Gravitational Wilson networks and conformal field theory

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Outline

- Large-c CFT and conformal blocks
- Wilson networks in AdS
- AdS/CFT correspondence

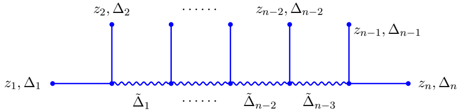
large-c CFT

Conformal correlation functions

- The n-point correlation function of $\mathcal{O}_{\Delta_i,\bar{\Delta}_i}(z_i,\bar{z}_i)$, $i=1,\dots,n$:

$$\langle \mathcal{O}_{\Delta_1,\bar{\Delta}_1}(z_1,\bar{z}_1)\dots\mathcal{O}_{\Delta_n,\bar{\Delta}_n}(z_n,\bar{z}_n)\rangle = \sum_{(\tilde{\Delta}_1,\bar{\tilde{\Delta}}_1),\dots,(\tilde{\Delta}_{n-3}\bar{\tilde{\Delta}}_{n-3})} C_{12\tilde{1}}\dots C_{\tilde{n}-3n-1n}\,\mathcal{F}\,\bar{\mathcal{F}}$$

(Holomorphic) conformal blocks $\mathcal{F}(z_1,\dots,z_n|\Delta_1,\dots,\Delta_n;\tilde{\Delta}_1,\dots,\tilde{\Delta}_{n-3}|c)$ in the OPE comb channel (there are many others)



- The 4-point conformal block ($z_1=\infty$, $z_2=1$, $z_3=z<1$, $z_4=0$):

$$\mathcal{F}(z|\Delta_i,\tilde{\Delta},c) = z^{\tilde{\Delta}-\Delta_1-\Delta_2}\sum_{N=0}^{\infty}F_N z^N \sim 1+\frac{(\tilde{\Delta}-\Delta_1+\Delta_2)(\tilde{\Delta}-\Delta_4+\Delta_3)}{2\tilde{\Delta}}z+F_2z^2+\dots$$

where

$$F_2 = \frac{(\tilde{\Delta}+\Delta_2-\Delta_1)(\tilde{\Delta}+\Delta_2-\Delta_1+1)(\tilde{\Delta}+\Delta_3-\Delta_4)(\tilde{\Delta}+\Delta_3-\Delta_4+1)}{4\tilde{\Delta}(2\tilde{\Delta}+1)}+$$
$$+2\left(\frac{\Delta_1+\Delta_2}{2}+\frac{3(\Delta_1-\Delta_2)^2}{2(1+2\tilde{\Delta})}+\frac{(\tilde{\Delta}-1)\tilde{\Delta}}{2(1+2\tilde{\Delta})}\right)\left(c+\frac{2\tilde{\Delta}(8\tilde{\Delta}-5)}{(1+2\tilde{\Delta})}\right)^{-1}\left(\frac{\Delta_3+\Delta_4}{2}+\frac{3(\Delta_4-\Delta_3)^2}{2(1+2\tilde{\Delta})}+\frac{(\tilde{\Delta}-1)\tilde{\Delta}}{2(1+2\tilde{\Delta})}\right)$$

Large-c CFT

- Different large-c regimes of conformal blocks depend on the behavior of Δ_i and $\tilde{\Delta}_i$:
  - $\Delta, \tilde{\Delta} = \mathcal{O}(c^0)$: light operators
  - $\Delta, \tilde{\Delta} = \mathcal{O}(c^1)$: heavy operators
  - $\Delta, \tilde{\Delta} = \mathcal{O}(c^\alpha)$: α-heavy operators, $\alpha \geq 0$
- E.g. Kac dimensions of degenerate operators:

$$\Delta_{r,s} = \frac{c-1}{24} + \frac{1}{4}(a_+ r + a_- s)^2\,, \quad \text{where } \; a_\pm = \frac{\sqrt{1-c} \pm \sqrt{25-c}}{\sqrt{24}}\,, \qquad r,s \in \mathbb{N}$$

Large-c expansion:

$$\Delta_{r,s} = \frac{1}{24}c\left(1-r^2\right) + \frac{1}{24}\left(13r^2 - 12rs - 1\right) + \frac{3(r-s)(r+s)}{2c} + O(c^{-2})$$

Note that at $r=1$ one has $-\Delta_{1,s} = \frac{s-1}{2}$ (i.e. degenerate $sl(2,\mathbb{R})$ modules)

- E.g. the twist operators in the replica trick:

$$\Delta_n = \frac{c}{24}\left(n - \frac{1}{n}\right)\,, \qquad n \in \mathbb{N}$$

