Graded superextensions of orthosymplectic supersymmetry $\mathfrak{osp}(1|2n;\mathbb{C})$ and their real (anti-) de Sitter forms

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Abstract

It is shown how, using the Clifford algebra, it is possible to turn an ordinary commutator into an anticommutator and vice versa. This is illustrated on ordinary bosons and fermions. Next, we consider Lie superalgebras, which are superextensions of the complex Lie algebra $so(5, C)$ and its real forms: de Sitter $so(1, 4)$ and anti de Sitter $so(2, 3)$. There are two types of such superextensions: with $\mathbb{Z}_2$- and $\mathbb{Z}_2 \times \mathbb{Z}_2$- gradations. The Lie $\mathbb{Z}_2$-superalgebras are the usual Lie seperalgebras, and the Lie $\mathbb{Z}_2 \times \mathbb{Z}_2$-superalgebras were presented by the speaker in 2014. It is shown how these superalgebras with different gradations are related to each other using the Clifford algebra. For the first time, the correct concept of compact real forms is introduced for both $\mathbb{Z}_2$- and $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded superextensions of so(5;C). For this purpose, we use a cliffonic dressing with a Hermitian imaginary unit (the usual imaginary unit is an anti-Hermitian one). We consider also a superextension in the case of the superalgebra $osp(1|2n)$

Contents:

1. *Introduction*

2. Clifford algebra and cliffonic dressing

By definition the Clifford algebra $Cl_n(p, q)$ of the rank $n = p + q$, with the signature (p, q), is a real or complex unital (with a unite) associative algebra generated by the generators c_i, $(i = 1, 2, \dots, n)$, which will be called the *cliffons*, with the defining relations

$$\{c_i, c_j\} := c_i c_j + c_j c_i = \eta_{ij} e, \quad i, j = 1, \dots n, \tag{1}$$

where e is identity element, $\eta = ||\eta_{ij}|| = \mathrm{diag}(1, \dots, 1, -1, \dots, -1)$ is the diagonal $(n \times n)$-matrix with its first p entries equal to 1 and the last q entries equal to -1 on the diagonal. Sometimes we will use an involutive antiautomorphysm (or automorphism) $(^{*})$ which acts on the generating cliffons as follows: $(c_i^{*})^{*} = c_i$, $(c_i c_j \cdots c_k)^{*} = c_k^{*} \cdots c_j^{*} c_i^{*}$ (or $(c_i c_j \cdots c_k)^{*} = c_i^{*} c_j^{*} \cdots c_k^{*}$) for $i, j, \dots, k = 1, 2, \dots, n$.
The Clifford (cliffonic[1]) algebras $Cl_n(p, q)$ with the different stars are different and they have the different applications. We consider here a special $Cl_n(p, q)$-application called *cliffonic dressing*. Let $\mathfrak{g}$ and $\mathfrak{g}'$ be two different non-isomorphic (real or complex) Lie (super)algebras (or associative (super)algebras) of the same dimension. Cliffonic dressing is an exact (non-degenerate) special homomorphism $\mathfrak{g} \mapsto Cl_n(p, q) \otimes \mathfrak{g}'$ for a some clifonic algebra $Cl_n(p, q)$. In other words, we get the realization of algebra $\mathfrak{g}$ by "dressing" another equidimensional algebra $\mathfrak{g}'$ with the help of $Cl_n(p, q)$ cliffons.

[1] Below we will also use the name "cliffonic algebra" by analogy with bosonic and fermionic associative algebras.

(2a) *Cliffonic dressing for the construction of real forms*

Let's consider a cliffonic algebra $Cl_{\bar{n}} := Cl_n(0,n)$ with a negative signature $\eta = ||\eta_{ij}|| = \mathrm{diag}(-1,-1,\dots,-1)$, and with the following real cliffons: $(c_i c_j \cdots c_k)^* = c_i^* c_j^* \cdots c_k^* = c_i c_j \cdots c_k$ for $i,j,\dots,k = 1,2,\dots,n$.
In the case of a one-cliffonic algebra $Cl_{\bar{1}}$, we put $J := c_1$ and we have $J = -1$ and $J^* = J$. The element J with such properties is named the operator of a fundamental symmery in the theory of Krein spaces.
It is easy to see that the cliffonic algebras $Cl_n(n,0)$ and $Cl'_n(p,q)$ with different signatures and the same star are connected to each other by means of $Cl_{\bar{1}}$-dressing, namely,

$$
\begin{aligned}
c'_i &= ec_i, \quad i = 1,\dots,p,\\
c'_i &= Jc_i, \quad i = p+1,\dots,n,
\end{aligned}
\tag{2}
$$

