$

$$Q_2(\lambda) \leftrightarrow \exp\left(i\lambda\,(x_1 + x_2 - y_1 - y_2) - e^{x_1 - y_1} - e^{x_2 - y_2} - e^{y_1 - x_2}\right)$$

$$Q_3(\lambda) \leftrightarrow \exp\left(i\lambda \sum_{k=1}^{3}(x_k - y_k) - e^{x_1 - y_1} - e^{x_2 - y_2} - e^{x_3 - y_3} - e^{y_1 - x_2} - e^{y_2 - x_3} \right)$$

Construction of Q-operator

Motivation from classical Baecklund transformation

V.A. Fock, On canonical transformation in classical and quantum mechanics (in Russian), Vestnik Leningradskogo universiteta 19 (1959) 67

V. Pasquier and M. Gaudin M, The periodic Toda chain and a matrix generalization of the Bessel function recursion relations, J. Phys. A: Math. Gen. 25 (1992) 5243–5252

E. K. Sklyanin Baecklund transformations and Baxter's Q-operator arXiv:nlin/0009009

Operator $A(u) = A(u|x_i, \partial_{x_i})$ is function of quantum canonical variables x_i and $p_i = -i\partial_{x_i}$. Relation of commutativity

$$A(u)\,Q(\lambda) = Q(\lambda)\,A(u)$$

is equivalent to the kernel identity for the integral kernel $Q(x, y)$

$$A(u|x_i, \partial_{x_i})\,Q_\lambda(x, y) = A(u|y_i, -\partial_{y_i})\,Q_\lambda(x, y)$$

Let us restore $\hbar$ for the moment $p_i = -i\hbar\partial_{x_i}$ and consider semiclassical approximation for integral kernel $Q(x, y) = \exp \frac{1}{\hbar}\,F(x, y)$

$$A(u|x_i, -i\hbar\partial_{x_i})\,e^{\frac{i}{\hbar}\,F(x,y)} = A(u|y_i, i\hbar\partial_{y_i})\,e^{\frac{i}{\hbar}\,F(x,y)} \xrightarrow{\hbar\to 0}$$

$$A(u|x_i, \partial_{x_i}F) = A(u|y_i, -\partial_{y_i}F)$$

Construction of Q-operator

Canonical transformation $(P,y) \to (p,x)$ in classical mechanics

$$\{P_i, y_k\} = \delta_{ik} \to \{p_i, x_k\} = \delta_{ik}$$

Generating function $F(x,y)$

$$P_i = -\frac{\partial F(x,y)}{\partial y_i} \quad ; \quad p_i = \frac{\partial F(x,y)}{\partial x_i}$$

$$A(u|x_i, \partial_{x_i} F) = A(u|y_i, -\partial_{y_i} F) \quad \leftrightarrow \quad A(u|x_i, p_i) = A(u|y_i, P_i)$$

Guess that semiclassical approximation is exact, forget about $\hbar$ and try ansatz $Q(x,y) = \exp F(x,y)$, where $F(x,y)$ is generating function of classical Baecklund transformation (Toda, Gaudin)

$$F(x,y) = i\lambda \sum_{k=1}^{n} (x_k - y_k) - \sum_{k=1}^{n} e^{x_k - y_k} - \sum_{k=1}^{n-1} e^{y_k - x_{k+1}}$$

Simplest example of kernel identity

$$A(u|x_i,\partial_{x_i})\,Q_\lambda(x,y) = A(u|y_i,-\partial_{y_i})\,Q_\lambda(x,y)$$

$$Q_\lambda(x_1,x_2|y_1,y_2) = \exp\left(i\lambda\left(x_1+x_2-y_1-y_2\right) - e^{x_1-y_1} - e^{x_2-y_2} - e^{y_1-x_2}\right)$$

$$\begin{pmatrix} u+i\partial_{x_2} & e^{-x_2} \\ -e^{x_2} & 0 \end{pmatrix}\begin{pmatrix} u+i\partial_{x_1} & e^{-x_1} \\ -e^{x_1} & 0 \end{pmatrix} \to$$