- *Three* types of conformal blocks:
  - Global conformal block — all operators are light (this is the $sl(2,\mathbb{R})$ block; in CFT$_d$ all conformal blocks are global)
  - Classical conformal block — all operators are heavy ($\mathcal{F} \sim \exp\left(c\, F_{cl}(\Delta/c, \tilde{\Delta}/c)\right)$, Zamolodchikov 1988)
  - Heavy-light blocks interpolate between these two extreme regimes

Global conformal symmetry in the large-c

• Virasoro algebra commutation relations

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n,0}\ , \qquad m, n \in \mathbb{Z}$$

Primary operators transform as

$$[L_m, \mathcal{O}_\Delta(z)] = z^m(z\partial_z + (m+1)\Delta)\mathcal{O}_\Delta(z)$$

• Inönu-Wigner contraction for the Virasoro algebra, where the deformation parameter is c^{-1}.

Rescaled Virasoro generators:

$$L_{0,\pm1} \to l_{0,\pm1} = L_{0,\pm1}\ , L_m \to a_m = L_m/c\ , \quad |m| \geq 2$$

The contracted Virasoro algebra splits into $sl(2)$ algebra and the inf-dim Abelian algebra $\mathcal{A}$,

$$[l_m, l_n] = (m-n)l_{m+n}\ , \qquad\qquad [a_m, a_n] = 0$$

In the limit $c \to \infty$, keeping the conformal dimension Δ finite we find that the primary operator transforms as

$$[l_m, \mathcal{O}_\Delta(z)] = z^m(z\partial_z + (m+1)\Delta)\mathcal{O}_\Delta(z)\ , \qquad [a_m, \mathcal{O}_\Delta(z)] = 0$$

i.e. $\mathcal{O}_\Delta(z)$ are $sl(2)$ primary operators and a_m-singlets.

• Global blocks can be calculated via $sl(2,\mathbb{R})$ matrix elements.

Global conformal blocks

The n-point global conformal block in the comb channel (Ferrara et al 1976, K.A., Belavin 2015, Rosenhaus 2018):

$$\mathcal{F}_{\Delta,\tilde{\Delta}}(z) = \mathcal{L}_{\Delta}(z)\, G_{\Delta,\tilde{\Delta}}(\chi(z))$$

- leg factor

$$\mathcal{L}_{\Delta}(z) = \left(\frac{z_{23}}{z_{12}z_{13}}\right)^{h_1} \left(\frac{z_{n-2,n-1}}{z_{n-2,n}z_{n-1,n}}\right)^{h_n} \prod_{i=1}^{n-2} \left(\frac{z_{i,i+2}}{z_{i,i+1}z_{i+1,i+2}}\right)^{h_{i+1}}$$

- cross-ratios

$$\chi_i = \frac{z_{i,i+1}z_{i+2,i+3}}{z_{i,i+2}z_{i+1,i+3}}\,, \quad 1 \leq i \leq n-3$$

- n-point (bare) conformal block

$$G_{\Delta,\tilde{\Delta}} = \left(\prod_{i=1}^{n-3} \chi_i^{\tilde{\Delta}_i}\right) F_K \left[\begin{array}{c} \Delta_1 - \Delta_2 + \tilde{\Delta}_1, \ldots, \Delta_n - \Delta_{n-1} + \tilde{\Delta}_{n-3} \\ \tilde{\Delta}_1, \ldots, \tilde{\Delta}_{n-3} \end{array} \middle| \chi_1, \ldots, \chi_{n-3}\right]$$

- and F_K is the comb function (Rosenhaus 2018)

$$F_K \left[\begin{array}{c} a_1, b_1, \ldots, b_{k-1}, a_2 \\ c_1, \ldots, c_k \end{array} \middle| x_1, \ldots, x_k\right] = \sum_{l_1,\ldots,l_k=0}^{\infty} \frac{(a_1)_{l_1}(b_1)_{l_1+l_2}(b_2)_{l_2+l_3}\cdots(b_{k-1})_{l_{k-1}+l_k}(a_2)_{l_k}}{(c_1)_{l_1}\cdots(c_k)_{l_k}} \frac{x_1^{l_1}}{l_1!}\cdots\frac{x_k^{l_k}}{l_k!}$$

- $(x)_n = \Gamma(x+n)/\Gamma(n)$ are Pochhammer symbols
- Gauss ${}_2F_1$ and Appell F_2 functions
- the blocks = eigenfunctions of $sl(2,\mathbb{R})$ Casimir equations in each exchange channel