or the inverse $Cl_{\bar{1}}$-dressing

$$
\begin{aligned}
c_i &= ec'_i, \quad i = 1,\dots,p,\\
c_i &= Jc'_i, \quad i = p+1,\dots,n,
\end{aligned}
\tag{3}
$$

where c_i $(i = 1,\dots,n)$ are the cliffons of $Cl_n(n,0)$ and c'_i $(i = 1,\dots,n)$ are the cliffons of $Cl'_n(p,q)$ and everywhere we use short notation $ec_i := e \otimes c_i$, $Jc_i := J \otimes c_i$. In the following sections we will use the cliffonic dressing to construct real de Sitter and anti-de Sitter superalgebras.

In this special case we will use the cliffonic algebra $Cl_n := Cl_n(n, 0)$ with a positive signature $\eta = ||\eta_{ij}|| = \mathrm{diag}(1, 1, \ldots, 1)$. We will apply the Cl_n-dressing to the well-known bosonic and fermionic associative algebras. Recall that the bosonic algebra B_Λ $(\Lambda \in \mathbb{N})$ generated by the creation b_i^+ and annihilation b_i $(i = 1, 2, \ldots, \Lambda)$ operators with the following defining relations:

$$\begin{aligned} &[b_i, b_j] = [b_i^+, b_j^+] = 0, \\ &[b_i, b_j^+] = \delta_{ij} I \end{aligned} \tag{4}$$

for $i, j = 1, 2, \ldots, \Lambda$. Analogously, the fermionic algebra F_Λ generated by the creation a_i^+ and annihilation a_i $(i = 1, 2, \ldots, \Lambda)$ operators with the following defining relations:

$$\begin{aligned} &\{a_i, a_j\} = \{a_i^+, a_j^+\} = 0, \\ &\{a_i, a_j^+\} = \delta_{ij} I \end{aligned} \tag{5}$$

for $i, j = 1, 2, \ldots, \Lambda$.

Now we applay the Cl_n-cliffonic dressing to the bosonic and fermion algebras. Let $[\lambda_1, \lambda_2, \ldots, \lambda_n]$, where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 1$, be the partition of a closed integer interval $[1, \Lambda]$ where $\Lambda = \lambda_1 + \lambda_2 + \cdots + \lambda_n$. We set for the the bosonic algebra B_Λ:

$$\begin{array}{llll} \tilde{b}_{\lambda_1 j} = c_1 b_j, & \tilde{b}^+_{\lambda_1 j} = c_1 b^+_j, & j = 1, 2, \ldots, \lambda_1, & \\ \tilde{b}_{\lambda_2 j} = c_2 b_j, & \tilde{b}^+_{\lambda_2 j} = c_2 b^+_j, & j = \lambda_1 + 1, \ldots, \lambda_1 + \lambda_2, & \\ \ldots \quad \ldots & \ldots \quad \ldots & \ldots \quad \ldots \quad \ldots & \\ \tilde{b}_{\lambda_n j} = c_n b_j, & \tilde{b}^+_{\lambda_n j} = c_n b^+_j, & j = \Lambda_{n-1} + 1, \ldots, \Lambda_{n-1} + \lambda_n, & \end{array} \tag{6}$$

and also $\tilde{I} = eI$, where $\Lambda_{n-1} := \sum_{s=1}^{n-1} \lambda_s$. It is easy to check that the new generators (6) satisfy the following relations (instead of (29)) for the given partition $[\lambda_1, \lambda_2, \ldots, \lambda_n]$:

$$\{\tilde{b}_{\lambda_r i}, \tilde{b}_{\lambda_{r'} j}\} = \{\tilde{b}^+_{\lambda_r i}, \tilde{b}^+_{\lambda_{r'} j}\} = \{\tilde{b}_{\lambda_r i}, \tilde{b}^+_{\lambda_{r'} j}\} = 0 \tag{7}$$

for $r \neq r'$, and

$$\begin{aligned} &[\tilde{b}_{\lambda_r i}, \tilde{b}_{\lambda_r j}] = [\tilde{b}^+_{\lambda_r i}, \tilde{b}^+_{\lambda_r j}] = 0, \\ &[\tilde{b}_{\lambda_r i}, \tilde{b}^+_{\lambda_r j}] = \delta_{ij} \tilde{I}, \end{aligned} \tag{8}$$

for $r = 1, 2, \ldots, n$ and $i, j = \Lambda_r + 1, \Lambda_r + 2, \ldots, \Lambda_r + \lambda_{r+1}$, where $\Lambda_r := \sum_{s=1}^{r} \lambda_s$.

