From equations in coordinate space on Feynman integrals to Picard-Fuchs and back

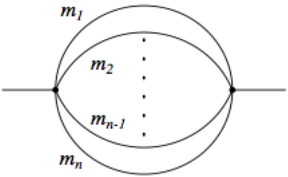
Maxim Reva

MIPT, ITEP

reva.ma@phystech.edu

Presentation Overview

Banana diagram



	Coordinate space		Momentum space
0 loop	$(\Box + m^2)G_m = 0$	$\overset{\text{Fourier}}{\Longleftrightarrow}$	$(p^2 - m^2)F\left[G_m\right] = 0$
n-1 loop	$(???)\prod_{i=1}^{n} G_{m_i} = 0$	$\overset{???}{\Longleftrightarrow}$	$(???)F\left[\prod_{i=1}^{n} G_{m_i}\right] = 0$

General construction scheme

Single line satisfies:

$$(\Box + m^2) G_m(\vec{x}) = 0$$

The equation implies that G is the function of modulus $|\vec{x}|$ what allows to move to more convenient $\Lambda = \vec{x}\partial_{\vec{x}} = x\partial_x$ operators:

$$\left(\Lambda^2 + (D-2)\Lambda + x^2 m^2\right) G_m(x) = 0$$

This relation allows to write equation on G_m^n in rather simple way:

$$\Lambda^k G_m = \sum_{i=0}^{k} a_{k,i} x^{2i} \Lambda G_m + \sum_{i=1}^{k} b_{k,i} x^{2i} G_m$$

One loop example

As Λ^2 can be expressed through Λ, it's reasonable t distinguish 4 "basis" functions:

$$I_{0,0} = G_1 G_2, \qquad I_{1,0} = (\Lambda G_1) G_2, \qquad I_{0,1} = G_1 (\Lambda G_2), \qquad I_{1,1} = (\Lambda G_1)(\Lambda G_2)$$

Then acting by Λ on $I_{0,0}$ we get the linear system:

$$\begin{cases} \Lambda I_{0,0} = I_{1,0} + I_{0,1} \\ \Lambda^2 I_{0,0} = 2I_{1,1} - (D-2)I_{1,0} - (D-2)I_{0,1} - (m_1^2 + m_2^2)x^2 I_{0,0} \\ \Lambda^3 I_{0,0} = B_1 I_{1,1} + B_2 I_{1,0} + B_3 I_{0,1} + B_4 I_{0,0} \\ \Lambda^4 I_{0,0} = C_1 I_{1,1} + C_2 I_{1,0} + C_3 I_{0,1} + C_4 I_{0,0} \end{cases}$$

One loop example

It's convenient to express this system in matrix form:

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & I_{0,0} \\ 0 & 1 & 1 & 0 & \Lambda I_{0,0} \\ -(m_1^2+m_2^2)x^2 & -(D-2) & -(D-2) & 2 & \Lambda^2 I_{0,0} \\ B_1 & B_2 & B_3 & B_4 & \Lambda^3 I_{0,0} \\ C_1 & C_2 & C_3 & C_4 & \Lambda^4 I_{0,0} \end{pmatrix}$$

The kernel of this matrix is of the form $(I_{0,0}, I_{1,0}, I_{0,1}, I_{1,1}, -1)$, which implies

$$\det A = 0$$

One loop example

$$
\begin{aligned}
\det A = &\Big\{\Lambda^4 + 2(2D-5)\Lambda^3 + \Big(2x^2(m_1^2+m_2^2) + (D-2)(5D-16)\Big)\Lambda^2 + \\
&+ 2\Big((m_1^2+m_2^2)x^2(2D-3) + (D-4)(D-2)^2\Big)\Lambda + x^2\Big((m_1^2-m_2^2)^2x^2 + \\
&+ 2(D-1)(D-2)(m_1^2+m_2^2)\Big)\Big\} I_{0,0} = 0
\end{aligned}
$$

Equal mass case

The case of equal masses implies new identity:

$$I_{1,0} = I_{0,1}$$

Then the linear system reduces to:

$$A = \begin{pmatrix} 1 & 0 & 0 & I_{0,0} \\ 0 & 1 & 0 & \Lambda^1 I_{0,0} \\ -2m^2x^2 & 2-D & 2 & \Lambda^2 I_{0,0} \\ 2(D-4)m^2x^2 & (D-2)^2-4m^2x^2 & -6(D-2) & \Lambda^3 I_{0,0} \end{pmatrix}$$