Gravity in AdS

Topological gravities in lower dimensions

There are no local PDoF:

$$\text{vanishing Weyl tensor:} \quad C_{mn,kl} = 0$$

Indeed,

$$R_{mn,kl} = C_{mn,kl} + \text{on-shell terms}$$

- The cosmological constant $\Lambda \neq 0$
- $d = 3$: the metric g_{mn} and EOM $R_{mn} + \Lambda g_{mn} = 0$ – Einstein theory
- $d = 2$: the metric g_{mn}, the scalar ϕ and EOM $R + \Lambda = 0$ – Jackiw-Teitelboim theory

The frame formulations:

- $d = 3$: $S_{CS} = \int_{M^3} \mathrm{Tr}(dAA + A^3)$, where A is $o(2,2)$-connection (Achucarro, Townsend 1986, Witten 1988)
- $d = 2$: $S_{BF} = \int_{M^2} \mathrm{Tr}(BF)$, where A is $o(2,1)$-connection, $F = dA + A^2$ and B – 0-form (Fukuyama, Kamimura 1985)
- The chiral factorization: $o(2,2) \approx o(2,1) \oplus o(2,1)$ and $o(2,1) \approx sl(2,\mathbb{R})$

The common EOM:

$$F = dA + A^2 = 0 \ : \qquad \text{gravitational flat connections}$$

AdS_2 spacetime

- Any solution is locally AdS
- From now on, assuming the chiral factorization we will be discussing AdS_2 flat connections
  - The gauge algebra is $sl(2,\mathbb{R})$:
  $$[J_m, J_n] = (m-n)J_{m+n}$$
  where $n, m = 0, \pm 1$
  - The local coordinates on AdS_2: $x^\mu = (\rho, z)$, where $\rho, z \in \mathbb{R}$
  - The AdS_2 solution (Banados 1998):
  $$A = e^{-\rho J_0}(J_1 dz)e^{\rho J_0} + J_0 d\rho \equiv A^m_\mu dx^\mu J_m$$
  - The associated metric $g_{\mu\nu} = e_\mu \cdot e_\nu$, where $(e_\mu, \omega_\mu) = A_\mu$:
  $$ds^2 = e^{2\rho}dz^2 + d\rho^2$$
  with the conformal boundary at $\rho = \infty$ (actually, there are two conformal boundaries)

A primer on $sl(2,\mathbb{R})$ representations $\mathcal{R}_j$

- Finite-dimensional series. $\mathcal{R}_j = \mathcal{D}_j$ with weights $j \in \mathbb{N}_0/2$, $\dim \mathcal{D}_j = 2j+1$. The standard ladder basis is given by

$$\{\mathcal{D}_j \ni |j,m\rangle : \; J_0\,|j,m\rangle = m\,|j,m\rangle \;,\; m = -j, -j+1, \ldots, j-1, j\}$$

where the highest-weight (HW) vector $|j,j\rangle$ is defined by

$$J_0\,|j,j\rangle = j\,|j,j\rangle \qquad J_{-1}\,|j,j\rangle = 0$$

- Negative discrete series. $\mathcal{R}_j = \mathcal{D}_j^-$ with weights $j \in \mathbb{R}$, $\dim \mathcal{D}_j = \infty$. The basis is given by

$$\{\mathcal{D}_j^- \ni |j,m\rangle : \; J_0\,|j,m\rangle = m\,|j,m\rangle \;,\; m = j, j-1, j-2, \ldots, -\infty\}$$

where $|j,j\rangle$ is a HW vector and m is generally non-integer.

- If $j \in \mathbb{N}_0/2$ then the respective module contains a singular vector (light Kac dimensions) so that

$$\mathcal{D}_j^- / \mathcal{S}_{-j-1} \approx \mathcal{D}_j$$

where $\mathcal{S}_{-j-1} \subset \mathcal{D}_j^-$ is the singular subspace.

In both types of modules $\mathcal{R}_j$ the action of $sl(2,\mathbb{R})$ algebra is defined as

$$J_0\,|j,m\rangle = m\,|j,m\rangle$$

$$J_1\,|j,m\rangle = \sqrt{(m+j)(j-m+1)}\,|j,m-1\rangle \equiv M(j,m-1)\,|j,m-1\rangle$$

$$J_{-1}\,|j,m\rangle = -\sqrt{(m+j+1)(j-m)}\,|j,m+1\rangle \equiv -M(j,m)\,|j,m+1\rangle$$

The zeros of $M(j,m)$ define the passage from $\mathcal{D}_j^-$ to $\mathcal{D}_j$ since they correspond to singular vectors.

Gravitational Wilson line

The basic object is a Wilson line:

$$W_j[L] = \mathbb{P}\exp - \int_L A_j$$

where

#1 L – a path in AdS_2 from x_1 to x_2 and $\mathbb{P}$ is the path-ordering operator

#2 A_j takes values in $sl(2,\mathbb{R})$ module $\mathcal{R}_j$, fin-dim or inf-dim.