In the case of a complete cliffonic cross-dressing, when $n = \Lambda$, $\Lambda_r = r$ $(r = 1, 2, \ldots, \Lambda)$, the formulas (6) take the form

$$\tilde{b}_j = c_j b_j, \quad \tilde{b}_j^+ = c_j b_j^+, \quad j = 1, 2, \ldots, \Lambda. \tag{9}$$

These cross-dressing generators satisfy the defining relations:

$$\{\tilde{b}_i, \tilde{b}_j\} = \{\tilde{b}_i^+, \tilde{b}_j^+\} = \{\tilde{b}_i, \tilde{b}_j^+\} = 0, \tag{10}$$

for $i \neq j$, $(i, j = 1, 2, \ldots, \Lambda)$, and

$$[\tilde{b}_i, \tilde{b}_i^+] = \tilde{I} \tag{11}$$

for $i = 1, 2, \ldots, \Lambda$.

In the case of a complete cliffonic cross-dressing, when $n = \Lambda$, $\Lambda_r = r$ $(r = 1, 2, \dots, \Lambda)$, the formulas (6) take the form

$$\tilde{a}_j = c_j a_j, \quad \tilde{a}_j^+ = c_j a_j^+, \quad j = 1, 2, \dots, \Lambda. \tag{12}$$

The cross-dressing generators (15) satisfy the defining relations:

$$[\tilde{a}_i, \tilde{a}_j] = [\tilde{a}_i^+, \tilde{a}_j^+] = [\tilde{a}_i, \tilde{a}_j^+] = 0 \tag{13}$$

for $i \neq j$, $(i, j = 1, 2, \dots, \Lambda)$, and

$$\begin{aligned} &\{\tilde{a}_i, \tilde{a}_i\} = \{\tilde{a}_i^+, \tilde{a}_i^+\} = 0, \\ &\{\tilde{a}_i, \tilde{a}_i^+\} = \tilde{I} \end{aligned} \tag{14}$$

for $i = 1, 2, \dots, \Lambda$.

(2c) *Cliffons and fermions.*

Take once again the standard fermionic algebra F_n generated by the creation and annihilation operators a_i^+, a_i $(i = 1, 2, \dots, n)$

$$\{a_i, a_j\} = \{a_i^+, a_j^+\} = 0, \quad \{a_i, a_j^+\} = \delta_{ij} I, \quad i = 1, 2, \dots, n. \tag{15}$$

Let us introduce the new generators:

$$c_{2i-1} := (a_i + a_i^+), \quad c_{2i} := \imath(a_i - a_i^+), \quad i = 1, 2, \dots, n. \tag{16}$$

These new generators c_i $(i = 1, 2, \dots, 2n)$ satisfy the Clifford algebra defining relations

$$\{c_i, c_j\} := c_i c_j + c_j c_i = \eta_{ij} e, \quad i, j = 1, \dots 2n, \tag{17}$$

where e is identity element, $\eta = ||\eta_{ij}|| = \mathrm{diag}(1, 1, \dots, 1)$ is the diagonal $(2n \times 2n)$-matrix.

(3a) The complex $\mathbb{Z}_2$-graded superalgebra

The complex $\mathbb{Z}_2$-graded Lie superalgebra (LSA) $\mathfrak{g}$, as a linear space over $\mathbb{C}$, is a direct sum of two graded components

$$\mathfrak{g} = \bigoplus_{a=0,1} \mathfrak{g}_a = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \tag{1}$$

with a bilinear operation (the general Lie bracket), $[\![\cdot,\cdot]\!]$, satisfying the identities:

$$\deg([\![x_a, y_b]\!]) = \deg(x_a) + \deg(x_b) = a + b \pmod 2, \tag{2}$$

$$[\![x_a, y_b]\!] = -(-1)^{ab}[\![y_b, x_a]\!], \tag{3}$$

$$[\![x_a, [\![y_b, z]\!]]\!] = [\![[\![x_a, y_b]\!], z]\!] + (-1)^{ab}[\![y_b, [\![x_a, z]\!]]\!], \tag{4}$$