Repeating the same argument we get the equation

$$\det A = \left(\Lambda^3 + 3(D-2)\Lambda^2 + \left(2(D-2)^2 + 4m^2x^2\right)\Lambda + 4(D-1)m^2x^2\right) I_{0,0} = 0$$

Momentum space

As we are interested in “maximal cut” solutions, the Green function reads:

$$G = \int e^{ipx} f(p) \delta(p^2 - m^2) dp$$

So for n-loop integral we have:

$$G^n = \int e^{ipx} f(p) \delta(\sum_i k_i - p) \prod_i \delta(k_i^2 - m_i^2) \prod_i dk_i dp = \int e^{ipx} f(p) I_n dp$$

Using Feynman trick and integrating over k_i we arrive at

$$I_n = \int_{\mathbb{R}^n} \prod_{i=1}^{n} d\alpha_i \delta \left(1 - \sum_{i=1}^{n} \alpha_i\right) \frac{U^{\frac{n}{2}(2-D)}}{F^{1-\frac{(n-1)(D-2)}{2}}}$$

$$U = \prod_{i=1}^{n} \alpha_i \sum_{i=1}^{n} \alpha_i^{-1}, \quad F = \left(\prod_{i=1}^{n} \alpha_i \sum_{i=1}^{n} \alpha_i^{-1}\right) \left(\sum_{i=1}^{n} m_i^2 \alpha_i\right) - p^2 \prod_{i=1}^{n} \alpha_i$$

Momentum space

As the integrand is homogeneous we can move to projective form

$$I_n = \int\limits_{\Gamma^n} \frac{U^{\frac{n}{2}(2-D)}}{F^{1-\frac{(n-1)(D-2)}{2}}}\,\omega$$

where $\Gamma_n = \left\{(\alpha_1,\dots,\alpha_n) \in \mathbb{CP}^{n-1} | (\alpha_1,\dots,\alpha_n) \in \mathbb{R}^n\right\}$ and
$\omega = \sum_{i=1}^{n}(-1)^{i+1}\alpha_i d\alpha_1 \wedge \dots \wedge \widehat{d\alpha_i} \wedge \dots \wedge d\alpha_n$.
Using Hopf fibration we move the integrration to complex domain

$$I_n = \frac{1}{2\pi i}\int\limits_{\widetilde{\Gamma}_n} \frac{U^{\frac{n}{2}(2-D)}}{F^{1-\frac{(n-1)(D-2)}{2}}}\prod_{i=1}^{n} d\widetilde{\alpha}_i$$

with $\widetilde{\Gamma}_n = \left\{(\widetilde{\alpha}_1,\dots,\widetilde{\alpha}_n) \in \mathbb{C}^n \mid \widetilde{\alpha}_i = \alpha_i e^{i\phi},\ \alpha_i \in \Gamma_n,\ \phi \in S^1\right\}$.

General construction scheme

As $\widetilde{\Gamma}_n$ is closed the differential equation can be easily found

$$\sum_{i=1}^{k} a_i(t) \frac{\partial^i}{\partial t^i} I_n = \widehat{PF} \cdot I_n = 0 \quad \Longleftrightarrow \quad \widehat{PF} \cdot \left(\frac{U^{\frac{n}{2}(2-D)}}{F^{1-\frac{(n-1)(D-2)}{2}}} \prod_{i=1}^{n} d\widetilde{\alpha}_i \right) = d\beta$$

For convenience we put $t = p^2$.
The right hand side can be expanded as

$$d\beta = \sum_i \frac{\partial g_i}{\partial \alpha_i}, \qquad g_i = f_i \frac{U^a}{F^b}$$

where f_i are homogeneous polynomials of certain degree.

One loop example

$$U = \alpha_1 + \alpha_2, \qquad F = (\alpha_1 + \alpha_2)(m_1^2\alpha_1 + m_2^2\alpha_2) - t\alpha_1\alpha_2$$

In different mass case:

$$\Big\{ 2t(t-(m_1-m_2)^2)(t-(m_1+m_2)^2)\frac{\partial}{\partial t} + \\ + \left((D-2)(m_1^2-m_2^2)^2 - 2(m_1^2+m_2^2)t + (4-D)t^2\right)\Big\} \frac{U^{2-D}}{F^{2-\frac{D}{2}}} = \\ = 2\left(m_2^2(t+m_1^2-m_2^2)\frac{\partial}{\partial\alpha_1} + m_1^2(t+m_2^2-m_1^2)\frac{\partial}{\partial\alpha_2}\right)\frac{U^{3-D}}{F^{2-\frac{D}{2}}}$$

