No-go theorem for higher-spin geometry

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Plan of the talk

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Introduction and motivation

Light-cone approach and chiral higher-spin theory

Light-cone approach: a point particle in a chiral higher-spin background

Side observation: double-copy structure for point particles in SDYM and SDGR backgrounds

Conclusions and future work

Introduction and motivation

What do I mean by geometry?

Geometry is a branch of mathematics concerned with properties of space such as the *distance*, *shape*, *size* and relative *position* of figures

[Wikipedia]

For example, *(pseudo)-Riemannian geometry* allows one to define *lengths, straight lines (geodesics), angles, volumes, parallel transport, curvature of space-time* etc in terms of *metric*

What do I mean by higher-spin geometry?

Higher-spin theories are theories of massless higher-spin fields

These involve (modified) gravity in the spin-2 sector

Geometry for the spin-2 sector is the pseudo-Riemannian geometry built in terms of the metric

Higher-spin symmetries mix non-trivially fields of different spins

Accordingly, when higher-spin fields are present the geometric notions available in the pseudo-Riemannian geometry should non-trivially depend on higher-spin fields

What do I mean by higher-spin geometry?

Definition:

Higher-spin geometry is an extension of the pseudo-Riemannian geometry to higher-spin backgrounds, which is consistent with higher-spin symmetries and dynamics

Geometry and point particles

All geometric notions can be derived from point particle's motion in given backgrounds

Examples:

Length is the on-shell action

Angles are defined in terms of lengths

Volumes are defined in terms of lengths

Space-time curvature is defined in terms of tidal forces

Thus, the key goal is to consistently couple a point particle to the higher-spin background

«Background» means that particle's back reaction can be ignored

Chiral higher-spin backgrounds

Chiral higher-spin theories are natural higher-spin extension of SDYM and SDGR to higher-spins

[Metsaev '91; DP, Skvortsov '16]

Properties:

Defined in 4d Minkowski space, not real in the Lorentzian signature, integrable, scattering is trivial

SDYM and SDGR form closed sectors of YM and GR, the same holds for chiral higher-spin theory

Key advantage of Chiral HS: known in a closed form, very simple

Concrete problem

We will explore the coupling of a point-particle to the background of the chiral higher-spin theory

Motivation

Extensions of the Riemannian geometry are interesting in their own right

In the actual Universe we can measure lengths etc. So if higher-spin theories have anything to do with reality, it should be possible to extend the geometric notions to the higher-spin case

The Riemannian geometry + GR have some difficulties, e. g. geodesic incompleteness. Higher-spin theories + higher-spin geometry can potentially resolve these problems

Previous work

De Wit and Freedman

Metric-like formalism. The action of a point particle was found at the leading order in higher-spin fields by requiring HS gauge invariance and reparametrisation invariance

At this order interactions of higher-spin fields are irrelevant

$$S = -m \int d\tau \sqrt{-\eta_{\mu\nu}\dot{x}^\mu \dot{x}^\nu}\Big(1 + \sum_{s=0}^{\infty} c_s \varphi_{\mu_1 \ldots \mu_s} \frac{\dot{x}^{\mu_1} \ldots \dot{x}^{\mu_s}}{(\sqrt{-\eta_{\mu\nu}\dot{x}^\mu \dot{x}^\nu})^s} + \ldots\Big)$$

[de Wit, Freedman '80]

Solution to all orders

A general reparametrization-invariant action in the Hamiltonian form

$$S = \int d\tau (p_\mu \dot{x}^\mu - \lambda H(p,x))$$

is symmetric with respect to

$$\delta H(p,x) = [\varepsilon(p,x), H(p,x)] + a(p,x)H(p,x) \qquad (*)$$

Expanding around the Minkowski space background

$$H(p,x) = \frac{1}{2}\eta^{\mu\nu}p_\mu p_\nu + h(p,x)$$

the first term in (*) reproduces the Fronsdal HS transformations for h. In the Lagrangian form

$$S = -m\int d\tau \sqrt{-\eta_{\mu\nu}\dot{x}^\mu \dot{x}^\nu}\Big(1 + \frac{e}{m^2}\sum_{s=0}^{\infty}\varphi_{\mu_1\dots\mu_s}\frac{\dot{x}^{\mu_1}\dots\dot{x}^{\mu_s}}{(\sqrt{-\eta_{\mu\nu}\dot{x}^\mu\dot{x}^\nu})^s} + \dots\Big)$$

[Segal '00]

Problem

The problem is that there is no reasonable higher-spin theory known with symmetry (*)

The best one has achieved so far is the *conformal higher-spin theory*. It, however, requires the Moyal deformation of (*)

$$\delta H(p,x) = [\varepsilon(p,x), H(p,x)]_\star + \{a(p,x), H(p,x)\}_\star \qquad (**)$$

[Tseytlin '02; Segal '02]