where the elements x_a and y_b are homogeneous, $x_a \in \mathfrak{g}_a$, $x_b \in \mathfrak{g}_b$, and the element $z \in \mathfrak{g}$ is not necessarily homogeneous. The grading function $\deg(\cdot)$ is defined for homogeneous elements of the subspaces $\mathfrak{g}_0$ and $\mathfrak{g}_1$ modulo 2, $\deg(\mathfrak{g}_0) = 0$, $\deg(\mathfrak{g}_1) = 1$. The first identity (2) is called the grading condition, the second identity (3) is called the symmetry property and the condition (4) is the Jacobi identity. From (2) and (3) it is follows that the general Lie bracket $[\![\cdot,\cdot]\!]$ for homogeneous elements posses two value: commutator $[\cdot,\cdot]$ and anticommutator $\{\cdot,\cdot\}$. From (2) it is follows that $\mathfrak{g}_0$ is a Lie subalgebra in $\mathfrak{g}$, and $\mathfrak{g}_1$ is a $\mathfrak{g}_0$-module:

$$[\mathfrak{g}_0, \mathfrak{g}_0] \subseteq \mathfrak{g}_0, \quad [\mathfrak{g}_0, \mathfrak{g}_1] \subseteq \mathfrak{g}_1, \quad (\{\mathfrak{g}_1, \mathfrak{g}_1\} \subseteq \mathfrak{g}_0). \tag{5}$$

(3b) The complex $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded superalgebra

The complex $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded LSA $\tilde{\mathfrak{g}}$, as a linear space, is a direct sum of four graded components

$$\tilde{\mathfrak{g}} = \bigoplus_{\mathbf{a}=(a_1,a_2)} \tilde{\mathfrak{g}}_{\mathbf{a}} = \tilde{\mathfrak{g}}_{(0,0)} \oplus \tilde{\mathfrak{g}}_{(1,1)} \oplus \tilde{\mathfrak{g}}_{(1,0)} \oplus \tilde{\mathfrak{g}}_{(0,1)} \tag{6}$$

with a bilinear operation $[\![\cdot,\cdot]\!]$ satisfying the identities (grading, symmetry, Jacobi):

$$\deg([\![x_{\mathbf{a}}, y_{\mathbf{b}}]\!]) = \deg(x_{\mathbf{a}}) + \deg(x_{\mathbf{b}}) = \mathbf{a} + \mathbf{b} = (a_1 + b_1, a_2 + b_2), \tag{7}$$

$$[\![x_{\mathbf{a}}, y_{\mathbf{b}}]\!] = -(-1)^{\mathbf{ab}}[\![y_{\mathbf{b}}, x_{\mathbf{a}}]\!], \tag{8}$$

$$[\![x_{\mathbf{a}}, [\![y_{\mathbf{b}}, z]\!]]\!] = [\![[\![x_{\mathbf{a}}, y_{\mathbf{b}}]\!], z]\!] + (-1)^{\mathbf{ab}}[\![y_{\mathbf{b}}, [\![x_{\mathbf{a}}, z]\!]]\!], \tag{9}$$

where the vector $(a_1 + b_1, a_2 + b_2)$ is defined $\bmod (2,2)$ and $\mathbf{ab} = a_1 b_1 + a_2 b_2$. Here in (6)-(8) $x_{\mathbf{a}} \in \tilde{\mathfrak{g}}_{\mathbf{a}}$, $x_{\mathbf{b}} \in \tilde{\mathfrak{g}}_{\mathbf{b}}$, and the element $z \in \tilde{\mathfrak{g}}$ is not necessarily homogeneous. From (6) and (7) it is follows that the general Lie bracket $[\![\cdot,\cdot]\!]$ for homogeneous elements posses two value: commutator $[\cdot,\cdot]$ and anticommutator $\{\cdot,\cdot\}$ as well as in the previous $\mathbb{Z}_2$-case. From (7) it is follows that $\tilde{\mathfrak{g}}_{(0,0)}$ is a Lie subalgebra in $\tilde{\mathfrak{g}}$, and the subspaces $\tilde{\mathfrak{g}}_{(1,1)}$, $\tilde{\mathfrak{g}}_{(1,0)}$ and $\tilde{\mathfrak{g}}_{(0,1)}$ are $\tilde{\mathfrak{g}}_{(0,0)}$-modules:

$$[\tilde{\mathfrak{g}}_{(0,0)}, \tilde{\mathfrak{g}}_{(0,0)}] \subseteq \tilde{\mathfrak{g}}_{(0,0)}, \quad [\tilde{\mathfrak{g}}_{(0,0)}, \tilde{\mathfrak{g}}_{(a,b)}] \subseteq \tilde{\mathfrak{g}}_{(a,b)}, \tag{10}$$