$$\begin{pmatrix} u-\lambda-i\left(e^{x_2-y_2}-e^{y_1-x_2}\right) & e^{-x_2} \\ -e^{x_2} & 0 \end{pmatrix}\begin{pmatrix} u-\lambda-ie^{x_1-y_1} & e^{-x_1} \\ -e^{x_1} & 0 \end{pmatrix}$$

$$\begin{pmatrix} u-i\partial_{y_2} & e^{-y_2} \\ -e^{y_2} & 0 \end{pmatrix}\begin{pmatrix} u-i\partial_{y_1} & e^{-y_1} \\ -e^{y_1} & 0 \end{pmatrix} \to$$

$$\begin{pmatrix} u-\lambda-ie^{x_2-y_2} & e^{-y_2} \\ -e^{y_2} & 0 \end{pmatrix}\begin{pmatrix} u-\lambda-i\left(e^{x_1-y_1}-e^{y_1-x_2}\right) & e^{-y_1} \\ -e^{y_1} & 0 \end{pmatrix}$$

Iterative construction of eigenfunctions $\Psi_{\lambda_1\ldots\lambda_n}(x)$

$$\Psi_{\lambda_1\ldots\lambda_n}(x_1,\ldots,x_n) = \int_{\mathbb{R}^{n-1}} dy_1\cdots dy_{n-1}\,\Lambda_{\lambda_n}(x_1,\ldots,x_n|y_1,\ldots,y_{n-1})\,\Psi_{\lambda_1\ldots\lambda_{n-1}}(y_1,\ldots,y_{n-1})$$

Integral kernel

$$\Lambda_\lambda(x_1,\ldots,x_n|y_1,\ldots,y_{n-1}) = \exp\left(i\lambda\left(\sum_{k=1}^{n} x_k - \sum_{k=1}^{n-1} y_k\right) - \sum_{k=1}^{n-1} e^{x_k-y_k} - \sum_{k=1}^{n-1} e^{y_k-x_{k+1}}\right)$$

Integral operators

$$\Lambda_1(\lambda) \leftrightarrow \exp\left(i\lambda x_1\right)$$

$$\Lambda_2(\lambda) \leftrightarrow \exp\left(i\lambda\left(x_1+x_2-y_1\right) - e^{x_1-y_1} - e^{y_1-x_2}\right)$$

$$\Lambda_3(\lambda) \leftrightarrow \exp\left(i\lambda\left(x_1+x_2+x_3-y_1-y_2\right) - e^{x_1-y_1} - e^{x_2-y_2} - e^{y_1-x_2} - e^{y_2-x_3}\right)$$

Iterative construction of eigenfunctions $\Psi_{\lambda_1 \dots \lambda_n}(x)$

$\Psi_{\lambda_1}(x_1) = e^{i\lambda_1 x_1} = \Lambda_1(\lambda_1)\, 1$

$$A_1(u) = u - p_1 = u + i\frac{\partial}{\partial x_1} \quad ; \quad A_1(u)\, e^{i\lambda_1 x_1} = (u - \lambda_1)\, e^{i\lambda_1 x_1}$$

$\Psi_{\lambda_1\lambda_2}(x_1\,, x_2) = \Lambda_2(\lambda_2)\, \Lambda_1(\lambda_1)\, 1$

$$\Psi_{\lambda_1\lambda_2}(x_1\,, x_2) \;=\; \int_{\mathbb{R}} dy_1 \,\exp\left(i\lambda_2\,(x_1 + x_2 - y_1) - e^{x_1 - y_1} - e^{y_1 - x_2}\right) e^{i\lambda_1 y_1}$$

$$A_2(u)\, \Psi_{\lambda_1\lambda_2}(x_1\,, x_2) = (u - \lambda_1)\,(u - \lambda_2)\, \Psi_{\lambda_1\lambda_2}(x_1\,, x_2)$$

