SPECTRUM AND POINT SYMMETRIES OF N IDENTICAL 1D-OSCILLATORS

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OUTLINE

- The statement of the problem
- New symmetrized coordinates
- Symmetric and antisymmetric basis functions
- Close-coupling equations
- The metastable states with complex values of energy
- Resume

The statement of the problem

The Schrödinger equation for the problem of penetration of A identical spinless quantum particles in Oscillator units

$$\left[-\sum_{i=1}^{A} \frac{\partial^2}{\partial x_i^2} + \sum_{j=2}^{A} \sum_{i=1}^{j-1} \frac{1}{A}(x_{ij})^2 + \sum_{i=1}^{A} V(x_i) - E \right] \Psi(x_1, ..., x_A; E) = 0.$$
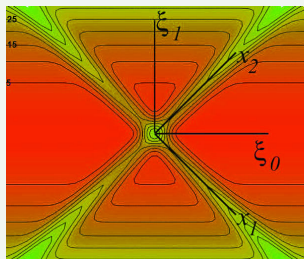
The problem under consideration is to find the solutions of SE that are totally symmetric (or antisymmetric) with respect to the permutations of A particles, i.e. the permutations of coordinates $x_i \leftrightarrow x_j$ at $i, j = 1, ..., A$, or symmetry operations of permutation group S_n.

Barrier potential in configuration space $A = 2$
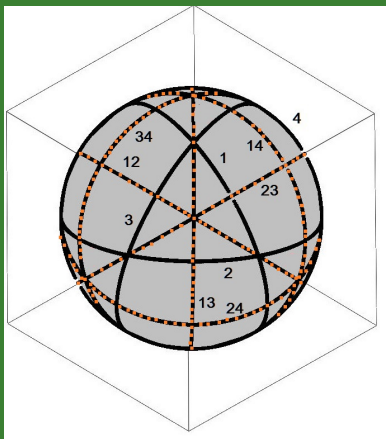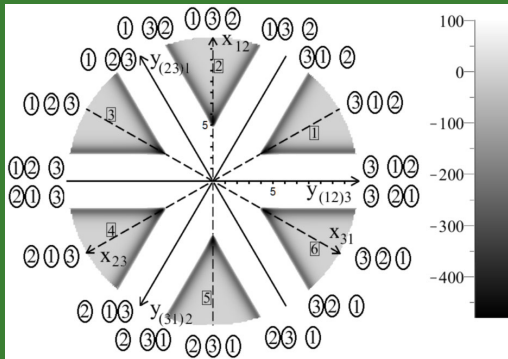
Example of interaction potential in center-of-mass plane at $A = 3$.
Intersections in $\mathbb{R}^4$ of the coordinate spaces $\mathbb{R}^3$ (labelled 1, 2, 3, 4) and the spaces $\mathbb{R}^3$ of pair collisions (labelled 12, etc.) with the sphere S^2 in the center-of-mass space $\mathbb{R}^3$.

Solution of a Three-Body Problem in One Dimension

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The problem of three equal particles interacting pairwise by inversecube forces ("centrifugal potential") in addition to linear forces ("harmonical potential") is solved in one dimension.

1. INTRODUCTION

It has been known for some time that the one-dimensional three-body problem with linear ("harmonical") and inverse-cube ("centrifugal") pair forces is separable,[1,2] but apparently there has been no attempt at its solution. In this paper this problem is solved in the case of equal particles: The complete energy spectrum is determined, and all the corresponding eigenfunctions are explicitly written out.

The particles may satisfy Boltzmann, Bose, or Fermi statistics; in fact, the nature of the problem is such that the type of statistics does not modify the energy spectrum and affects the wavefunctions only in a trivial way. The problem which obtains from that described above eliminating the inverse-cube force between two pairs (so that it acts only between one pair) is also solved.

In Sec. 2 we discuss the two-body problem with the same "oscillator plus centrifugal" forces. This treatment is useful both as a preliminary to the solution and as a model for the interpretation of the three-body problem, which is discussed in Sec. 3. The last section contains comments on possible

of-mass (CM) and relative coordinates,[4]

$$R^{(2)} = \tfrac{1}{2}(x_1 + x_2), \qquad (2.2a)$$
$$x = 2^{-\frac{1}{2}}(x_1 - x_2), \qquad (2.2b)$$

and eliminating the center-of-mass motion, we get

$$\left[-\frac{d^2}{dx^2} + \tfrac{1}{4}\omega^2 x^2 + \tfrac{1}{2}gx^{-2} - E \right]\psi = 0, \quad (2.3)$$

where now E is the energy in the CM system.

The physically acceptable solutions of this equation (in the interval $0 \le x < \infty$) are

$$\psi_n(x) = x^{a+\frac{1}{2}} \exp\left(-\tfrac{1}{4}\omega x^2\right)L_n^a(\tfrac{1}{2}\omega x^2), \quad n = 0, 1, 2, \cdots, \qquad (2.4)$$

with

$$a = \tfrac{1}{2}(1 + 2g)^{\frac{1}{2}}. \qquad (2.5)$$

Here L_n^a is a generalized Laguerre polynomial, defined as in Ref. 5. By changing the sign of a, namely, by taking the negative determination of the square root in Eq. (2.5), one would still obtain a solution of the Schrödinger equation, but it would not be an acceptable one owing to its behavior at $x = 0$.[6]

The corresponding energy levels are

SET OF CO-ORDINATE SYSTEMS WHICH DIAGONALIZE THE KINETIC ENERGY OF RELATIVE MOTION*

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A simple scheme is given whereby one can write down any one of a large number of possible sets of co-ordinates, to use in an N-particle problem, which have the property of expressing the relative kinetic energy of the system in diagonal form. This gives a Schrödinger equation without cross-derivatives. These sets of co-ordinates can be visualized in terms of certain "mobile" models. It is easy to construct a "mobile" which leads to a co-ordinate set appropriate to a particular physical problem.

* As E. P. Wigner has pointed out to us in private correspondence, if the masses of the four particles are equal, then there is a co-ordinate system which treats each of the four particles in the same manner:

$$(Q_1)_w = \frac{\sqrt{m}}{2} [r_1 + r_2 - r_3 - r_4],$$

$$(Q_2)_w = \frac{\sqrt{m}}{2} [r_1 - r_2 + r_3 - r_4],$$

$$(Q_3)_w = \frac{\sqrt{m}}{2} [r_1 - r_2 - r_3 + r_4],$$

$$(Q_4)_w = \frac{\sqrt{m}}{2} [r_1 + r_2 + r_3 + r_4].$$

Comparing with our co-ordinates of Fig. 4, b,

$$(Q_1)_w = Q_3,$$
$$(Q_2)_w = -(Q_1 + Q_2)/\sqrt{2},$$
$$(Q_3)_w = (Q_2 - Q_1)/\sqrt{2},$$
$$(Q_4)_w = Q_4.$$

There are similar relations between the Wigner co-ordinates and our co-ordinates of Fig. 4, d. Wigner states that if the number of particles is a power of 2 and the masses are equal, there is a co-ordinate system of high symmetry in which all particles are treated equally. Unfortunately the form of the potential energy in the collision of diatomic molecules would make these Wigner co-ordinates inconvenient.