It should be noted that $\tilde{\mathfrak{g}}_{(0,0)} \oplus \tilde{\mathfrak{g}}_{(1,1)}$ is a Lie subalgebra in $\tilde{\mathfrak{g}}$ and the subspace $\tilde{\mathfrak{g}}_{(1,0)} \oplus \tilde{\mathfrak{g}}_{(0,1)}$ is a $\tilde{\mathfrak{g}}_{(0,0)} \oplus \tilde{\mathfrak{g}}_{(1,1)}$-module, and moreover $\{\tilde{\mathfrak{g}}_{(1,1)}\tilde{\mathfrak{g}}_{(1,0)}\} \subset \tilde{\mathfrak{g}}_{(0,1)}$ and vice versa $\{\tilde{\mathfrak{g}}_{(1,1)}, \tilde{\mathfrak{g}}_{(0,1)}\} \subset \tilde{\mathfrak{g}}_{(1,0)}$.

Let us take the complex $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie superalgebra (LSA) $\mathfrak{g}$, as a linear space over $\mathbb{C}$:

$$\tilde{\mathfrak{g}} = \bigoplus_{\mathbf{a}=(a_1,a_2)} \tilde{\mathfrak{g}}_{\mathbf{a}} = \tilde{\mathfrak{g}}_{(0,0)} \oplus \tilde{\mathfrak{g}}_{(1,1)} \oplus \tilde{\mathfrak{g}}_{(1,0)} \oplus \tilde{\mathfrak{g}}_{(0,1)} \tag{11}$$

and apply to it the following cross-dressing

$$\begin{aligned} \tilde{\mathfrak{g}}' &= \left(e\tilde{\mathfrak{g}}_{(0,0)} \oplus c_1 c_2 \tilde{\mathfrak{g}}_{(1,1)}\right) \oplus \left(c_1 \tilde{\mathfrak{g}}_{(1,0)} \oplus c_2 \tilde{\mathfrak{g}}_{(0,1)}\right) \\ &= \tilde{\mathfrak{g}}'_0 \oplus \tilde{\mathfrak{g}}'_1. \end{aligned} \tag{12}$$

Because of the Lie symmetries AdS_5 and dS_5 are real forms of the simple complex Lie algebra $\mathfrak{so}(5;\mathbb{C}) \simeq \mathfrak{sp}(4;\mathbb{C})$ it is natural to belive that AdS_5 and dS_5 $(N=1)$ supersymmeries are the real forms of the Lie superalgebra $\mathfrak{osp}(1|4;\mathbb{C})$ that is a minimal superextension of $\mathfrak{sp}(4;\mathbb{C})$.

3. Bosonic realization of the complex $\mathbb{Z}_2$-graded superalgebra $\mathfrak{osp}(1|4;\mathbb{C})$

Let Bo_2 be the standard bosonic algebra generated by two oscillators b_i^+, b_i $(i=1,2)$:

$$[b_i, b_j] = [b_i^+, b_j^+] = 0, \quad [b_i, b_j^+] = \delta_{ij} I. \tag{13}$$

Let's put

$$X_i = \frac{1}{\sqrt{2}} b_i^+, \quad X_{-i} = \frac{1}{\sqrt{2}} b_i, \tag{14}$$

$$X_{ii} = \{X_i, X_i\} = (b_i^+)^2, \quad X_{-i-i} = \{X_{-i}, X_{-i}\} = (b_i)^2, \tag{15}$$

$$X_{i-i} = \{X_i, X_{-i}\} = \frac{1}{2}(b_i^+ b_i + b_i b_i^+), \tag{16}$$

$$X_{1-2} = \{X_1, X_{-2}\} = b_1^+ b_2, \quad X_{2-1} = \{X_2, X_{-1}\} = b_2^+ b_1, \tag{17}$$

$$X_{12} = \{X_1, X_2\} = b_1^+ b_2^+, \quad X_{-2-1} = \{X_{-2}, X_{-1}\} = b_2 b_1. \tag{18}$$

These formulas represent the well-known realization of the $\mathbb{Z}_2$-graded orthosymplectic superalgebra $\mathfrak{osp}(1|4;\mathbb{C})$ in the terms of the $\mathbb{Z}_2$-graded bosonic algebra Bo_2.
So we can say that: *A two-oscillator boson system over a complex field $\mathbb{C}$, where bosons in different oscillators commute, generates an orthosymplectic complex contragredient $\mathbb{Z}_2$-graded superalgebra $\mathfrak{osp}(1/4;\mathbb{C})$. This superalgebra is a maximally-generated finite-dimensional contragredient $\mathbb{Z}_2$-graded Lie superalgebra for a given two-oscillator boson system, and the all graded Lie superalgebra Bo_2 realizes a universally enveloping superalgebra $U(\mathfrak{osp}(1/4;\mathbb{C}))$.*