#3 a gauge transformation: $A \to gA\,g^{-1} + gdg^{-1}\,, \quad W_j[L] \to g(x_2)W_j[L]g^{-1}(x_1)$

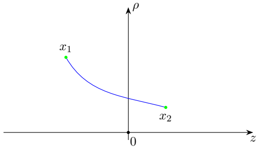
#4 a path transitivity: $W_j[L_1 + L_2] = W_j[L_2]W_j[L_1]$

#5 The main property:

$$W_j[L] = W_j[x_1, x_2]$$

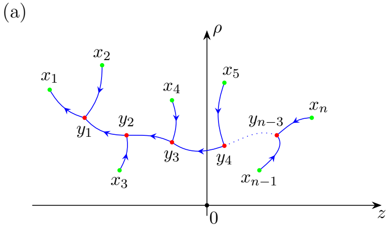
i.e. for a flat connection it depends on $x_{1,2}$ only! In our case:

$$W_j[x_1, x_2] = e^{-\rho_2 J_0}\, e^{z_{12} J_1}\, e^{\rho_1 J_0}\,, \qquad \text{where } x_i = (z_i, \rho_i)$$



Gravitational Wilson networks, I

Let us now compose Wilson lines into a network:



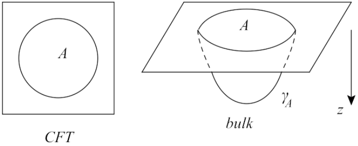
There are:

1. n endpoints (green dots, coordinates x_i)
2. n external legs (Wilson lines)
3. $n-3$ internal legs (Wilson lines)
4. $n-3$ three-valent vertices (red dots, coordinates y_i)
5. Each Wilson line carries its own $\mathcal{R}_j$

Motivation, finally

Why conformal blocks and networks?

- Ryu-Takayanagi entanglement entropy formula (2006): $S_A = \min_{\gamma_A} \frac{\text{area}(\gamma_A)}{4G_N}$



- Replica trick for the entanglement entropy:

$$\text{Renyi entropies} \quad S_A^{(n)} = \frac{1}{1-n} \log \text{Tr}\rho_A^n \quad \text{and} \quad S_A = \lim_{n\to 1} S_A^{(n)}$$

When A consists of N disjoint intervals, the Renyi entropy can be realized as a 2N-point conformal correlation function of the twist operators (Calabrese, Cardy 2009):

$$\text{Tr}\rho_A^n \sim \langle \mathcal{O}_1 \ldots . \mathcal{O}_{2N} \rangle \,, \qquad \Delta(\mathcal{O}_i) \sim c$$

The S_A in CFT_2 is calculated in the large-c regime by the Zamolodchikov conformal block (Hartman 2013)

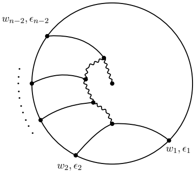
$$\sim \exp\left(c\, F_{cl}\right)$$

Why conformal blocks and networks?

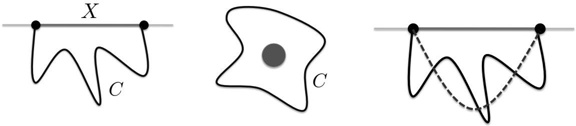
The Brown-Henneaux relation $c \sim 1/G_N$

- Geodesic networks vs large-c conformal blocks (Fitzpatrick et al 2013, Hijano et al 2015, K.A., Belavin 2015, Datta et al 2016)

$$\text{area}(\gamma_A) \sim F_{cl}$$

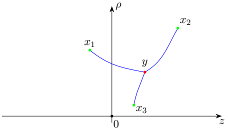


- Metric vs frame formulation of gravity theory and Wilson lines/loops vs geodesics (Amon et al 2016, Castro et al 2018)



AdS, back again

3-valent intertwiner



- The 3-valent intertwiner:

$$Inv(\mathcal{R}_{j_1} \otimes \mathcal{R}_{j_2} \otimes \mathcal{R}_{j_3}) \ni I_{j_1 j_2 j_3} : \quad \mathcal{R}_{j_2} \otimes \mathcal{R}_{j_3} \to \mathcal{R}_{j_1}$$

- The invariance property:

$$I_{j_1 j_2 j_3}\, U_{j_2}\, U_{j_3} = U_{j_1}\, I_{j_1 j_2 j_3}$$

where U_j are $SL(2,\mathbb{R})$ operators of the corresponding representations.