GROUP THEORY OF HARMONIC OSCILLATORS

(III). States with Permutational Symmetry

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Abstract: This article continues the analysis of the problem of n particles in a common harmonic oscillator potential that was initiated in two previous papers under the same general title. The first objective of the paper is to give an analytic procedure for the explicit construction of the states in the $U_{3n} \supset \mathscr{R}_3 \times U_n$, $\mathscr{R}_3 \supset \mathscr{R}_2 \supset \mathscr{R}_1$, $U_n \supset U_{n-1} \supset \ldots \supset U_1$ chain of subgroups, where the $3n$ dimensional unitary group U_{3n} is the symmetry group of the Hamiltonian while $\mathscr{R}_3$ is the symmetry group of the harmonic oscillator, $\mathscr{R}_3$ is the ordinary rotation group, and U_n is the unitary group in n dimensions associated with the particle indices. The second and main objective of this paper is to construct states with definite permutational symmetry. After taking out the centre-of-mass motion the states given in terms of $n-1$ relative Jacobi vectors will be a basis for irreducible representations of the unitary group U_{n-1} and its orthogonal subgroup O_{n-1}. The characterization of the states is completed with the help of the irreducible representations of the symmetric group S_n, which, through its representations $D^{[\alpha], 1}(S_n)$, is a subgroup of O_{n-1}. This implies that the states transform irreducibly under the groups in the chain $U_n \supset U_{n-1} \supset O_{n-1} \supset S_n$ rather than under those in the chain $U_n \supset U_{n-1} \supset \ldots \supset U_1$. The states classified in this way contain as particular cases, those of both the shell and the cluster model. Explicit expressions are given for two, three and four particles.

referred to the system $\dot{S} = (\dot{e}_1 \dot{e}_2 \dot{e}_3)$ of coordinates. We shall use rather the system $\ddot{S} = (\ddot{e}_1 \ddot{e}_2 \ddot{e}_3)$ which, in turn, corresponds to relative coordinates

$$\ddot{\eta}^1 = \tfrac{1}{2}(\eta^1 + \eta^3 - \eta^2 - \eta^4),$$
$$\ddot{\eta}^2 = \tfrac{1}{2}(\eta^1 + \eta^4 - \eta^2 - \eta^3),$$
$$\ddot{\eta}^3 = \tfrac{1}{2}(\eta^1 + \eta^2 - \eta^3 - \eta^4). \tag{6.9}$$

NUCLEI
Theory

Resonant Tunneling of a Few-Body Cluster Through Repulsive Barriers*

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L. L. Hai[1], **V. L. Derbov**[3], **A. Góźdź**[4], **and P. M. Krassovitskiy**[5]

Abstract—A model for quantum tunnelling of a cluster comprised of A identical particles, interacting via oscillator-type potential, through short-range repulsive barrier potentials is introduced for the first time in symmetrized-coordinate representation and numerically studied in the s-wave approximation. A constructive method for symmetrizing or antisymmetrizing the $(A-1)$-dimensional harmonic oscillator basis functions in the new symmetrized coordinates with respect to permutations of coordinates of A identical particles is described. The effect of quantum transparency, manifesting itself in nonmonotonic resonance-type dependence of the transmission coefficient upon the energy of the particles, their number $A = 2, 3, 4$ and the type of their symmetry, is analyzed. It is shown that the total transmission coefficient demonstrates the resonance behavior due to the existence of barrier quasi-stationary states, embedded in the continuum.

1. INTRODUCTION

During a decade the mechanism of quantum penetration of two bound particles through repulsive barriers, manifested in [1], attracts attention from both theoretical and experimental viewpoints in relation with such problems as near-surface quantum diffusion of molecules [2–5], fragmentation in producing very neutron-rich light nuclei [6–9], and heavy-ion collisions through multidimensional barriers [10–16]. In a general formulation of the scattering problem for ions having different masses a benchmark model

interaction, through a repulsive potential barrier. We assume that the spin part of the wave function is known, so that only the spatial part of the wave function is to be considered, which may be symmetric or antisymmetric with respect to a permutation of A identical particles [21–24]. The initial problem is reduced to penetration of a composite system with the internal degrees of freedom, describing an $(A-1) \times d$-dimensional oscillator, and the external degrees of freedom, describing the center-of-mass motion of A particles in d-dimensional Euclidean space. For simplicity, we restrict our consideration to the

so that Eq. (1) takes the form

$$\left[-\frac{\partial^2}{\partial y_0^2} + \sum_{i=1}^{A-1} \left(-\frac{\partial^2}{\partial y_i^2} + (y_i)^2 \right) \right.$$

$$\left. + U(y_0, \ldots, y_{A-1}) - E \right] \Psi(y_0, \ldots, y_{A-1}) = 0,$$

$$U(y_0, \ldots, y_{A-1})$$

$$= \sum_{i,j=1; i<j}^{A} U^{\mathrm{pair}}(x_{ij}(y_1, \ldots, y_{A-1}))$$

$$+ \sum_{i=1}^{A} V(x_i(y_0, \ldots, y_{A-1})),$$

which, as follows from Eq. (2), is *not invariant* with respect to permutations $y_i \leftrightarrow y_j$ at $i, j = 1, \ldots, A-1$.

The construction of desirable solutions of Eq. (1) in the form of linear combinations of the solutions of Eq. (3), totally symmetric (antisymmetric) with respect to permutations of coordinates $x_i \leftrightarrow x_j$ (at $i, j = 1, \ldots, A$) of A identical particles is implemented using various special procedures (see, e.g., [26–35]).

Symmetrized Coordinates

As will be shown below, a simple and clear way to construct the states keeping the symmetry (antisymmetry) under the permutations of A initial Cartesian coordinates, which we refer as S (A) states, is to use the symmetrized relative coordinates rather than the Jacobi coordinates.

The transformation from the Cartesian coordinates to one of the possible choices of symmetrized ones ξ_i has the form

$$\xi_0 = \frac{1}{\sqrt{A}} \left(\sum_{t=1}^{A} x_t \right), \qquad (3)$$

$$\xi_s = \frac{1}{\sqrt{A}} \left(x_1 + \sum_{t=2}^{A} a_0 x_t + \sqrt{A} x_{s+1} \right),$$
$$s = 1, \ldots, A-1,$$

$$x_1 = \frac{1}{\sqrt{A}} \left(\sum_{t=0}^{A-1} \xi_t \right),$$

$$x_s = \frac{1}{\sqrt{A}} \left(\xi_0 + \sum_{t=1}^{A-1} a_0 \xi_t + \sqrt{A} \xi_{s-1} \right),$$
$$s = 2, \ldots, A,$$

or, in the matrix form,

$$\begin{pmatrix} \xi_0 \\ \xi_1 \\ \xi_2 \\ \vdots \\ \xi_{A-2} \\ \xi_{A-1} \end{pmatrix} = C \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{A-1} \\ x_A \end{pmatrix}, \qquad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{A-1} \\ x_A \end{pmatrix} = C^{-1} \begin{pmatrix} \xi_0 \\ \xi_1 \\ \xi_2 \\ \vdots \\ \xi_{A-2} \\ \xi_{A-1} \end{pmatrix},$$