The corresponding cross-dressing bosonic algebra $\tilde{Bo}_2$ is generated by the following elements $\tilde{b}_i^+$, $\tilde{b}_i$ $(i=1,2)$:

$$\{\tilde{b}_1,\tilde{b}_2\}=\{\tilde{b}_1^+,\tilde{b}_2^+\}=0,\quad [\tilde{b}_i,\tilde{b}_i^+]=I\quad (i=1,2). \tag{19}$$

Let's put

$$\tilde{X}_i=\frac{1}{\sqrt{2}}\tilde{b}_i^+,\quad \tilde{X}_{-i}=\frac{1}{\sqrt{2}}\tilde{b}_i, \tag{20}$$

$$\tilde{X}_{ii}=\{\tilde{X}_i,\tilde{X}_i\}=(\tilde{b}_i^+)^2,\quad \tilde{X}_{-i-i}=\{\tilde{X}_{-i},\tilde{X}_{-i}\}=(\tilde{b}_i)^2, \tag{21}$$

$$\tilde{X}_{i-i}=\{\tilde{X}_i,\tilde{X}_{-i}\}=\frac{1}{2}(\tilde{b}_i^+\tilde{b}_i+\tilde{b}_i\tilde{b}_i^+), \tag{22}$$

$$\tilde{X}_{1-2}=[\tilde{X}_1,\tilde{X}_{-2}]=\tilde{b}_1^+\tilde{b}_2,\quad \tilde{X}_{2-1}=[\tilde{X}_2,\tilde{X}_{-1}]=\tilde{b}_2^+\tilde{b}_1, \tag{23}$$

$$\tilde{X}_{12}=[\tilde{X}_1,\tilde{X}_2]=\tilde{b}_1^+\tilde{b}_2^+,\quad \tilde{X}_{-2-1}=[\tilde{X}_{-2},\tilde{X}_{-1}]=\tilde{b}_2\tilde{b}_1. \tag{24}$$

These formulas represent the realization of the $\mathbb{Z}_2\times\mathbb{Z}_2$-graded orthosymplectic superalgebra $\tilde{\mathfrak{osp}}(1|2,2;\mathbb{C})$ in the terms of the cross-dressing bosonic algebra $\tilde{Bo}_2$. So we can say that: *A two-oscillator boson system over a complex field $\mathbb{C}$, where bosons in different oscillators anticommute, generates a complex contragredient orthosymplectic $\mathbb{Z}_2\times\mathbb{Z}_2$-graded superalgebra $\tilde{\mathfrak{osp}}(1|2,2;\mathbb{C})$. This superalgebra is a maximally-generated finite-dimensional contragredient $\mathbb{Z}_2\times\mathbb{Z}_2$-graded Lie superalgebra for a given two-oscillator boson system, and the total graded Lie superalgebra $\tilde{Bo}_2$ realizes a universally enveloping superalgebra $U(\tilde{\mathfrak{osp}}(1|2,2;\mathbb{C}))$.*

4a. Clifford algebra of quaternionic type

Let us to consider the Clifford algebra of quaternionic type $C_{\bar{2}}$. It generates by the cliffonnic elements $\{e,\, c_1,\, c_2, c_1c_2\}$. We use the standard quatenionic notations:

$$i = c_1, \quad j = c_2, \quad k = c_1c_2 = ij, \tag{25}$$

These generators satisfy the relations:

$$i^2 = j^2 = k^2 = -1. \tag{26}$$

There are two type of quaternion involutions: anti-Hermitian $(^{+})$ and Hermitian $(^{\star})$:

$$i^{+}_{-H} = -i_{-H}, \quad j^{+}_{-H} = -j_{-H}, \quad k^{+}_{-H} = -k_{-H} \tag{27}$$

$$i^{\star}_{H} = i_{H}, \quad j^{\star}_{H} = j_{H}, \quad k^{\star}_{H} = k_{-H} \tag{28}$$

4b. Real forms of the boson algebra

$$[b, b^{+}] = I, \tag{29}$$

$$(b)^{+} = b^{+}, \quad (b^{+})^{+} = b \quad (non-compact \quad form), \tag{30}$$

$$(b)^{\star} = i_H b^{+}, \quad (b^{+})^{\star} = -i_H b \quad (compact \quad form) \tag{31}$$

for Hermitian i_H:

$$i_H^{\star} = i_H, \quad i_H^2 = -1. \tag{32}$$

4c. Real forms of the orthosymplectic supersymmetry $\mathfrak{osp}(1|2;\mathbb{C})$