- The basic idea is an invariant contraction: $I_{a\alpha A}\, X^a\, Y^\alpha\, Z^A$
- To have a non-trivial intertwiner the weights of three representations must be constrained. The Clebsch-Gordon series

$$\mathcal{R}_{j_2} \otimes \mathcal{R}_{j_3} = \bigoplus_{j_1} \mathcal{R}_{j_1}$$

If $\mathcal{D}_{j_1}$ does arise in the CG series then the intertwiner is just a projector, otherwise it is zero. In components, the CG coefficient takes the form

$$|j_2, m\rangle \otimes |j_3, n\rangle = \sum_k \Big(\langle j_1, k|\, I_{j_1 j_2 j_3}\, |j_2, m\rangle \otimes |j_3, n\rangle \Big)\, |j_1, k\rangle$$

where the summation domain depends on the type of modules $\mathcal{R}_{j_i}$.

3j symbol

The intertwiner as 3j symbol:

$$\langle j_1, k|\, I_{j_1 j_2 j_3}\, |j_2, m\rangle \otimes |j_3, n\rangle \equiv [I_{j_1 j_2 j_3}]^k{}_{mn} = (-1)^{j_1 - k} \begin{pmatrix} j_1 & j_2 & j_3 \\ -k & m & n \end{pmatrix}$$

The explicit form of the 3j symbol (e.g. in Varshalovich et all 1987)

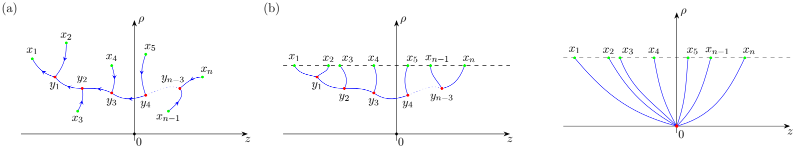
$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = \delta_{m_1+m_2+m_3,0} \frac{\sqrt{(j_3 - j_1 + j_2)!}\sqrt{(-j_3 + j_1 + j_2)!}\sqrt{(j_3 + j_1 + j_2 + 1)!}}{(j_3 + j_1 + j_2 + 1)!\sqrt{(j_3 + j_1 - j_2)!}}$$

$$\times \frac{\sqrt{(j_3 - m_3)!}\sqrt{(j_1 - m_1)!}}{\sqrt{(j_3 + m_3)!}\sqrt{(j_1 + m_1)!}\sqrt{(j_2 - m_2)!}\sqrt{(j_2 + m_2)!}} \frac{(-)^{j_1 + m_2 - m_3}(2j_3)!\,(j_3 + j_2 + m_1)!}{(j_3 - j_1 + j_2)!(j_3 - m_3)!}$$

$$\times\ {}_3F_2\left(-j_3 + m_3, -j_3 - j_1 - j_2 - 1, -j_3 + j_1 - j_2; -2j_3, -j_3 - j_2 - m_1; 1\right)$$

- Note that weights j_1, j_2, j_3 satisfy the selection rules (triangle inequalities).
- We assume that $j! = \Gamma(j+1)$ for $j \in \mathbb{R}$ (Holman et all 1966).

Gravitational Wilson networks: AdS vertex functions



- The Wilson line network operator:

$$\widehat{W}^{j_1\cdots j_n}_{\tilde{j}_1\cdots\tilde{j}_{n-3}}(\mathbf{x},\mathbf{y}) := \Big(W_{j_1}[y_1,x_1]I_{j_1j_2\tilde{j}_1}\,W_{\tilde{j}_1}[y_2,y_1]I_{\tilde{j}_1j_3\tilde{j}_2}\ \ldots\ W_{\tilde{j}_{n-3}}[y_{n-2},y_{n-3}]I_{\tilde{j}_{n-3}j_{n-1}j_n}\Big)$$
$$\times\Big(W_{j_2}[x_2,y_1]\ldots W_{j_{n-1}}[x_{n-1},y_{n-2}]W_{j_n}[x_n,y_{n-3}]\Big)$$

where the sets of endpoints and vertices are $\mathbf{x} = (x_1,\ldots,x_n)$ and $\mathbf{y} = (y_1,\ldots,y_{n-3})$.

- Let us now associate to each endpoint a particular cap state $|a_i\rangle \in \mathcal{R}_{j_i}$. Then, one introduces the AdS vertex function

$$\mathcal{V}_{j\tilde{j}}(\mathbf{x},\mathbf{y}) \equiv \langle a_1|\,\widehat{W}^{j_1\cdots j_n}_{\tilde{j}_1\cdots\tilde{j}_{n-3}}(\mathbf{x},\mathbf{y})\,|a_2\rangle\otimes|a_3\rangle\otimes\cdots\otimes|a_n\rangle\,,$$

- Using the intertwiner invariance property and the path transitivity one can set $\mathbf{y}=0$ (see Fig. 2 and Fig. 3)

$$\mathcal{V}_{j\tilde{j}}(\mathbf{x},\mathbf{y}) \to \mathcal{V}_{j\tilde{j}}(\mathbf{z},\rho)$$

where $\mathbf{z} = (z_1,\ldots,z_n)$ are the boundary points and ρ labels the line which will be finally pulled at (conformal) infinity, $\rho\to\infty$.