Symmetry $\lambda_1 \rightleftarrows \lambda_2 \quad \longleftrightarrow \quad y_1 \to x_1 + x_2 - y_1$

$$A(\lambda_2) \leftrightarrow \begin{pmatrix} \lambda_2 + i\partial_{x_2} & e^{-x_2} \\ -e^{x_2} & 0 \end{pmatrix} \begin{pmatrix} \lambda_2 + i\partial_{x_1} & e^{-x_1} \\ -e^{x_1} & 0 \end{pmatrix} \to$$

$$\begin{pmatrix} ie^{y_1 - x_2} & e^{-x_2} \\ -e^{x_2} & 0 \end{pmatrix} \begin{pmatrix} -ie^{x_1 - y_1} & e^{-x_1} \\ -e^{x_1} & 0 \end{pmatrix}$$

$A(\lambda_2)\Psi_{\lambda_1\lambda_2}(x) = 0 \to$ by symmetry $\lambda_1 \rightleftarrows \lambda_2 \to A(\lambda_1)\Psi_{\lambda_1\lambda_2}(x) = 0 \to$

$$A(u)\Psi_{\lambda_1\lambda_2}(x_1\,, x_2) = (u - \lambda_1)(u - \lambda_2)\Psi_{\lambda_1\lambda_2}(x_1\,, x_2)$$

Iterative construction of eigenfunctions $\Psi_{\lambda_1\ldots\lambda_n}(x)$

$$\Psi_{\lambda_1\ldots\lambda_n}(x_1,\ldots,x_n)=\Lambda_n(\lambda_n)\cdots\Lambda_2(\lambda_2)\,\Lambda_1(\lambda_1)\,1$$
$$A(u)\Psi_{\lambda_1\ldots\lambda_n}(x)=(u-\lambda_1)\cdots(u-\lambda_n)\Psi_{\lambda_1\ldots\lambda_n}(x)$$
$$Q_n(\lambda)\,\Psi_{\lambda_1\ldots\lambda_n}(x)=\Gamma(i\lambda-i\lambda_1)\cdots\Gamma(i\lambda-i\lambda_n)\,\Psi_{\lambda_1\ldots\lambda_n}(x)$$

Symmetry $\lambda_i\rightleftarrows\lambda_j\longleftrightarrow$

$$\Lambda_k(\lambda_k)\,\Lambda_{k-1}(\lambda_{k-1})=\Lambda_k(\lambda_{k-1})\,\Lambda_{k-1}(\lambda_k)$$

$A(\lambda_n)\Psi_{\lambda_1\ldots\lambda_n}(x)=0\rightarrow$ by symmetry $\lambda_i\rightleftarrows\lambda_j\rightarrow A(\lambda_j)\Psi_{\lambda_1\ldots\lambda_n}(x)=0\rightarrow$

$$A(u)\Psi_{\lambda_1\ldots\lambda_n}(x)=(u-\lambda_1)\cdots(u-\lambda_n)\Psi_{\lambda_1\ldots\lambda_n}(x)$$

Calculation of eigenvalue of Q-operator is based on relation

$$Q_n(\lambda)\,\Lambda_n(\mu)=\Gamma(i\lambda-i\mu)\,\Lambda_n(\mu)\,Q_{n-1}(\lambda)$$

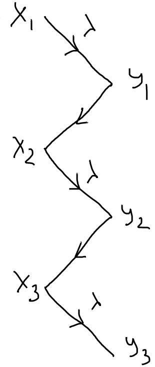
$$Q_n(\lambda)\,\Lambda_n(\lambda_n)\,\Lambda_{n-1}(\lambda_{n-1})\,\cdots\,\Lambda_1(\lambda_1)\,1=$$
$$\Gamma(i\lambda-i\lambda_n)\,\Lambda_n(\lambda_n)\,Q_{n-1}(\lambda)\,\Lambda_{n-1}(\lambda_{n-1})\,\cdots\,\Lambda_1(\lambda_1)\,1=$$
$$\Gamma(i\lambda-i\lambda_n)\,\Gamma(i\lambda-i\lambda_{n-1})\,\Lambda_n(\lambda_n)\,\Lambda_{n-1}(\lambda_{n-1})\,Q_{n-2}(\lambda)\,\cdots\,\Lambda_1(\lambda_1)\,1$$