$$C = \frac{1}{\sqrt{A}} \begin{pmatrix} 1 & 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & a_1 & a_0 & a_0 & \cdots & a_0 & a_0 \\ 1 & a_0 & a_1 & a_0 & \cdots & a_0 & a_0 \\ 1 & a_0 & a_0 & a_1 & \cdots & a_0 & a_0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & a_0 & a_0 & a_0 & \cdots & a_1 & a_0 \\ 1 & a_0 & a_0 & a_0 & \cdots & a_0 & a_1 \end{pmatrix},$$

where $a_0 = 1/(1 - \sqrt{A}) < 0$, $a_1 = a_0 + \sqrt{A}$. The inverse coordinate transformation is performed using the same matrix $C^{-1} = C$, $C^2 = I$, i.e. $C = C^T$ is a symmetric orthogonal matrix with the eigenvalues $\lambda_1 = -1$, $\lambda_2 = 1$, ..., $\lambda_A = 1$ and $\det C = -1$. At
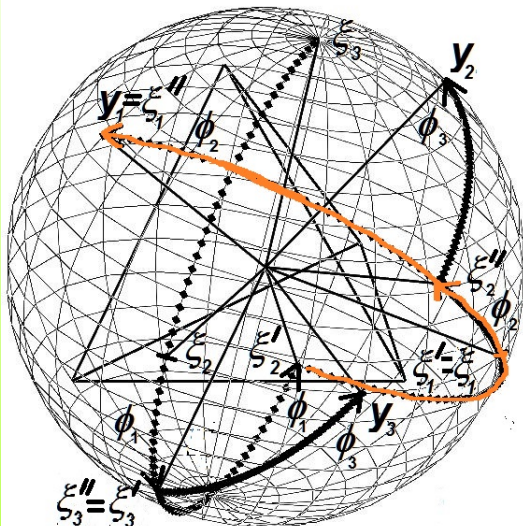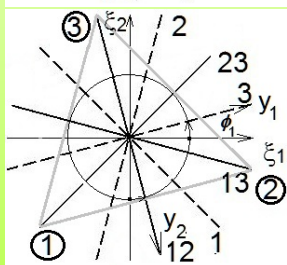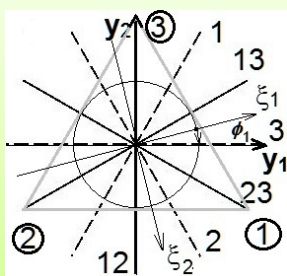
The two systems of coordinates, (y_1, y_2) and (ξ_1, ξ_2), are related via via counterclockwise rotation by the angle $\varphi = \pi/12$.

The two systems of coordinates, (y_1, y_2, y_3) and (ξ_1, ξ_2, ξ_3), are related via via three counterclockwise rotations by the angles $\varphi_1 = 3\pi/4$, $\varphi_2 = \pi - \arctan\sqrt{2}$, and $\varphi_3 = \pi/3$.

Interacting electrons in a magnetic field in a center-of-mass free basis*

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Abstract
We present an extension of the spin-adapted configuration-interaction method (SACI) for the computation of four electrons in a quasi two-dimensional quantum dot. By a group-theoretical decomposition of the basis set and working with relative and center-of-mass (cm) coordinates we obtain an analytical identification of all spurious cm states of the Coulomb-interacting electrons. We find a substantial reduction in the basis set used for numerical computations. At the same time we increase the accuracy compared to the standard SACI due to the absence of distortions caused by an unbalanced cut-off of cm excitations.

Keywords: few-electron quantum dot, variational principle, block-diagonal basis sets

find: all class operators equation (A.2) have diagonal representations, see table A.2. This proves:

Prop. All basis states of the irreps of the subgroup D_{2d} in the chosen representations are basis states of irreducible representations of the bigger group $S(4)$.

This remarkable result allows in table A.3 , up to certain ambiguities, to almost avoid the use and projection with Young operators for the bigger group $S(4)$. In tables A.4, A.5 we use it in relation with the full scheme of groups including $SU(3) > O(3, R)$ and subgroups to assign orbital symmetry to the oscillator states.

Of two states separated by |, one and only one can belong to the listed tableau. The states [4], [1⁴] are identified as eigenstates under the transposition $T(2, 3)$ with eigenvalue ± 1 respectively. If a state is not reproduced under $T(2, 3)$, it necessarily belongs to $f = [22]$ and spin $S = 0$. We conclude that the states equation (A.1) yield all the bases of the orbital Young tableaus.

Appendix B. Symmetrized relative coordinates for $n > 4$ electrons and their permutations

The efficiency of the tetrahedral coordinates raises the question if similar relative coordinates exist for $n > 4$. As a generalization of the tetrahedral coordinates from [13], new symmetrized coordinates for n particles were proposed by Gusev et al [8]. The matrix that gives the n new coordinates $(\eta_0, \eta_1, ..., \eta_{n-1})$ in terms of the old ones $(x_1, x_2 ..., x_n)$ reads

$$C = \frac{1}{\sqrt{n}} \begin{bmatrix} 1 & 1 & 1 & ... & 1 & 1 \\ 1 & b & a & a & ... & a & a \\ 1 & a & b & a & ... & a & a \\ \cdot & \cdot & \cdot & \cdot & ... & \cdot & \cdot \\ 1 & a & a & a & ... & b & a \\ 1 & a & a & a & ... & a & b \end{bmatrix}.$$

$$C = C^T, \ C^{-1} = C, \ C^2 = I,$$
$$a = \left[1 - \sqrt{n}\right]^{-1}, \ b = a + \sqrt{n}. \tag{B.1}$$

The cm coordinate is included as η_0. We shall explore the properties of these coordinates under the action of

Exact Eigenfunctions of N-body System with Quadratic Pair Potential[*]

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Abstract   We obtain the energy spectrum and all the corresponding eigenfunctions of N-body Bose and Fermi systems with Quadratic Pair Potentials in one dimension. The original first excited state or energy level is disappeared in one dimension, which results from the operation of symmetry or antisymmetry of identical particles. In two and higher dimensions, we give the energy spectrum and the analytical ground state wave functions and the degree of degeneracy. By comparison, we refine Avinash Khare's results by making some items in his article precisely.

1 Introduction

Operating on quantum many-body systems provides a way to understand the world, so hunting an exactly solved quantum many-body model becomes more important, especially for pair interaction models. Over a period of more than seventy years, there has been much successes in dealing with quantum many-body problems, but only a few models can be solved exactly. Calogero–Sutherland (CS) model[1−5] is one celebrated example of solvable many-body problems. It has been wide applications in quantum chaos and fractional statistics.[6] Up to now, Calogero–Sutherland model is still not a completely solved problem. Calogero[1,7] and Khare[8−9] made a big step forward that they gave partial exact solutions of the Calogero–Sutherland model, such as the Boson and Fermion ground state and radial excitations over it.

As a special case of CS model, one-dimensional system with quadratic pair potential was studied by Post,[10] who obtained the ground state energy. Levy–Leblond[5] obtained the energy spectrum in one dimension and the ground state energy in three dimension. Ma[11] considered three particle system and he pointed out that some states were disappeared by antisymmetric operations. In general, one-dimensional energy spectrum and three-dimensional ground state energy and partial eigenfunctions had been achieved the main results and methods to solve this system can be referred to Refs. [12−15]. As we know, exact eigenfunctions have not been given and eigenfunctions are important for us to compute many kinds of correlations, which motivate us to restudy this

ground-state eigenfunctions and the degree of degeneracy. Especially in one dimension, we obtain all the excited-state eigenfunctions.