In equal mass case:

$$\left\{ t(t-4m^2)\frac{\partial}{\partial t} - (D-4)t - 4m^2 \right\} \frac{U^{2-D}}{F^{2-\frac{D}{2}}} = m^2\left(\frac{\partial}{\partial\alpha_1} + \frac{\partial}{\partial\alpha_2}\right)\frac{U^{3-D}}{F^{2-\frac{D}{2}}}$$

Calabi-Yau case

$$I_n(D=2) = \int_{\Gamma_n} \frac{\omega}{F} \quad \xrightarrow{\text{Res}} \quad X: \ F = 0$$

The integral becomes the period of CY X, possibly singular. This allows to derive the additional property on differential operator annihilating integral.

Let $P = \sum_{i=0}^{n} a_i \frac{\partial^i}{\partial t^i}$, $P \cdot I_n = 0$ and its formal adjoint $P^* = \sum_{i=0}^{n} (-1)^i \frac{\partial^i}{\partial t^i} a_i$, then the following relation holds:

$$Pf(t) = (-1)^{\deg P} f(t) P^*$$

where $f(t)$ is some function depending on operator.

$$f(2a_{n-1} - na_n') + na_n f' = 0$$

Fourier transform

The Fourier transform from x to p space:

$$\hat{\Lambda} \cong -D - 2t\frac{\partial}{\partial t}$$
$$x^2 \cong -2D\frac{\partial}{\partial t} - 4t\frac{\partial^2}{\partial t^2}$$

And from p to x:

$$t \cong \partial^2 - \frac{(D-1)}{|x|}\partial$$
$$\Theta = t\frac{\partial}{\partial t} \cong -\frac{1}{2}(D + \hat{\Lambda})$$

Fourier transform

coordinate		Fourier transform	momentum
Minimal in $\Lambda \sim \frac{\partial}{\partial X}$	can be Large in X	$\Longrightarrow$	Large in $\frac{\partial}{\partial t}$
			$\downarrow$ factorization
$\uparrow$ factorization			
Large in $\frac{\partial}{\partial X} \sim \Lambda$		$\Longleftarrow$	Minimal in $\frac{\partial}{\partial t}$ = PF can be Large in t

One loop, equal masses, arbitrary D

coordinate	Fourier transform	momentum
bi-degree $(3,2)$	$\Longrightarrow$	bi-degree $(3,3)$
$x \cdot \uparrow$		$\downarrow$ left div. by $\left(D + 2t\frac{\partial}{\partial t}\right)\frac{\partial}{\partial t} \simeq x^2$
bi-degree $(3, 2-1)$	$\Longleftarrow$	bi-degree $(1,2)$

One loop, different masses, arbitrary D

coordinate	Fourier transform	momentum
bi-degree $(4,3)$	$\Longrightarrow$	bi-degree $(4,4)$
left div. by $x^{-3}(x\frac{\partial}{\partial x}-1)$ $\uparrow$		$\downarrow$ left div. by $\left(D+2t\frac{\partial}{\partial t}\right)\frac{\partial^2}{\partial t^2}$
bi-degree $(5,4-3)$	$\Longleftarrow$	bi-degree $(1,3)$

Two loop, equal masses, arbitrary D

coordinate	Fourier transform	momentum
bi-degree $(4,3)$	$\Longrightarrow$	bi-degree $(4,4)$
left div. by $x^{-3}\left(x\frac{\partial}{\partial x}-1\right)\left(x\frac{\partial}{\partial x}+D-3\right)\uparrow$		$\downarrow$ left div. by $\left(D+2t\frac{\partial}{\partial t}\right)\frac{\partial}{\partial t}$
bi-degree $(6,5-3)$	$\Longleftarrow$	bi-degree $(2,3)$

Two loop, different masses, $D = 2$

coordinate	Fourier transform	momentum
bi-degree $(8, 11)$	$\Longrightarrow$	bi-degree $(12, 10)$
factorization $\uparrow$		$\downarrow$ factorization
bi-degree $(14, 13 - 11)$	$\Longleftarrow$	bi-degree $(2, 7)$

References

V. Mishnyakov, A. Morozov, and M. Reva.
On factorization hierarchy of equations for banana feynman amplitudes.
2023.

V. Mishnyakov, A. Morozov, and M. Reva.
From equations in coordinate space to picard-fuchs and back (to be published).
2024.

The End

Questions? Comments?