Q- and Λ-operators

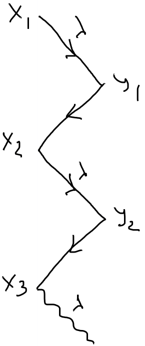
$x \xrightarrow{\lambda} y \qquad \exp\left[i\lambda(x-y) - e^{x-y}\right]$

$x \longrightarrow y \qquad \exp - e^{x-y} \qquad\qquad x \overset{\lambda}{\sim\!\sim} \qquad \exp i\lambda x$

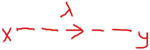
Q-operator



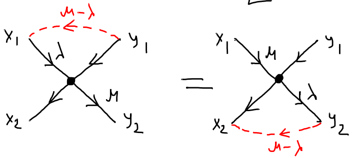
Λ-operator

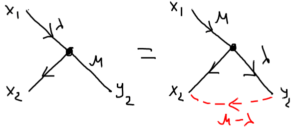


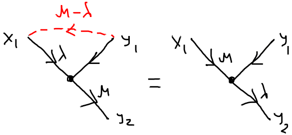
Commutativity of Q-operators

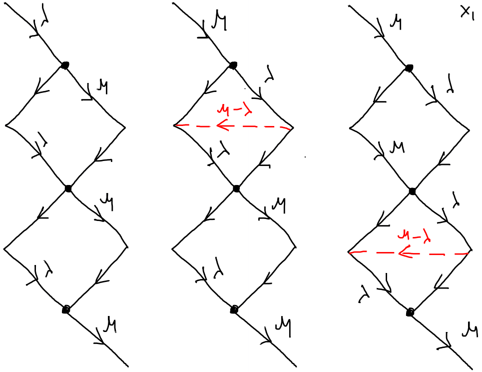


$$\left[1 + e^{x-y}\right]^{-i\lambda}$$



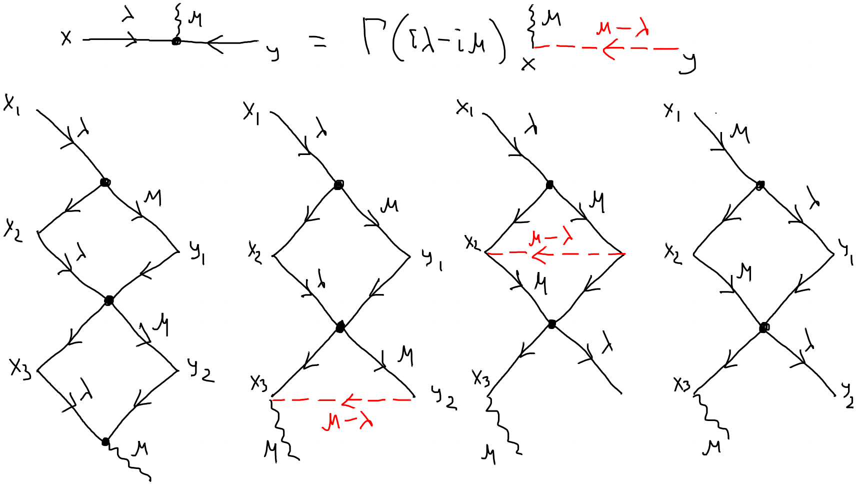






$$Q(\lambda)\,Q(\mu) = Q(\mu)\,Q(\lambda)$$

$Q\Lambda$-relation



$$Q_3(\lambda)\,\Lambda_2(\mu) = \Gamma(i\lambda - i\mu)\,\Lambda_3(\mu)\,Q_2(\lambda)$$