In history, quadratic pair potential model is not only an interesting many-body model, but has a significant relationship with shell model of nuclei[10,16] and baryons in the quark model[17] as well. They use quadratic pair potential model as a phenomenological model to explain some phenomena of nuclei and quark.

In this article, we will first exhibit the exact eigenvalues and the corresponding eigenfunctions of N-body Bose and Fermi systems with quadratic pair potentials in one dimension. Second, we will give our findings of Fermi system with quadratic pair potentials in two and higher dimensions. At last, our results will be compared with the existing results of Khare.[9]

2 N-Body System with Quadratic Pair Potentials in One Dimension

The Hamiltonian of one-dimensional N-body problems with quadratic pair potentials is

$$H = \sum_{i=1}^{N} -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x_i^2} + \sum_{i<j}^{N}\frac{1}{2}m\omega^2(x_i - x_j)^2. \quad (1)$$

With the Jacobian coordinates

$$\xi_i = \begin{cases} \dfrac{1}{i}\sum_{j=1}^{i} x_j - x_{i+1}, & 1 \leqslant i \leqslant N-1, \\ \dfrac{1}{N}(x_1 + x_2 + \cdots + x_n), & i = N, \end{cases} \quad (2)$$

where ξ_i is the distance between the center of mass of the

functions of N-particle Fermi system could be obtained

$$\psi_k^F = C_k \prod_{i>j}^N x_{ij} \exp\left(-\frac{m\omega}{2\sqrt{N}h}\sum_{i>j}^N x_{ij}^2\right) \sum_{i=0}^{(k+\epsilon(k))/2} \left[\sum_{j=0}^{2i+1-\epsilon(k)} \left(\frac{4N\sqrt{N}m\omega}{(N-1)h}\right)^i \frac{(-1)^{i+j}\sigma_1^j V_{2i-j+1-\epsilon(k)}}{((k+\epsilon(k))/2-i)!j!(N+2i-j-\epsilon(k))!N^j}\right], (15)$$

where σ_i is the elementary symmetric polynomial. For N-variable polynomials,

$$\sigma_i = \begin{cases} 1, & i=0, \\ \sum_{j_1<j_2<\cdots<j_i}^N x_{j_1}x_{j_2}\cdots x_{j_i}, & 1\leqslant i\leqslant N, \\ 0, & \text{else}, \end{cases} (16)$$

$$V_i = \begin{cases} 1, & i=0, \\ \begin{vmatrix} \sigma_1 & \sigma_2 & \sigma_3 & \cdots & \sigma_i \\ \sigma_0 & \sigma_1 & \sigma_2 & \cdots & \sigma_{i-1} \\ 0 & \sigma_0 & \sigma_1 & \cdots & \sigma_{i-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \sigma_1 \end{vmatrix}, & i\neq 0, \end{cases} (17)$$

$$\epsilon(k) = \begin{cases} 0, & k \text{ is even}, \\ 1, & k \text{ is odd}. \end{cases} (18)$$

The Fermi exchange anti-symmetry is reflected in the factor $\prod_{i<j}^N x_{ij}$, and it is lucky for us to see this factor appears in all eigenfunctions. The rest are all functions of the elementary exchange symmetric polynomials.

From (iii), we know that $\{1,2,3,\ldots,N-2,N+i\}$ is one kind of combinations standing for the i-th excited states of N-particle system. The exact eigenvalues corresponding to ψ_0 and ψ_k of the Fermi system with quadratic pair potentials in one dimension are

$$E_i = \begin{cases} \frac{1}{2}(N^2-1)\sqrt{N}\hbar\omega, & i=0, \\ E_0 + (i+1)\hbar\sqrt{N}\omega, & \text{else}. \end{cases} (19)$$

3 Solutions of D-Dimensional Fermi System

The Hamiltonian is

$\{1,2,\ldots,N-2,N+k\}$, and its eigenfunction is

$$= H_x + H_y + \cdots + H_\eta, (20)$$

with

$$r = x\vec{e}_x + y\vec{e}_y + \cdots + \eta\vec{e}_\eta$$

the D-dimensional coordinate vector. For Bose system, the ground state wave function is just like Eq. (11) except transforming x_{ij} to r_{ij}. For Fermi system, things are a bit more complicated. We have only obtained the ground state energy and eigenfunction through the summary of the calculated results about fixed N. The correctness of all the following equations can be verified by checking whether they meet the stationary Schrödinger equation or not.

The ground state energy is

$$E_{0,f}^{(D)} = \hbar\omega\sqrt{N}\left[\left(K+\frac{D}{2}\right)N - \frac{1}{(D+1)!}\prod_{i=0}^D(K+i)-\frac{D}{2}\right], (21)$$
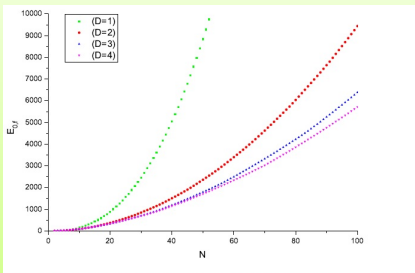
where K is an integer obeying

$$\frac{1}{D!}\prod_{i=0}^{D-1}(K+i) \leqslant N-1 < \frac{1}{D!}\prod_{i=1}^D(K+i), (22)$$

the result of Levy-Leblond[5] is a special case for $D=3$ here. For large N, the result is asymptotically changes to

$$E_{0,f}^{(D)} \xrightarrow{N\to\infty} \hbar\omega\sqrt{N}\left\{\frac{DN}{D+1}[D!(N-1)]^{1/D} + O(N)\right\}. (23)$$

This result is the same as Eq. (20) in Khare's article,[9] except the last small amount term.

The corresponding ground state eigenfunction is

Plots of the ground state energy $E_{0,f}$ with particle number N in different dimensions.

From calculation Zh. Wang et al is following that all states of the system are non-degenerate in one dimension, and this result is against with that of J. M. Levy-Leblond.

WANG Zhao-Liang, WANG An-Min, YANG Yang, and LI Xue-Chao Exact Eigenfunctions of N-body System with Quadratic Pair Potential Commun. Theor. Phys. 58 (2012) 639–644 Vol. 58, No. 5, November 15, 2012

J.M. Levy-Leblond, Generalized uncertainty relations for many-fermion system Phys. Lett. A 26 (1968) 540.

As is showing below in our approach we have degenerate states start from A=3 identical particle case.

SE in the symmetrized coordinates

$$\left[-\frac{\partial^2}{\partial \xi_0^2} + \sum_{i=1}^{A-1} \left[-\frac{\partial^2}{\partial \xi_i^2} + (\xi_i)^2 \right] + \sum_{i=1}^{A} V(x_i(\xi_0, ..., \xi_{A-1})) - E \right] \Psi(\xi_0, ..., \xi_{A-1}; E) = 0,$$

which is invariant w.r.t. perm. $\xi_i \leftrightarrow \xi_j$ at $i, j = 1, ..., A-1$ (instead of Jacobi coords.) as follows from the invariance SE w.r.t. perm. $x_i \leftrightarrow x_j$ at $i, j = 1, ..., A$ is preserved.