Let's put

$$X_{+}=\frac{1}{\sqrt{2}}b^{+},\quad X_{-}=\frac{1}{\sqrt{2}}b, \tag{33}$$

$$X_{++}=\{X_{+},X_{+}\}=(b^{+})^{2},\quad X_{--}=\{X_{-},X_{-}\}=(b)^{2}, \tag{34}$$

$$X_{+-}=\{X_{+},X_{-}\}=\frac{1}{2}(b^{+}b+bb^{+}). \tag{35}$$

This is the bosonic realization of the superalgebra $\mathfrak{osp}(1|2;\mathbb{C})$.
It is not difficult to verify that the operation $(^{+})$ is involutive on the superalgebra $\mathfrak{osp}(1|2;\mathbb{C})$ and it gives the non-compact real form $\mathfrak{osp}(1|\mathfrak{sp}(2;\mathbb{R}))$:

$$X_{+}^{+}=X_{-},\quad X_{-}^{+}=X_{+}, \tag{36}$$

$$X_{++}^{+}=X_{--},\quad X_{--}^{+}=X_{++},\quad X_{+-}^{+}=X_{+-}. \tag{37}$$

Similarly, in the case of evolution $(^{\star})$ we obtain a compact real form $\mathfrak{osp}(1|\mathfrak{sp}(2))$:

$$X_{+}^{\star}=-i_{H}X_{-},\quad X_{-}^{\star}=i_{H}X_{+}, \tag{38}$$

$$X_{++}^{\star}=-X_{--},\quad X_{--}^{\star}=-X_{++},\quad X_{+-}^{\star}=X_{+-}. \tag{39}$$

4d. Real forms of the orthosymplectic supersymmetry $\mathfrak{osp}(1|4;\mathbb{C})$

$$\mathfrak{osp}(1|4;\mathbb{C}) = \mathfrak{osp}_1(1|2;\mathbb{C}) \oplus \mathcal{P}_{12} \oplus \mathfrak{osp}_2(1|2;\mathbb{C}), \tag{40}$$

where the linear envelope

$$\mathcal{P}_{12} = \mathrm{Lin}\{X_{12}, X_{-2-1}, X_{1-2}, X_{2-1}\} \tag{41}$$

is called the complex space of a curved four-momentum.
The orthosymplectic $\mathbb{Z}_2$-graded supersubalgebras $\mathfrak{osp}_i(1|2;\mathbb{C})$ $(i = 1, 2)$ are generated by the bases (33)-(35).

Using the standard contraction procedure: $L_{\mu 4} = R\, P_\mu$ $(\mu = 0, 1, 2, 3)$, $Q_k \to \sqrt{R}\, Q_k$ and $\bar{Q}_k \to \sqrt{R}\, \bar{Q}_k$ $(k = 1, 2)$ for $R \to \infty$ we obtain the super-Poincarè algebra (standard and alternative) which is generated by $L_{\mu\nu}$, P_μ, Q_α, $Q_{\dot\alpha}$ where $\mu, \nu = 0, 1, 2, 3$; $\alpha, \dot\alpha = 1, 2$ with the relations (we write down only those which are changed in the standard and alternative Poincarè SUSY.

(I) *for the standard Poincarè SUSY:*

$$[P_\mu, Q_\alpha] \;=\; [P_\mu, \bar{Q}_{\dot\alpha}] = 0\;, \quad \{Q_\alpha, \bar{Q}_{\dot\alpha}\} \;=\; 2\sigma^{\mu}_{\alpha\beta} P_\mu, \tag{42}$$

(II) *for the altrenative Poincarè SUSY:*

$$\{P_\mu, Q_\alpha\} \;=\; \{P_\mu, \bar{Q}_{\dot\alpha}\} = 0\;, \quad [Q_\alpha, \bar{Q}_{\dot\alpha}] \;=\; 2\sigma^{\mu}_{\alpha\beta} P_\mu. \tag{43}$$

Let us consider the supergroups associated to the $\mathbb{Z}_2$- and $\mathbb{Z}_2\times\mathbb{Z}_2$-graded Poincarè superalgebras. A group element g is given by the exponential of the super-Poincarè generators, namely

$$g(x^\mu,\omega^{\mu\nu},\theta^\alpha,\bar\theta^{\dot\alpha}) \;=\; \exp(x^\mu P_\mu+\omega^{\mu\nu}M_{\mu\nu}+\theta^\alpha Q_\alpha+\bar Q_{\dot\alpha}\bar\theta^{\dot\alpha}). \tag{1}$$