Calculation of the scalar product

$$\int_{\mathbb{R}^n} \overline{\Psi_{\lambda_1\ldots\lambda_n}(x)}\,\Psi_{\lambda'_1\ldots\lambda'_n}(x)\,\prod_{j=1}^{n}\frac{dx_j}{2\pi} = \mu_n^{-1}(\lambda)\,\delta_{sym}(\lambda-\lambda')$$

Step one – regularization

$$\int_{\mathbb{R}^n} \overline{\Psi_{\lambda_1\ldots\lambda_n}(x)}\,\Psi_{\lambda'_1\ldots\lambda'_n}(x)\,\prod_{j=1}^{n}\frac{dx_j}{2\pi} =$$

$$\lim_{x_0\to-\infty}\lim_{\varepsilon\to 0} e^{-i\lambda'_n x_0}\int_{\mathbb{R}^n} \overline{\Psi_{\lambda_1\ldots\lambda_n}(x)}\,\exp\left(i\lambda'_n x_0 - e^{x_0-x_1}\right)\Psi_{\lambda'_1+i\varepsilon\ldots\lambda'_n+i\varepsilon}(x)\,\prod_{j=1}^{n}\frac{dx_j}{2\pi}$$

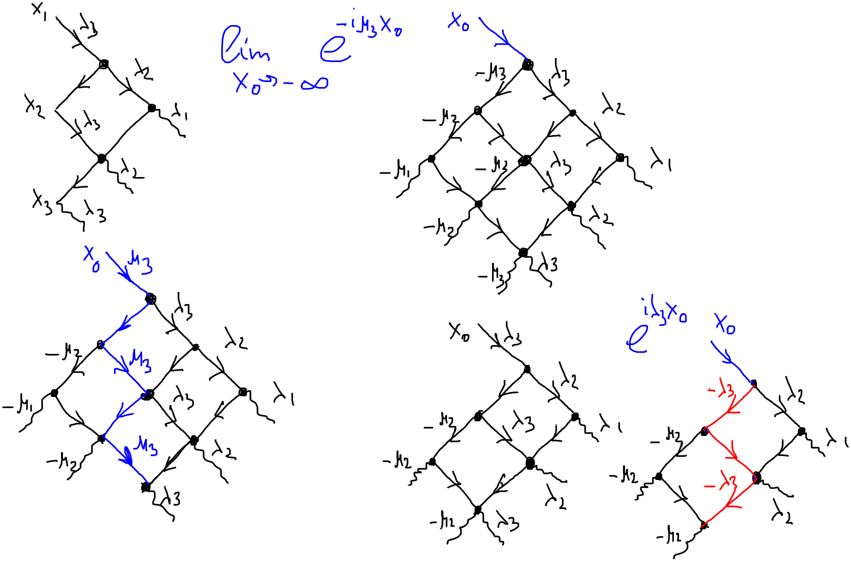
Step two – hidden Q-operators allow to calculate everything in a closed form

$$\prod_{i,k=1}^{n}\Gamma(i\lambda_i - i\lambda'_k + \varepsilon)\exp ix_0\sum_{k=1}^{n}\left(\lambda_k-\lambda'_k\right)$$

Last step – delta-sequence

$$\lim_{x_0\to-\infty}\lim_{\epsilon\to 0^+}\frac{e^{ix_0\sum_{k=1}^{n}(\lambda_k-\lambda'_k)}}{\prod_{i,k=1}^{n}\left(\lambda_i-\lambda'_k-i\varepsilon\right)} = \frac{(-1)^{\frac{n(n-1)}{2}}(2\pi i)^n n!}{\prod_{i<k}^{n}\left(\lambda_i-\lambda_k\right)^2}\,\delta_{sym}(\lambda,\lambda'),$$

Calculation of the scalar product



Dual representation for eigenfunctions

Iteration in x-space

$$\Psi_{\lambda_1\ldots\lambda_n}(x_1,\ldots,x_n) = \int_{\mathbb{R}^{n-1}} dy_1\cdots dy_{n-1}\,\Lambda_{\lambda_n}(x_1,\ldots,x_n|y_1,\ldots,y_{n-1})\,\Psi_{\lambda_1\ldots\lambda_{n-1}}(y_1,\ldots,y_{n-1})$$