Galerkin expansion in the symmetrized coordinates

$$\Psi_{i_o}(\xi_0)(\xi_0, ..., \xi_{A-1}) = \sum_{j=1}^{j_{\max}} \Phi_j(\xi_1, ..., \xi_{A-1}) \chi_{ji_o}(\xi_0),$$

The close-coupling Galerkin equations in symmetrized coordinates

$$\left[-\frac{d^2}{d\xi_0^2} + E_i - E \right] \chi_{ii_o}(\xi_0) + \sum_{j=1}^{j_{\max}} (V_{ij}(\xi_0)) \chi_{ji_o}(\xi_0) = 0,$$

$$V_{ij}(\xi_0) = \int d\xi_1 ... d\xi_{A-1} \Phi_i(\xi_1, ..., \xi_{A-1}) \left(\sum_{k=1}^{A} V(x_k(\xi_0, ..., \xi_{A-1})) \right) \Phi_j(\xi_1, ..., \xi_{A-1}),$$

Symbolic-numerical algorithm for generating cluster eigenfunctions: identical particles with pair oscillator interactions in 1D Euclidian space

Eq for $(A-1)$-dimensional oscillator with known eigenfunctions $\Phi_j(\xi_1, ..., \xi_{A-1})$ and eigenenergies E_j

$$\left[\sum_{i=1}^{A-1}\left[-\frac{\partial^2}{\partial \xi_i^2} + (\xi_i)^2\right] - E_j\right]\Phi_j(\xi_1, ..., \xi_{A-1}) = 0, \quad E_j = 2\sum_{k=1}^{A-1} i_k + A - 1,$$

where the indices i_k, $k = 1, ..., A-1$ take integer values $i_k = 0, 1, 2, 3, ...$

We define the SCR in the form of linear combinations of the conventional oscillator eigenfunctions $\bar{\Phi}_{[i_1, i_2, ..., i_{A-1}]}(\xi_1, ..., \xi_{A-1})$:

$$\Phi_j(\xi_1, ..., \xi_{A-1}) = \sum_{2\sum_{k=1}^{A-1} i_k + A - 1 = E_j} \beta_{[i_1, i_2, ..., i_{A-1}]}^{(j)} \bar{\Phi}_{[i_1, i_2, ..., i_{A-1}]}(\xi_1, ..., \xi_{A-1}),$$

$$\bar{\Phi}_{[i_1, i_2, ..., i_{A-1}]}(\xi_1, ..., \xi_{A-1}) = \prod_{k=1}^{A-1}\bar{\Phi}_{i_k}(\xi_k), \quad \bar{\Phi}_{i_k}(\xi_k) = \frac{\exp(-\xi_k^2/2)H_{i_k}(\xi_k)}{\sqrt[4]{\pi}\sqrt{2^{i_k}}\sqrt{i_k!}},$$

where $H_{i_k}(\xi_k)$ are Hermite polynomials.

Step 1. Symmetrization with respect to permutation of $A-1$ particles

The states, symmetric with respect to permutation of $A-1$ particles $i = [i_1, i_2, ..., i_{A-1}]$

$$\beta^{(i)}_{[i'_1, i'_2, ..., i'_{A-1}]} = \begin{cases} 1/\sqrt{N_\beta}, & [i'_1, i'_2, ..., i'_{A-1}] \text{ is a multiset permutation of } [i_1, i_2, ..., i_{A-1}], \\ 0, & \text{otherwise.} \end{cases}$$

Here $N_\beta = (A-1)!/\prod_{k=1}^{N_v} v_k!$ is the number of multiset permutations of $[i_1, i_2, ..., i_{A-1}]$, $N_v \leq A-1$ is the number of different values i_k in the multiset $[i_1, i_2, ..., i_{A-1}]$, and v_k is the number of repetitions of the given value i_k.

The states, antisymmetric with respect to permutation of $A-1$ particles

$$\Phi^a_j(\xi_1, ..., \xi_{A-1}) = \frac{1}{\sqrt{(A-1)!}} \begin{vmatrix} \bar{\Phi}_{i_1}(\xi_1) & \bar{\Phi}_{i_2}(\xi_1) & \cdots & \bar{\Phi}_{i_{A-1}}(\xi_1) \\ \bar{\Phi}_{i_1}(\xi_2) & \bar{\Phi}_{i_2}(\xi_2) & \cdots & \bar{\Phi}_{i_{A-1}}(\xi_2) \\ \vdots & \vdots & \ddots & \vdots \\ \bar{\Phi}_{i_1}(\xi_{A-1}) & \bar{\Phi}_{i_2}(\xi_{A-1}) & \cdots & \bar{\Phi}_{i_{A-1}}(\xi_{A-1}) \end{vmatrix},$$

i.e., $\beta^{(i)}_{[i'_1, i'_2, ..., i'_{A-1}]} = \varepsilon_{i'_1, i'_2, ..., i'_{A-1}}/\sqrt{(A-1)!}$ where $\varepsilon_{i'_1, i'_2, ..., i'_{A-1}}$ is a totally antisymmetric tensor.

Step 2. Symmetrization with respect to permutation of A particles

Case $A = 2$ ($\xi_1 = (x_2 - x_1)/\sqrt{2}$)

Function being even (or odd) with respect to ξ_1 appears to be symmetric (or antisymmetric) with respect to permutation of two particles, i.e. $x_2 \leftrightarrow x_1$.

Case $A \geq 3$

The functions, symmetric (or antisymmetric) with respect to permutations in Cartesian coordinates $x_i \leftrightarrow x_j$, $i, j = 1, ..., A$ become symmetric (or antisymmetric) with respect to permutations of symmetrized coordinates $\xi_i \leftrightarrow \xi_j$, at $i', j' = 1, ..., A-1$
$$\Phi(..., x_i, ..., x_j, ...) = \pm\Phi(..., x_j, ..., x_i, ...) \rightarrow \Phi(..., \xi_{i'}, ..., \xi_{j'}, ...) = \pm\Phi(..., \xi_{j'}, ..., \xi_{i'}, ...).$$

Here and below we use the above property of the symmetrized coordinates

$$x_{ij} \equiv x_i - x_j = \xi_{i-1} - \xi_{j-1} \equiv \xi_{i-1, j-1}, \quad i, j = 2, ..., A, \quad x_1 = \frac{1}{\sqrt{A}} \sum_{i'=0}^{A-1} \xi_{i'}.$$

Step 2. Symmetrization with respect to permutation of A particles

However, the converse is not true, because we deal with a projection map:

$$
\begin{pmatrix} \xi_1 \\ \xi_2 \\ \cdots \\ \xi_{A-1} \end{pmatrix} = \begin{pmatrix} 1 & a_1 & a_0 & a_0 & \cdots & a_0 & a_0 \\ 1 & a_0 & a_1 & a_0 & \cdots & a_0 & a_0 \\ 1 & a_0 & a_0 & a_1 & \cdots & a_0 & a_0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & a_0 & a_0 & a_0 & \cdots & a_1 & a_0 \\ 1 & a_0 & a_0 & a_0 & \cdots & a_0 & a_1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \cdots \\ x_{A-1} \\ x_A \end{pmatrix}
$$