Because the grading of the exponent is zero (0 or (00)) we have the following.
1). $\mathbb{Z}_2$-case: $\deg P=\deg x=0$, $\deg Q=\deg\bar Q=\deg\theta=\deg\bar\theta=1$. This means that

$$[x_\mu,\theta_\alpha] \;=\; [x_\mu,\bar\theta_{\dot\alpha}] \;=\; \{\theta_\alpha,\bar\theta_{\dot\beta}\} \;=\; \{\theta_\alpha,\theta_\beta\} \;=\; \{\bar\theta_{\dot\alpha},\bar\theta_{\dot\beta}\} \;=\; 0. \tag{2}$$

2). $\mathbb{Z}_2\times\mathbb{Z}_2$-case: $\deg P=\deg x=(11)$, $\deg Q=\deg\theta=(10)$, $\deg\bar Q=\deg\bar\theta=(01)$. This means that

$$\{x_\mu,\theta_\alpha\} \;=\; \{x_\mu,\bar\theta_{\dot\alpha}\} \;=\; [\theta_\alpha,\bar\theta_{\dot\beta}] \;=\; \{\theta_\alpha,\theta_\beta\} \;=\; \{\bar\theta_{\dot\alpha},\bar\theta_{\dot\beta}\} \;=\; 0. \tag{3}$$

One defines the superspaces as the coset spaces of the standard alternative super-Poincarè groups by the Lorentz subgroup, parameterized the coordinates x^μ, θ^α, $\bar\theta^{\dot\alpha}$, subject to the condition $\bar\theta^{\dot\alpha}=(\theta^\alpha)^*$.
We can define a superfield $\mathcal{F}$ as a function of superspace.

In the theory of supergravity (SUGRA), there has been the following unsolved problem for about 30 years. All physical reasonable solutions of SURGA models with cosmological constants Λ have been constructed for the case $\Lambda < 0$, i.e. for the anti-de Sitter metric

$$g_{ab} = \mathrm{diag}\,(1, -1, -1, -1, 1), \quad (a, b = 0, 1, 2, 3, 4) \tag{4}$$

with the space-time symmetry $\mathfrak{o}(3, 2)$. In the case $\Lambda > 0$, i.e. for the Sitter metric

$$g_{ab} = \mathrm{diag}\,(1, -1, -1, -1, -1), \quad (a, b = 0, 1, 2, 3, 4) \tag{5}$$

with the space-time symmetry $\mathfrak{o}(4, 1)$ no reasonable solutions have been find. For example, in SUGRA it was obtained the following relation

$$\Lambda = -3m^2, \tag{6}$$

where m is the massive parameter of gravitinos.
In my opinion these problems for the case $\Lambda > 0$ are connected with superextensions of anti-de Sitter $\mathfrak{o}(3, 2)$ and de Sitter $\mathfrak{o}(4, 1)$ symmetries. The symmetry $\mathfrak{o}(3, 2)$ has the superextension - the superalgebra $\mathfrak{osp}(3, 2|1)$. This is the usual $\mathbb{Z}_2$-graded superalgebra. In the case of $\mathfrak{o}(4, 1)$ such superextension does not exist. However the Lie algebra $\mathfrak{o}(4, 1)$ has another *alternative* superextension that is based on the $\mathbb{Z}_2 \times \mathbb{Z}_2$-grading and a preliminary analysis shows that we can construct the reasonable SUGRA models for the case $\Lambda > 0$.

Let us return to SUGRA formula which connects the cosmological constant Λ with the mass of gravitinos m

$$\Lambda = -3m^2. \tag{7}$$

We can rewrite the formula in terms of the time-component of the four-momenta, P_0:

$$\Lambda = -3P_0^2 < 0, \tag{8}$$

This formula is valid for $\mathbb{Z}_2$-graded case and we belive tbat it is valid for the $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded case also, that is

$$\Lambda = -3\tilde{P}_0^2 > O. \tag{9}$$

where $\tilde{P}_0$ is the time-component of the four-momenta for the $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded case. Because

$$\tilde{P}_0 = c_1 c_2 P_0 = c_1 c_2 m, \tag{10}$$

where c_1, c_2 are the real cliffons: $c_1^* = c_1$, $c_2^* = c_2$, $(c_1 c_2)^* = c_1 c_2$, $(c_1 c_2)^2 = -1$, therefore

$$0 < \Lambda = -3\tilde{P}_0^2 = 3m^2. \tag{11}$$

THANK YOU FOR YOUR ATTENTION