Other setups

Chern-Simons HS theories, Vasiliev's theories: connections on some bundles. Other type of geometry

Chern-Simons HS theories: black holes, conical defects, BH thermodynamics. Another story

[Kraus, Castro, Campoleoni et al]

Usually one one is content with saying that the interval is not HS invariant. As a result, e. g. location of BH horizon cannot be defined

[Sundell, Sezgin, Vasiliev, Didenko et al]

Light-cone formalism: generalities

Covariant vs light-cone formalism

Covariant formalism (e. g. Fronsdal's fields):

Fields are Lorentz tensors. Thus, Lorentz invariance is manifest

Lorentz tensors carry more dof than UIR's. Gauge invariance is required

Light-cone formalism:

There are no redundant (gauge) degrees of freedom

Lorentz invariance is not manifest, has to be checked/imposed

Can be connected to the result of gauge fixing of a covariant theory

Imposing Poincare symmetry

Any symmetry by the Noether theorem entails the associated conserved current

Integrating over a constant-time surface one obtains a conserved charge

In the Hamiltonian formalism conserved charges generate the action of symmetries on phase space via the Poisson (Dirac) bracket (time translation -> the Hamiltonian flow)

The charges commute with the Dirac bracket the same way as the associated generators with the Lie bracket

$$\big[Q[T_1], Q[T_2]\big]_D = Q\big[[T_1, T_2]_L\big] \qquad (***)$$

Solution of this equation defines a Poincare invariant theory. For example, the action in the Hamiltonian form can be easily found: the Hamiltonian is a conserved charge associated with time translations

[Dirac '49]

Constructing interacting Poincare invariant theories

One starts with a free Poincare invariant theory

$$\big[Q_2[T_1], Q_2[T_2]\big]_D = Q_2\big[[T_1, T_2]_L\big]$$

Then one deforms the theory with non-linear terms. The charges get deformed

$$Q_2[T] \to Q[T] = Q_2[T] + \delta Q[T]$$

Then (***) entails

$$\big[\delta Q[T_1], Q_2[T_2]\big]_D + \big[Q_2[T_1], \delta Q[T_2]\big]_D + \big[\delta Q[T_1], \delta Q[T_2]\big]_D = \delta Q\big[[T_1, T_2]_L\big]$$

It is to be solved order by order in perturbations

Light-cone formalism and Fronsdal fields

Light-cone gauge fixing

Starting with the Fronsdal action

$$S = -\frac{1}{2}\int d^4x \partial_\mu \varphi^{\nu_1 \dots \nu_s} \partial^\mu \varphi_{\nu_1 \dots \nu_s} + \dots$$

one imposes the light-cone gauge

$$\varphi^{+\mu_2 \dots \mu_s} = 0$$

From equations of motion one finds that the only two independent components of the field are

$$\Phi^s \equiv \varphi^{x(s)}, \qquad \Phi^{-s} \equiv \varphi^{\bar{x}(s)}$$

The action is

$$S = -\int d^4x \partial_\mu \Phi^{-s} \partial^\mu \Phi^s$$

Light-cone coordinates

$$x^+ = \frac{1}{\sqrt{2}}(x^3 + x^0), \qquad x^- = \frac{1}{\sqrt{2}}(x^3 - x^0)$$
$$x = \frac{1}{\sqrt{2}}(x^1 - ix^2), \qquad \bar{x} = \frac{1}{\sqrt{2}}(x^1 + ix^2)$$

$$ds^2 = 2dx^+dx^- + 2dxd\bar{x}$$

Poincare symmetry

The gauge-fixed theory is also Poincare invariant. Transformations induced from the Fronsdal theory

$$
\begin{aligned}
P^\mu \Phi^\lambda &= \partial^\mu \Phi^\lambda, \\
J^{\mu\nu}\Phi^\lambda &= (x^\mu \partial^\nu - x^\nu \partial^\mu + S^{\mu\nu})\Phi^\lambda,
\end{aligned}
$$

where

$$
\begin{aligned}
S^{+\mu}\Phi^\lambda &= 0, \\
S^{x\bar{x}}\Phi^\lambda &= -\lambda \Phi^\lambda, \\
S^{x-}\Phi^\lambda &= \lambda \frac{\partial}{\partial^+}\Phi^\lambda, \\
S^{\bar{x}-}\Phi^\lambda &= -\lambda \frac{\bar{\partial}}{\partial^+}\Phi^\lambda
\end{aligned}
$$