Thus, the functions, symmetric (or antisymmetric) with respect to permutations of symmetrized coordinates (i.e. by permutations $x_i \leftrightarrow x_j$ at $i, j = 2, ..., A$), are divided into two types, namely,
the physical symmetric (antisymmetric) solutions, symmetric (or antisymmetric) with respect to permutations $x_1 \leftrightarrow x_{j+1}$ at $j = 1, ..., A-1$
$\Phi(x_1, ..., x_{i+1}, ...) = \pm\Phi(x_{i+1}, ..., x_1, ...)$,
and the nonphysical solutions, $\Phi(x_1, ..., x_{i+1}, ...) \neq \pm\Phi(x_{i+1}, ..., x_1, ...)$, which should be eliminated.
This step is equivalent to only one permutation $x_1 \leftrightarrow x_2$, that simplifies its practical implementation.
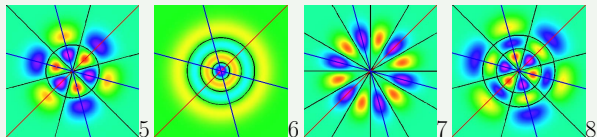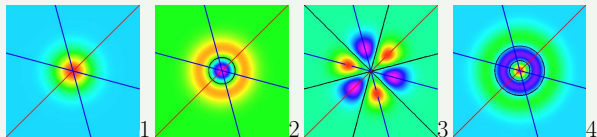
$$\Phi^B_{[i_1,i_2]}(\xi_1,\xi_2) = C_{km}(\rho^2)^{3m/2}\exp(-\rho^2/2)\cos(3m(\varphi+\pi/12))L_k^{3m}(\rho^2),$$
$$(\xi_1 = \rho\cos\varphi,\ \xi_2 = \rho\sin\varphi,\ k = 0,1,...,m = 0,1,....)$$
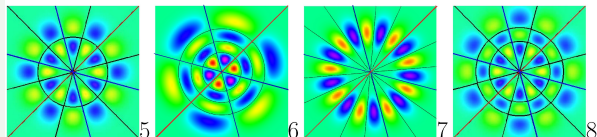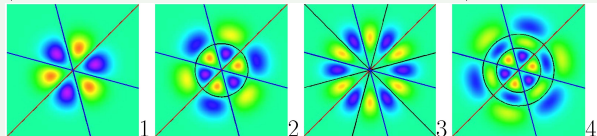
$$\Phi^F_{[i_1,i_2]}(\xi_1,\xi_2) = C_{km}(\rho^2)^{3m/2}\exp(-\rho^2/2)\sin(3m(\varphi+\pi/12))L_k^{3m}(\rho^2),$$
$$(\xi_1 = \rho\cos\varphi,\ \xi_2 = \rho\sin\varphi,\ k = 0,1,...,m = 1,2,...)$$

Profiles of the first eight oscillator symmetric (upper panels) and antisymmetric (lower panels) eigenfunctions Red line correspond to pair collision $x_2 = x_3$, and blue lines correspond to pair collisions $x_1 = x_2$ and $x_1 = x_3$ of projection $(x_1,x_2,x_3) \rightarrow (\xi_1,\xi_2)$. Our result about degeneracy of the basis agrees with J. M. Levy-Leblond, Physics Letters A 26 (1968) 540 and disagrees with: Zh.Wang et al., arXiv:1108.1607v4 [math-ph], Commun. Theor. Phys. 58 (2012) 639–644.
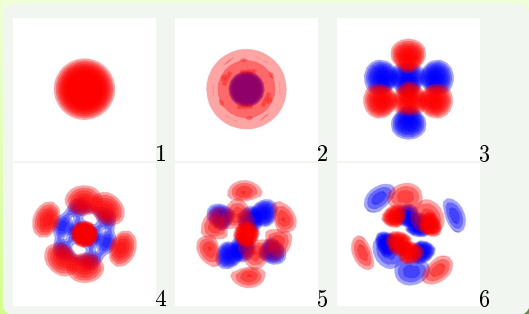
Profiles of the first six oscillator symmetric eigenfunctions $\Phi^{S}_{[i_1, i_2, i_3]}(\xi_1, \xi_2, \xi_3)$ at $A = 4$ in coordinate frame (ξ_1, ξ_2, ξ_3).
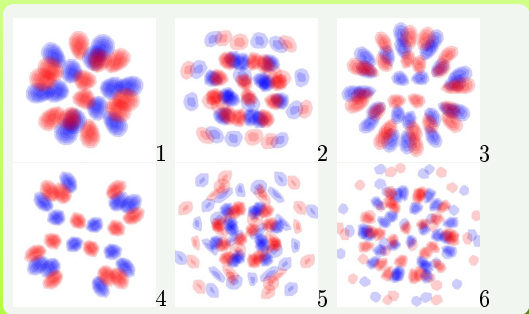
Profiles of the first six oscillator antisymmetric eigenfunctions $\Phi^{A}_{[i_1, i_2, i_3]}(\xi_1, \xi_2, \xi_3)$ at $A = 4$ in coordinate frame (ξ_1, ξ_2, ξ_3).

$[c_{40} P_4^0(\eta) + c_{44} P_4^4(\eta) \cos(4\varphi)]$

$[c_{32} P_3^2(\eta) \sin(2\varphi)]$
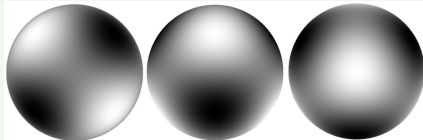
$[c_{60} P_6^0(\eta) + c_{64} P_6^4(\eta) \cos(4\varphi)]$
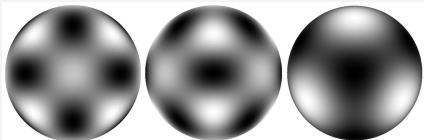
$[c_{72} P_7^2(\eta) \sin(2\varphi) + c_{76} P_7^6(\eta) \sin(6\varphi)]$
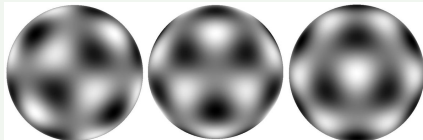
$[c_{80} P_8^0(\eta) + c_{84} P_8^4(\eta) \cos(4\varphi) + c_{88} P_8^8(\eta) \cos(8\varphi)]$