- By using the identity resolutions

$$\mathbb{I}_j = \sum_m |j, m\rangle \langle j, m|$$

the AdS vertex function can be represented as a matrix product

$$\mathcal{V}_{j\tilde{j}}(\mathbf{z}, \rho) = \sum_{\mathbf{m},\, \mathbf{p}} \langle j_1, m_1| I_{j_1 j_2 \tilde{j}_1} |j_2, m_2\rangle \otimes |\tilde{j}_1, p_1\rangle \langle \tilde{j}_1, p_1| I_{\tilde{j}_1 j_3 \tilde{j}_2} |j_3, m_3\rangle \otimes |\tilde{j}_2, p_2\rangle \cdots$$

$$\cdots \langle \tilde{j}_{n-3}, p_{n-3}| I_{\tilde{j}_{n-3} j_{n-1} j_n} |j_{n-1}, m_{n-1}\rangle \otimes |j_n, m_n\rangle \Big(\langle \tilde{a}_1 | j_1, m_1\rangle \langle j_2, m_2 | \tilde{a}_2\rangle \cdots \langle j_n, m_n | \tilde{a}_n\rangle \Big)$$

where

- x-dependent cap states

$$\langle \tilde{a}_1| = \langle a_1|\, W_{j_1}[0, x_1] \quad \text{and} \quad |\tilde{a}_i\rangle = W_{j_i}[x_i, 0]\, |a_i\rangle\,, \qquad i = 2, \ldots, n$$

- (magnetic) indices $\mathbf{m} = (m_1, \ldots, m_n)$ and $\mathbf{p} = (p_1, \ldots, p_{n-3})$
- there are $3n - 3$ independent (infinite) summations

- The AdS vertex function can be cast into the equivalent form:

$$\mathcal{V}_{j\tilde{j}}(\mathbf{z}, \rho) = \langle \tilde{a}_1|\, I_{j_1 \ldots j_n | \tilde{j}_1 \ldots \tilde{j}_{n-3}}\, |\tilde{a}_2\rangle \otimes \cdots \otimes |\tilde{a}_n\rangle$$

where the n-valent intertwiner is given by

$$I_{j_1 \ldots j_n | \tilde{j}_1 \ldots \tilde{j}_{n-3}} = I_{j_1 j_2 \tilde{j}_1}\, I_{\tilde{j}_1 j_3 \tilde{j}_2} \cdots I_{\tilde{j}_{n-3} j_{n-1} j_n} \quad : \qquad (n-2)\;\; 3j \text{ symbols in the comb channel}$$

with the generalized intertwiner invariance property

$$I_{j_1 \ldots j_n | \tilde{j}_1 \ldots \tilde{j}_{n-3}} = U_{j_1}^{-1}\, I_{j_1 \ldots j_n | \tilde{j}_1 \ldots \tilde{j}_{n-3}}\, U_{j_2} \cdots U_{j_n}$$

AdS/CFT

AdS/CFT correspondence: extrapolate dictionary

- The standard AdS/CFT: $Z_{AdS} = Z_{CFT}$ for two dual theories.

- For AdS_2 scalar quantum fields $\hat{\Phi}_i(\rho_i, z_i)$ with masses m_i the extrapolate dictionary gives

$$\lim_{\rho\to\infty} e^{\rho \sum \Delta_i} \langle \hat{\Phi}_1(\rho, z_1) \cdots \hat{\Phi}_n(\rho, z_n) \rangle_{AdS} = \langle \hat{\mathcal{O}}_1(z_1) \cdots \hat{\mathcal{O}}_n(z_n) \rangle_{CFT}$$

where

1. conformal dimensions Δ_i are related to masses as $m_i^2 = \Delta_i(\Delta_i - 1)$
2. all AdS fields are placed on the hypersurface $\rho = const \longrightarrow$ conformal boundary

- Extrapolate dictionary $\leftrightarrow$ HKLL reconstruction (Hamilton, Kabat, Lifschytz, Lowe 2006) as for three types of Witten diagrams

- In our context the AdS vertex functions are assumed to reproduce CFT correlation functions in the way which is essentially the same as the extrapolate dictionary relation.