Chiral higher-spin theory

Chiral higher-spin theory

Proceeding this way one can construct the chiral higher-spin theory

$$
\begin{aligned}
S &= S_2 - \int dx^+ H_3 \\
&= -\frac{1}{2}\int d^4x \sum_\lambda \partial_a \Phi^{-\lambda}\partial^a\Phi^\lambda + \sum_{\lambda_i} \frac{g}{(\lambda_1+\lambda_2+\lambda_3-1)!}\int d^4x \frac{(\ell\bar{\mathbb{P}})^{\lambda_1+\lambda_2+\lambda_3}}{\beta_1^{\lambda_1}\beta_2^{\lambda_2}\beta_3^{\lambda_3}}\Phi^{\lambda_1}\Phi^{\lambda_2}\Phi^{\lambda_3}.
\end{aligned}
$$

Here

$$
\beta_i \equiv \partial_i^+, \qquad \bar{\mathbb{P}} \equiv \bar\partial_1\partial_2^+ - \bar\partial_2\partial_1^+
$$

[Metsaev '91; DP, Skvortsov '16]

Individual vertices were found earlier

[Bengtsson, Bengtsson, Brink '83; Bengtsson, Bengtsson, Linden '86]

Light-cone formalism: point particle on a chiral higher-spin background

Key consistency condition

In the covariant approach (de Wit, Freedman; Segal):

One requires higher-spin gauge invariance and reparametrization invariance

In the light-cone formalism:

Reparametrization invariance and higher-spin gauge invariance are fixed. However, the Poincare symmetry is not manifest.

Poincare symmetry is the key consistency condition

Free point particle in the light-cone gauge

$$S = -m\int d\tau\sqrt{-\eta_{\mu\nu}\dot{x}^\mu\dot{x}^\nu} = -m\int dx^+\sqrt{-2\dot{x}^- - 2\dot{x}\dot{\bar{x}}}, \qquad \tau = x^+$$

No reparametrization invariance, no manifest Poincare symmetry any more

Conserved quantities associated with the Poincare symmetry

$$q_0[P^x] = -p^x, \qquad q_0[P^{\bar{x}}] = -p^{\bar{x}}, \qquad q_0[P^+] = -p^+, \qquad q_0[P^-] = H_p,$$
$$q_0[J^{x\bar{x}}] = \bar{x}p^x - xp^{\bar{x}}, \qquad q_0[J^{x+}] = -xp^+, \qquad q_0[J^{\bar{x}+}] = -\bar{x}p^+,$$
$$q_0[J^{x-}] = H_p x + p^x x^-, \qquad q_0[J^{\bar{x}-}] = H_p\bar{x} + p^{\bar{x}}x^-, \qquad q_0[J^{+-}] = p^+x^-.$$

where

$$H_p \equiv p_-\dot{x}^- + p_x\dot{x} + p_{\bar{x}}\dot{\bar{x}} - L = \frac{p_x p_{\bar{x}}}{p_-} + \frac{m^2}{2p_-}$$

Solution at the leading order

Eventually, one finds

$$H_{p|1} \equiv q_1[P^-] = \sum_{\lambda \geq 0} \left(C^\lambda \frac{\sigma_x^\lambda}{p_-} \Phi^\lambda(x) + C^{-\lambda} \frac{\sigma_{\bar{x}}^\lambda}{p_-} \Phi^{-\lambda}(x) \right)$$

where

$$\sigma_x \equiv p_x - \partial_x \frac{p^+}{\partial^+}, \qquad \sigma_{\bar{x}} \equiv p_{\bar{x}} - \partial_{\bar{x}} \frac{p^+}{\partial^+}$$

Locality and fake interaction were taken into account

Agrees with the result of de Wit and Freedman after gauge fixing

The second order analysis

After a tedious analysis one finds that there are no local solutions for H2, no mater how we choose couplings C

The second order analysis

In other words, there is no local Poincare invariant action for a point particle on a chiral higher-spin background!

Considering that the chiral higher-spin theory is an inevitable sector of any higher-spin theory in flat space and the way how point particles define geometry, we conclude:

There is no higher-spin geometry, at least, one, which is based on scalar point particles

Side observation: color-kinematics duality

SDYM and SDGR

For point particles on SDYM and SDGR backgrounds the 2nd order consistency conditions can be solved

Color factor

Kinematic factor

$$S_{SDYM} = S_{free} + g \int \frac{dx^+}{p^+} \theta_a^\dagger (T_{ab})^\alpha \theta_b \sigma_x \Phi^1_\alpha(x)$$

$$S_{SDGR} = S_{free} + g \int \frac{dx^+}{p^+} \sigma_x^2 \Phi^2(x)$$

Alternatively, it can be found by gauge fixing the known covariant actions

Here theta are point particle’s internal coordinates, T is a matrix of the associate representation of the color algebra

Quite remarkably, they both terminate at the leading order in field (recall square root in GR)

Quite manifestly has the double copy structure

SDYM and SDGR

Besides that, sigma's also define a representation of the area-preserving diffeomorphisms in the following way

$$[\sigma_x \varepsilon_1(x), \sigma_x \varepsilon_2(x)]_D = \sigma_x [\varepsilon_1(x), \varepsilon_2(x)]_{ap}$$

Where the area-preserving diffeos

$$[\varepsilon_1(x), \varepsilon_2(x)]_{ap} \equiv \partial_x \varepsilon_1(x) \partial_- \varepsilon_2(x) - \partial_- \varepsilon_1(x) \partial_x \varepsilon_2(x)$$

is the kinematic algebra of the SDYM theory

So, all typical feature of color-kinematics duality are explicit here in complete analogy with the field theory case: scalar field coupled to SDYM and SDGR

[Monteiro, O'Connell '11]

Why scalar point particles do not couple to higher-spin fields?