Iteration in λ-space

$$\Psi_{\lambda_1\ldots\lambda_n}(x_1,\ldots,x_n) = \int_{\mathbb{R}^{n-1}} \hat{\Lambda}_{x_n}(\lambda_1,\ldots,\lambda_n|\gamma_1,\ldots,\gamma_{n-1})\,\Psi_{\gamma_1\ldots\gamma_{n-1}}(x_1,\ldots,x_{n-1})\,\mu_{n-1}(\gamma)\prod_{j=1}^{n-1}\frac{d\gamma_j}{2\pi}$$

Orthogonality of eigenfunctions $\rightarrow$ $\hat{\Lambda}_{x_n}$ is defined by the scalar product of Ψ_n and Ψ_{n-1}

$$\hat{\Lambda}_{x_n}(\lambda_1,\ldots,\lambda_n|\gamma_1,\ldots,\gamma_{n-1}) = \int_{\mathbb{R}^{n-1}} \overline{\Psi_{\gamma_1\ldots\gamma_{n-1}}(x_1,\ldots,x_{n-1})}\,\Psi_{\lambda_1\ldots\lambda_n}(x_1,\ldots,x_n)\prod_{j=1}^{n}\frac{dx_j}{2\pi}$$

Dual representation for eigenfunctions

The scalar product of Ψ_n and Ψ_{n-1} is calculated in a closed form using Q-operators similar to previous scalar product calculation

$$\hat{\Lambda}_{x_n}(\lambda_1, \ldots, \lambda_n|\gamma_1, \ldots, \gamma_{n-1}) = \prod_{k=1}^{n}\prod_{j=1}^{n-1}\Gamma(i\lambda_k - i\gamma_j + \varepsilon)\, \exp ix_n \left(\sum_{j=1}^{n}\lambda_j - \sum_{j=1}^{n-1}\gamma_j\right)$$

$$\Psi_{\lambda_1\ldots\lambda_n}(x_1, \ldots, x_n) = \hat{\Lambda}_n(x_n)\cdots\hat{\Lambda}_2(x_2)\,\hat{\Lambda}_1(x_1)\,1$$

Integral operators

$$\hat{\Lambda}_1(x) \leftrightarrow \exp ix\lambda_1$$

$$\hat{\Lambda}_2(x) \leftrightarrow \Gamma(i\lambda_1 - i\gamma_1 + \varepsilon)\Gamma(i\lambda_2 - i\gamma_1 + \varepsilon)\, \exp ix\,(\lambda_1 + \lambda_2 - \gamma_1)$$

$$\Psi_{\lambda_1}(x_1) = e^{i\lambda_1 x_1} = \hat{\Lambda}_1(x_1)\,1 \longrightarrow \Psi_{\lambda_1\lambda_2}(x_1\,,x_2) = \hat{\Lambda}_2(x_2)\,\hat{\Lambda}_1(x_1)\,1 \longrightarrow \dots$$

$$\Psi_{\lambda_1\lambda_2}(x_1\,,x_2) = \int_{\mathbb{R}} d\gamma_1\, \exp ix_2\,(\lambda_1+\lambda_2-\gamma_1)$$
$$\Gamma(i\lambda_1 - i\gamma_1 + \varepsilon)\Gamma(i\lambda_2 - i\gamma_1 + \varepsilon)\, e^{i\gamma_1 x_1}$$