#1 The AdS vertex functions are not literally AdS scalar correlation functions

#2 the AdS vertex functions must be subject to particular spacetime symmetry criteria that mimic those satisfied by AdS scalar correlation functions

Spacetime invariance and cap states

• We require the AdS vertex functions to be invariant with respect to AdS_2 spacetime isometry transformations:

$$\mathcal{V}_{j\tilde{j}}(\mathbf{x}') = \mathcal{V}_{j\tilde{j}}(\mathbf{x}) \;, \qquad \mathbf{x}' = \mathbf{x}'(\mathbf{x}) \in SL(2,\mathbb{R})$$

The infinitesimal form of the symmetry condition is given by three global Ward identities

$$\sum_{i=1}^{n} \mathcal{J}_m^{(i)}\, \mathcal{V}_{j\tilde{j}}(x_1, \dots, x_i, \dots, x_n) = 0 \qquad m = 0, \pm 1$$

where $\mathcal{J}_m = \xi_m^{\mu} \partial_{\mu}$ are the Lie derivatives along the Killing vector fields $\xi_m(x)$ of the AdS_2 spacetime

$$\mathcal{J}_{-1} = \partial_z \;, \qquad \mathcal{J}_0 = z\partial_z - \partial_\rho \;, \qquad \mathcal{J}_1 = z^2\partial_z - 2z\partial_\rho - e^{-2\rho}\partial_z$$

The superscript i indicates that the derivative is taken with respect to the i-th coordinate.

• For general values of the radial coordinates ρ_i, $i = 1, \dots, n$, the system of PDEs has $2n - 3$ first integrals. The hypersurface the AdS vertex functions are parameterized as

$$\mathcal{V}_{j\tilde{j}}(\mathbf{x})\Big|_{\rho_1 = \dots = \rho_n = \rho} = \mathcal{V}_{j\tilde{j}}(\mathbf{z}, \rho) = \mathcal{V}_{j\tilde{j}}(q_{12}, \dots, q_{n-1,n})$$

where

$$q_{i,i+1} = (z_{i+1} - z_i)e^{\rho} \;, \qquad i = 1, \dots, n-1$$

Spacetime invariance and cap states

The Ward identities uniquely fix the form of the cap states $|a\rangle$:

Ishibashi state : $\quad (J_1 + J_{-1})\,|a\rangle = 0$

- At $j \neq \mathbb{N}_0$ there is a unique (up to a normalization) vector $|a\rangle \equiv |j\rangle\rangle \in \mathcal{D}_j^-$:

$$|j\rangle\rangle = \sum_{n=0}^{\infty} \prod_{k=1}^{n} \frac{(-)^n}{-4kj + 4k^2 - 2k}\,(J_1)^{2n}\,|j,j\rangle$$

- At $j \in \mathbb{N}_0$ the module $\mathcal{D}_j^-$ has a singular vector which additionally generates a new solution: the kernel of $J_1 + J_{-1}$ becomes two-dimensional and the two basis cap states read

$$|j\rangle\rangle_1 = \sum_{n=0}^{j} \prod_{k=1}^{n} \frac{(-)^n}{-4kj + 4k^2 - 2k}\,(J_1)^{2n}\,|j,j\rangle$$

$$|j\rangle\rangle_2 = \sum_{n=0}^{\infty} \prod_{k=1}^{n} \frac{(-)^n}{4kj + 4k^2 + 2k}\,(J_1)^{2n}\,|j,-j-1\rangle$$

- The case of finite-dimensional modules $\mathcal{D}_j$ with $j \in \mathbb{N}_0$: $|j\rangle\rangle = |j\rangle\rangle_1$

Wilsonian extrapolate relation

- The extrapolate dictionary:

$$\lim_{\rho\to\infty} e^{-\rho\sum_{i=1}^{n} j_i}\,\mathcal{V}_{j\tilde{j}}(\rho,\mathbf{z}) = C_{j\tilde{j}}\,\mathcal{F}_{h\tilde{h}}(\mathbf{z})$$

where weights and conformal dimensions are identified as $h_i = -j_i$ and $\tilde{h}_k = -\tilde{j}_k$.

- The normalization coefficients $C_{j\tilde{j}} \equiv C_{j_1\ldots j_n\tilde{j}_1\ldots\tilde{j}_{n-3}}$:

$$n=2: \quad C_{j_1 j_2} = \frac{\delta_{j_1 j_2}}{(2j_1+1)^{\frac{1}{2}}}\;; \qquad n=3: \quad C_{j_1 j_2 j_3} = \left[\frac{(2j_1)!(2j_2)!(2j_3)!}{\Delta(j_1,j_2,j_3)}\right]^{\frac{1}{2}}$$

$$n>3: \quad C_{j\tilde{j}} = C_{j_1 j_2 \tilde{j}_1}\left[\prod_{i=1}^{n-4} C_{\tilde{j}_i j_{i+2}\tilde{j}_{i+1}}\right] C_{\tilde{j}_{n-3} j_{n-1} j_n} \qquad \text{the comb channel}$$

where $\Delta(a,b,c) = (a+b+c+1)!(a+b-c)!(a+c-b)!(b+c-a)!$ is the modified triangle coefficient.