Hypothesis (almost a fact):

For a particle to be able to couple to a theory, its phase space should carry a representations of theories' global symmetry algebra

Analogous to: one-particle states should realise the global symmetry of a given QFT

Examples:

Massless scalar field in d dimensions forms a representation of the conformal higher-spin algebra in d dimensions. Accordingly, massless scalar can couple to background higher-spin fields

[Tseytlin '02; Segal '02]

Classical limit: a point particle can couple to a classical version of background conformal higher-spin fields

[Segal '00]

Massless scalar field in AdS of d dimensions *does not* carry a representation of the AdS massless higher-spin algebra in d dimensions. Accordingly, massless scalar cannot couple to background massless higher-spin fields in AdS

Classical limit: analogously, one expects that a scalar point particle cannot couple to massless higher-spin fields in AdS

In the core of out talk we found a counterpart of this statement in Minkowski space

Expectation: a tower of massless spinning point particles *can* couple to chiral higher spin background

This argument strongly suggests that:

The approach by Segal based on point particles *cannot* be amended so that it describes a massless higher-spin theory

Einstein gravity amplitudes can be extracted from those of conformal gravity by an appropriate projection of the Hilbert space (Maldacena'11). The same does not work for conformal and massless higher-spin theories

[Joung, Nakach, Tseytlin'15; Adamo, Hahnel, McLoughlin'16; Adamo, Nakach, Tseytlin'18]

Conclusions and future directions

Conclusions

A scalar point particle cannot be coupled to a chiral higher-spin background

One can regard this as a no-go for the extension of the familiar geometry to the higher-spin case

Point particle actions in self-dual Yang-Mills and gravity backgrounds have a manifest double-copy structure

Future directions

It seems reasonable to reconsider the problem for point particles with additional spinning degrees of freedom

Clarify how the double-copy and additional global symmetries emerge in the light-cone formalism from Poincare symmetry alone

Thank you!

Extras

Particle on HS background

Joint phase space

$$(\Phi(z^{\perp}), x^{\perp}, p_{\perp})$$

Joint Dirac bracket

$$[\cdot,\cdot] \equiv [\cdot,\cdot]_p + [\cdot,\cdot]_\Phi .$$

Conserved quantities in the joint system

$$\begin{aligned}
\mathcal{Q}[T] &\equiv Q[T] + q[T], \\
Q[T] &\equiv Q_2[T] + Q_3[T] + \dots, \\
q[T] &\equiv q_0[T] + q_1[T] + q_2[T] + \dots .
\end{aligned}$$

Consistency of the joint system

Consistency condition, as usual

$$\big[\mathcal{Q}[T_1], \mathcal{Q}[T_2]\big] = \mathcal{Q}\big[[T_1, T_2]\big].$$

Expand in Q and q. Take into account that field theory is Poincare invariant already

$$\big[q[T_1], q[T_2]\big]_p + \big[q[T_1], q[T_2]\big]_\Phi + \big[q[T_1], Q[T_2]\big]_\Phi + \big[Q[T_1], q[T_2]\big]_\Phi = q\big[[T_1, T_2]\big].$$

Dropping the back-reaction terms, one gets

$$\big[q[T_1], q[T_2]\big]_p + \big[q[T_1], Q[T_2]\big]_\Phi + \big[Q[T_1], q[T_2]\big]_\Phi = q\big[[T_1, T_2]\big]$$

Should be solved order by order

The second order analysis

The consistency condition becomes

$$\big[q_2[T_1], q_0[T_2]\big]_p + \big[q_1[T_1], q_1[T_2]\big]_p + \big[q_0[T_1], q_2[T_2]\big]_p + \big[q_2[T_1], Q_2[T_2]\big]_\Phi + \big[q_1[T_1], Q_3[T_2]\big]_\Phi$$
$$+\big[Q_2[T_1], q_2[T_2]\big]_\Phi + \big[Q_3[T_1], q_1[T_2]\big]_\Phi = q_2\big[[T_1, T_2]\big]$$

Here q2 is quadratic in higher-spin fields. Moreover, Q3 is the contribution from HS interactions.

After a tedious analysis one finds that there are no local solutions for H2, no mater how we choose couplings C