Eigenfunctions are constructed iteratively in λ-space

$$\Psi_n(x|\lambda) = \int_{\mathbb{R}^{n-1}} \prod_{k=1}^{n}\prod_{j=1}^{n-1}\Gamma(i\lambda_k - i\gamma_j + \varepsilon)\, e^{i(\Lambda-\gamma)x_n}\,\Psi_{n-1}(x|\gamma)\,\mu_{n-1}(\gamma)\prod_{j=1}^{n-1}\frac{d\gamma_j}{2\pi}$$

where in explicit notations we have: $\Lambda = \sum_{j=1}^{n}\lambda_j$, $\gamma = \sum_{j=1}^{n-1}\gamma_j$ and

$$\Psi_n(x|\lambda) = \Psi_n\,(x_1,\dots,x_n|\lambda_1,\dots,\lambda_n)$$
$$\Psi_{n-1}(x|\gamma) = \Psi_{n-1}\,(x_1,\dots,x_{n-1}|\gamma_1,\dots,\gamma_{n-1})$$

Gustafson integrals

$$\int_{\mathbb{R}^n} \prod_{k=1}^{n+1}\prod_{j=1}^{n} \Gamma(i\alpha_k - i\lambda_j)\Gamma(i\lambda_j - i\beta_k)\,\mu_n(\lambda)\prod_{j=1}^{n}\frac{d\lambda_j}{2\pi} = \frac{\prod_{k,j=1}^{n+1}\Gamma(i\alpha_j - i\beta_k)}{\Gamma\left(\sum_{j=1}^{n+1}(i\alpha_j - i\beta_j)\right)}$$

First reduction $\alpha_{n+1} = -ie^{-ix}L \;\; ; \;\; \beta_{n+1} = ie^{-iy}L \;\; ; \;\; L \to \infty$

$$\int_{\mathbb{R}^n} \prod_{k,j=1}^{n} \Gamma(i\alpha_k - i\lambda_j)\Gamma(i\lambda_j - i\beta_k)\, e^{i(x-y)\Lambda}\mu_n(\lambda)\prod_{j=1}^{n}\frac{d\lambda_j}{2\pi} =$$

$$e^{ix\beta}\, e^{-iy\alpha}\,\left(e^{-ix} + e^{-iy}\right)^{i\beta - i\alpha}\prod_{k,j=1}^{n}\Gamma(i\alpha_j - i\beta_k)$$

$$\Lambda = \sum_{k=1}^{n}\lambda_k \;\; ; \;\; \alpha = \sum_{k=1}^{n}\alpha_k \;\; ; \;\; \beta = \sum_{k=1}^{n}\beta_k$$

Gustafson integrals

$$\int_{\mathbb{R}^n} \prod_{k,j=1}^{n} \Gamma(i\alpha_k - i\lambda_j)\Gamma(i\lambda_j - i\beta_k)\, e^{i(x-y)\Lambda} \mu_n(\lambda) \prod_{j=1}^{n} \frac{d\lambda_j}{2\pi} =$$

$$e^{ix\beta}\, e^{-iy\alpha}\, \left(e^{-ix} + e^{-iy}\right)^{i\beta - i\alpha} \prod_{k,j=1}^{n} \Gamma(i\alpha_j - i\beta_k)$$

Second reduction $\alpha_n = -iL$; $\beta_n = iL$; $L \to \infty$

$$\int_{\mathbb{R}^n} \prod_{k=1}^{n-1} \prod_{j=1}^{n} \Gamma(i\alpha_k - i\lambda_j + \varepsilon)\Gamma(i\lambda_j - i\beta_k + \varepsilon)\, e^{i\Lambda(x-y)} \mu_n(\lambda) \prod_{j=1}^{n} \frac{d\lambda_j}{2\pi} =$$

$$= \frac{(2\pi)^{n-1}}{\mu_{n-1}(\alpha)}\, \delta(x-y)\, \delta_{sym}(\alpha - \beta)$$

Completeness and Gustafson integrals

The completeness relation has the following form

$$\int_{\mathbb{R}^n} \overline{\Psi_n(x|\lambda)}\,\Psi_n(y|\lambda)\,\mu_n(\lambda)\prod_{j=1}^{n}\frac{d\lambda_j}{2\pi} = \prod_{k=1}^{n}\delta(x_k - y_k)$$