$[c_{92} P_9^2(\eta) \sin(2\varphi) + c_{96} P_9^6(\eta) \sin(6\varphi)]$
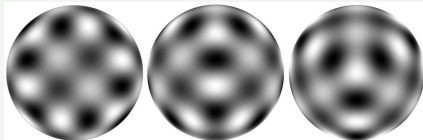
octahedral symmetry

tetrahedral symmetry

$\left[\sum_{i=2,6,10} c_{12i} P_{12}^i(\eta) \cos(i\varphi)\right.$
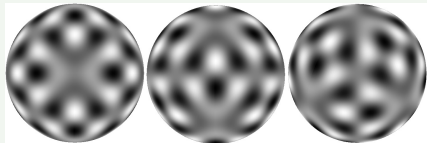
$\left[\sum_{i=4,8} c_{9i} P_9^i(\eta) \sin(i\varphi)\right.$
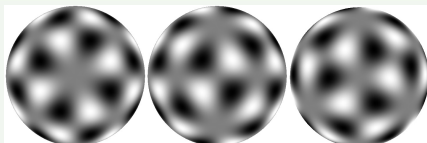
$\left[\sum_{i=2,6,10,14} c_{16i} P_{16}^i(\eta) \cos(i\varphi)\right.$
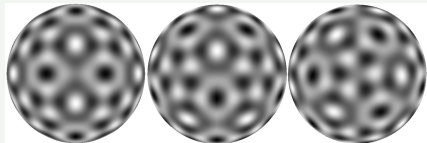
$\left[\sum_{i=4,8,12} c_{13i} P_{13}^i(\eta) \sin(i\varphi)\right.$

$\left[\sum_{i=2,6,10,14} c_{18i} P_{18}^i(\eta) \cos(i\varphi)\right.$
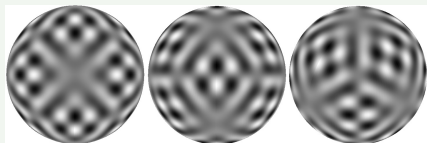
$\left[\sum_{i=4,8,12} c_{15i} P_{15}^i(\eta) \sin(i\varphi)\right.$
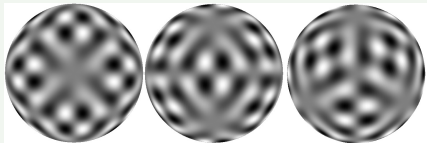
The degeneracy multiplicities p, $p_s = p_a$ and $p_S = p_A$ of s-, a-, S-, and A-eigenfunctions of the oscillator energy levels $\Delta E_j = E_j^\bullet - E_1^\bullet$, $\bullet = \emptyset, s, a, S, A$.

A	3	4	5	6	3	4	5	6	3	4	5	6
ΔE_j			p				$p_{s(a)}$				$p_{S(A)}$	
0	1	1	1	1	1	1	1	1	1	1	1	1
2	2	3	4	5	1	1	1	1	0	0	0	0
4	3	6	10	15	2	2	2	2	1	1	1	1
6	4	10	20	35	2	3	3	3	1	1	1	1
8	5	15	35	70	3	4	5	5	1	2	2	2
10	6	21	56	126	3	5	6	7	1	1	2	2
12	7	28	84	210	4	7	9	10	2	3	3	4
14	8	36	120	330	4	8	11	13	1	2	3	3
16	9	45	165	495	5	10	15	18	2	4	5	6
18	10	55	220	715	5	12	18	23	2	3	5	6

The close-coupling Galerkin equations in symmetrized coordinates

$$\left[-\frac{d^2}{d\xi_0^2} + E_i - E\right]\chi_{ii_o}(\xi_0) + \sum_{j=1}^{j_{\max}}(V_{ij}(\xi_0))\chi_{ji_o}(\xi_0) = 0,$$

$$V_{ij}(\xi_0) = \int d\xi_1...d\xi_{A-1}\Phi_i(\xi_1,...,\xi_{A-1})\left(\sum_{k=1}^{A}V(x_k(\xi_0,...,\xi_{A-1}))\right)\Phi_j(\xi_1,...,\xi_{A-1}),$$

Scattering problem (with real eigenvalues E)

$$\chi_{\xi_0 \to \pm\infty}^v = \begin{cases} \begin{cases} \mathbf{X}^{(+)}(\xi_0)\mathbf{T}_v, & \xi_0 > 0, \\ \mathbf{X}^{(+)}(\xi_0) + \mathbf{X}^{(-)}(\xi_0)\mathbf{R}_v, & \xi_0 < 0, \end{cases} & v = \to, \\ \begin{cases} \mathbf{X}^{(-)}(\xi_0) + \mathbf{X}^{(+)}(\xi_0)\mathbf{R}_v, & \xi_0 > 0, \\ \mathbf{X}^{(-)}(\xi_0)\mathbf{T}_v, & \xi_0 < 0, \end{cases} & v = \leftarrow, \end{cases}$$

where $\mathbf{R}_v$ and $\mathbf{T}_v$ are the reflection and transmission $N_o \times N_o$ matrices, N_o is number of open channels, v denote the initial direction of the particle motion, Open channels $i_o = 1,...,N_o$:

$$X_{ii_o}^{(\pm)}(\xi_0) = \frac{\exp\left(\pm\imath\left(p_{i_o}\xi_0\right)\right)}{\sqrt{p_{i_o}}}\delta_{ji_o}$$

Closed channels $i_c = N_o + 1, \ldots, N$: $\chi_{ii_c}(\xi_0) \to 0$

Metastable states (with complex eigenvalues $E = \Re E + \imath \Im E$, $\Im E < 0$)

Siegert boundary conditions

$$\left.\frac{d\boldsymbol{\chi}(\xi_0)}{d\xi_0}\right|_{\xi_0=\xi_0^t} = \mathcal{R}(\xi_0^t)\boldsymbol{\chi}(\xi_0^t),$$

$$t = \min, \max.$$

$$\mathcal{R}_{i_o i_o}(\xi_0^{\max}) = \imath p_{i_o},$$

$$\mathcal{R}_{i_o i_o}(\xi_0^{\min}) = -\mathcal{R}_{i_o i_o}(\xi_0^{\max}),$$

$$p_{i_o} = \sqrt{E - E_{i_o}},$$

The total transmission probabilities

The repulsive barrier is chosen to be a Gaussian potential $V(x_i) = \frac{\alpha}{\sqrt{2\pi}\sigma} \exp(-\frac{x_i^2}{\sigma^2})$.
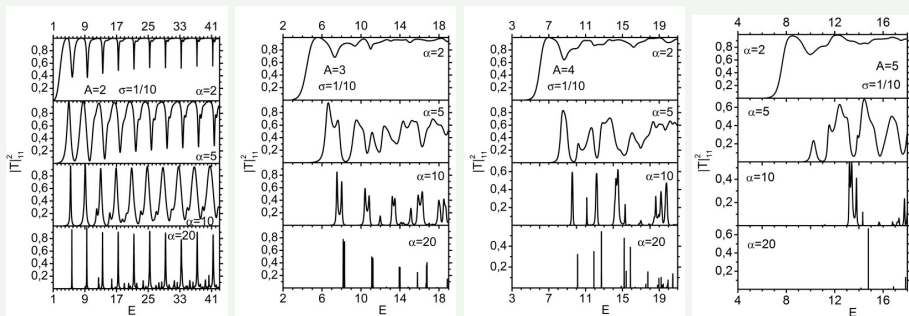
The total transmission probabilities $|T|^2_{11}$ vs energy E (in oscillator units) from the symmetric ground state of the system of $A = 2, 3, 4, 5$ of particles, coupled by the oscillator potential, through the repulsive Gaussian potential barriers $V(x_i) = \frac{\alpha}{\sqrt{2\pi}\sigma} \exp(-\frac{x_i^2}{\sigma^2})$ at $\sigma = 0.1$ and $\alpha = 2, 5, 10, 20$.