Eigenfunctions are constructed iteratively in λ-space

$$\Psi_n(x|\lambda) = \int_{\mathbb{R}^{n-1}} \prod_{k=1}^{n}\prod_{j=1}^{n-1}\Gamma(i\lambda_k - i\gamma_j + \varepsilon)\, e^{i(\Lambda-\gamma)x_n}\,\Psi_{n-1}(x|\gamma)\,\mu_{n-1}(\gamma)\prod_{j=1}^{n-1}\frac{d\gamma_j}{2\pi}$$

where in explicit notations we have: $\Lambda = \sum_{j=1}^{n}\lambda_j$, $\gamma = \sum_{j=1}^{n-1}\gamma_j$ and

$$\Psi_n(x|\lambda) = \Psi_n\left(x_1,\dots,x_n|\lambda_1,\dots,\lambda_n\right)$$
$$\Psi_{n-1}(x|\gamma) = \Psi_{n-1}\left(x_1,\dots,x_{n-1}|\gamma_1,\dots,\gamma_{n-1}\right)$$

Substitution inside completeness relation of explicit iterative expressions for $\overline{\Psi_n(x|\lambda)}$ and $\Psi_n(y|\lambda)$ gives.

Completeness and Gustafson integrals

$$\int_{\mathbb{R}^n} \overline{\Psi_n(x|\lambda)}\, \Psi_n(y|\lambda)\, \mu_n(\lambda) \prod_{j=1}^{n} \frac{d\lambda_j}{2\pi} =$$

$$\int_{\mathbb{R}^{n-1}} \overline{\Psi_{n-1}(x|\alpha)}\, \Psi_{n-1}(y|\beta)\, e^{i(\alpha-\beta)x_n} \mu_{n-1}(\beta)\, \mu_{n-1}(\alpha) \prod_{j=1}^{n-1} \frac{d\alpha_j}{2\pi} \frac{d\beta_j}{2\pi}$$

$$\int_{\mathbb{R}^n} \prod_{k=1}^{n-1} \prod_{j=1}^{n} \Gamma(i\alpha_k - i\lambda_j + \varepsilon)\Gamma(i\lambda_j - i\beta_k + \varepsilon)\, e^{i\Lambda(y_n - x_n)} \mu_n(\lambda) \prod_{j=1}^{n} \frac{d\lambda_j}{2\pi} =$$

$$= \delta(x_n - y_n) \int_{\mathbb{R}^{n-1}} \overline{\Psi_{n-1}(x|\alpha)}\, \Psi_{n-1}(y|\alpha)\, \mu_{n-1}(\alpha) \prod_{j=1}^{n-1} \frac{d\alpha_j}{2\pi}$$

where we used reduced Gustafson integral in the form

$$\int_{\mathbb{R}^n} \prod_{k=1}^{n-1} \prod_{j=1}^{n} \Gamma(i\alpha_k - i\lambda_j + \varepsilon)\Gamma(i\lambda_j - i\beta_k + \varepsilon)\, e^{i\Lambda(y_n - x_n)} \mu_n(\lambda) \prod_{j=1}^{n} \frac{d\lambda_j}{2\pi} =$$

$$= \frac{(2\pi)^{n-1}}{\mu_{n-1}(\alpha)}\, \delta(x_n - y_n)\, \delta_{sym}(\alpha - \beta)$$

Instead of conclusion – naive point of view

- Gauss-Givental representation. Everything is governed by local identities or, equivalently, Yang-Baxter relation.
- Mellin-Barnes representation. There is not local relations but instead there exists Gustafson integral.
- Toda, XXX-spin chains $\leftrightarrow$ local relations + Gustafson integrals
- XXZ-modular magnets, q-deformed gl_n Toda system $\leftrightarrow$ local relations + Rains integrals

  Gus Schrader, Alexander Shapiro, On b-Whittaker functions arXiv:1806.00747

Paradox or lack of understanding?

Ruijsenaars hyperbolic system $\leftrightarrow$ no local relations and special famous integrals but everything works in a very similar way