Sub-barrier transmission
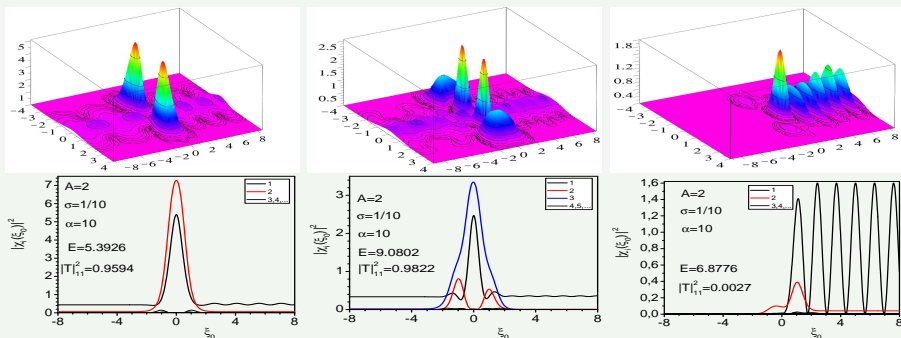
The probability densities $|\Psi(\xi_0, \xi_1)|^2$ of functions and their components $|\chi_i(\xi_0)|^2$ of functions of symmetric ground state for $A = 2$ identical particles.

Sub-barrier transmission

$A = 3$, $\sigma = 1/10$, $\alpha = 20$

l	E_l^S	$|T|_{11}^2$	m	E_m^M
1	8.175	0.775	1	$8.175 - i5.1(-3)$
	8.306	0.737	2	$8.306 - i5.0(-3)$
2	11.111	0.495	3	$11.110 - i5.6(-3)$
	11.229	0.476	4	$11.229 - i5.5(-3)$
3	12.598	0.013	5	$12.598 - i6.4(-3)$
			6	$12.599 - i6.3(-3)$

$A = 4$, $\sigma = 1/10$, $\alpha = 20$

l	E_l^S	$|T|_{11}^2$	m	E_m^M
1	10.121	0.321	1	$10.119 - i4.0(-3)$
			2	$10.123 - i4.0(-3)$
2	11.896	0.349	3	$11.896 - i6.3(-5)$
3	12.713	0.538	4	$12.710 - i4.5(-3)$
	12.717	0.538	5	$12.720 - i4.5(-3)$

$A = 5$, $\sigma = 1/10$, $\alpha = 20$.

l	E_l^S	$|T|_{11}^2$	m	E_m^M
1	11.794	$1.6(-4)$	1	$11.794 - i1.3(-3)$
			2	$11.794 - i1.3(-3)$
2	14.166	0.014	3	$14.166 - i1.1(-3)$
			4	$14.166 - i1.1(-3)$
3	14.764	0.666	5	$14.764 - i6.6(-6)$
	14.774	0.666	6	$14.774 - i5.6(-6)$

Over-barrier transmission

E_i^S	$|T|_{11}^2$	E_m^M
5.8228	0.3794	
9.6479	0.3779	$9.614 - i0.217$
13.5548	0.4765	$13.505 - i0.144$
13.9648	$0.8536(|T|_{33}^2)$	$14.018 - i0.286$
17.4512	0.4874	$17.445 - i0.103$
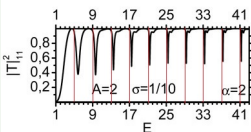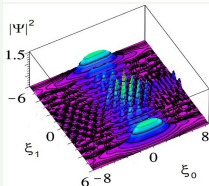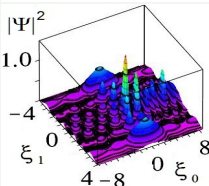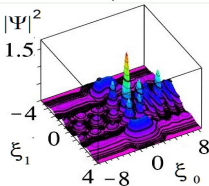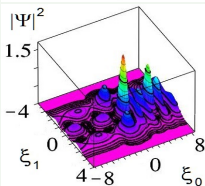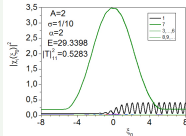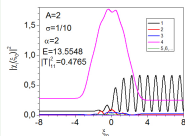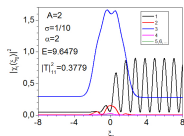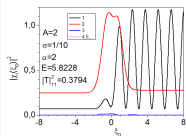
red lines are threshold energies

Over-well transmission
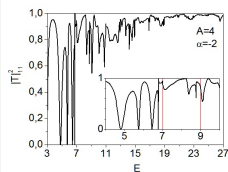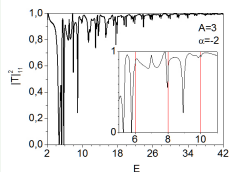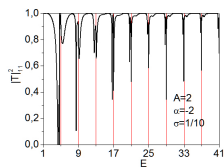
E_i^{th}	$E_m^M(A=2)$	E_i^{th}	$E_m^M(A=3)$	E_i^{th}	$E_m^M(A=4)$
	-0.3588		{-0.2605,1.5082}		{-0.1938, 1.7084 2.7046}
1	$4.4348-\imath0.2572$	2	$5.3307-\imath0.0620$	3	$5.7747-\imath0.0742$
	$4.6764-\imath0.0058$		$5.7911-\imath0.0621$		$6.4441-\imath0.1050$
5	$8.5158-\imath0.0506$	6	$6.9922-\imath0.0751$		$6.7934-\imath0.0033$
	$8.7675-\imath0.1261$		$7.9457-\imath0.0565$	7	$8.3668-\imath0.0651$
9	$12.6009-\imath0.1215$	8	$8.9601-\imath0.0588$		$8.7797-\imath0.0080$
	$12.7330-\imath0.0142$		$9.4950-\imath0.2251$	9	$9.4050-\imath0.1995$
13	$16.6841-\imath0.0364$		$9.8617-\imath0.0852$		$9.9926-\imath0.1225$
	$16.7050-\imath0.0914$	10	$11.4173-\imath0.1678$		$10.0755-\imath0.0676$

Over-well transmission

Resume

- Quantum tunnelling of a cluster comprised of several identical particles, coupled via the oscillator-type potential, through short-range repulsive barrier potentials is studied in the s-wave approximation of the symmetrised-coordinate representation.

- A procedure is described that allows construction of states, symmetric or asymmetric with respect to permutations of A identical particles, from the harmonic oscillator basis functions expressed via the newly introduced symmetrized coordinates [Lecture Notes in Computer Science 8136, 155–168 (2013).].

- The description of quantum tunneling (and channeling) of clusters of several identical particles through the barriers and wells in a coupled-channel approximation of symmetrized-coordinate representation of harmonic oscillator basis symmetric or antisymmetric w.r.t. the permutation of particles is presented.

- Efficiency of the proposed approach and computer codes (KANTBP, KANTBP 3.0 & KANTBP 4M) is demonstrated by analysis of metastable states with complex values of energy of composite systems leading to a quantum transparency effect of the barriers and wells in dependence on number of identical particles and type of symmetry of their states.

- The proposed model can be used as a benchmark to test different methods of calculating the metastable states of composite systems of several identical particles and confinement induced resonances in optical traps.

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Thank you for your attention!