- AdS vertex functions are defined up to multiplicative constants. Non-vanishing and real $C_{j_1 j_2 j_3}$ yield triangle inequalities:

$$1)\;\; j_1, j_2 \in \mathbb{N}_0/2, j_3 \in \mathbb{Z}: \qquad |j_1 - j_2| \leq |j_3| \leq j_1 + j_2$$

$$2)\;\; \text{in other cases}: \qquad j_3 \leq j_1 + j_2$$

Wilson line matrix elements and their asymptotics

- We recall that

$$\mathcal{V}_{j\tilde{j}}(\mathbf{z},\rho) = \sum_{m_1,\dots,m_n} \langle j_1, m_1 |\, I_{j_1\dots j_n|\tilde{j}_1\dots\tilde{j}_{n-3}}\, |j_2, m_2\rangle \otimes |j_n, m_n\rangle \Big(\langle \tilde{a}_1 | j_1, m_1\rangle \, \langle j_2, m_2 | \tilde{a}_2 \rangle \cdots \langle j_n, m_n | \tilde{a}_n\rangle \Big)$$

The x-dependence is in the blue terms only: $\langle \tilde{a} | j, m\rangle = \langle a|\, W_j[0,x]\, |j,m\rangle$ and $\langle j, m|\tilde{a}\rangle = \langle j, m|\, W_j[x,0]\, |a\rangle$

- Denote $q = -ze^{\rho}$. The left and right Wilson matrix elements are given by:

$$\langle j, m|\tilde{a}\rangle \sim e^{+\rho m}\,(q+i)^{j-m}\,{}_2F_1\left(-j, m-j; m+1 \Big| \frac{q-i}{q+i}\right)$$

$$\langle \tilde{a}|j, m\rangle \sim e^{-\rho m}\,(q+i)^{j}(q-i)^{m}\,{}_2F_1\left(-j, m-j; m+1 \Big| \frac{q-i}{q+i}\right)$$

- The radius of convergence of $\langle \tilde{a}|j,m\rangle$ equals one, i.e. $|q| < 1$. In terms of ρ-coordinate one has $\rho < -\log|z|$, which means that for arbitrary z the radius of convergence in ρ goes to zero. Nonetheless, the function can be analytically continued past $|q| = 1$ thereby making the large-ρ expansion possible.

- The asymptotic ($\rho \to \infty$ i.e. $q \to \infty$) Wilson matrix elements:

$$\langle \tilde{a}|j,m\rangle \approx e^{-\rho m} q^{j+m} \sim e^{\rho j}\, z^{j+m} \qquad \langle j,m|\tilde{a}\rangle \approx e^{\rho m} q^{j-m} \sim e^{\rho j}\, z^{j-m}$$

- There are singular vector subleading contributions.

Asymptotic conformal invariance

• Near the boundary the Ward identities for AdS vertex functions go to the Ward identities for CFT correlation functions. One directly finds how the AdS_2 Killing generators are restricted on the boundary:

$$\mathcal{J}_n \langle j, m|\tilde{a}\rangle = \mathcal{L}_n \langle j, m|\tilde{a}\rangle + O(e^{\rho(j-1)})$$

$$\mathcal{J}_n \langle \tilde{a}|j, m\rangle = \mathcal{L}_n \langle \tilde{a}|j, m\rangle + O(e^{\rho(j-1)})$$

where

$$\mathcal{L}_n = z^{n+1}\partial_z - j(n+1)z^n\,, \qquad n = 0, \pm 1$$

which is the standard realization of $sl(2,\mathbb{R})$ on CFT primary fields of conformal dimension $h = -j$.

• Substituting these relations into the AdS Ward identities:

$$\sum_{i=1}^{n} \mathcal{L}_m^{(i)}\, \mathcal{V}_{j\tilde{j}}(x_1, \ldots, x_i, \ldots, x_n)\Big|_{\rho_1 = \ldots = \rho_n = \rho} = 0 + O\big(e^{\rho\left(\sum_{i=1}^{n} j_i - 1\right)}\big)\ ,$$

Taking the limit $\rho \to \infty$ and using the extrapolate dictionary identification one finds out that the above relation goes into the $sl(2,\mathbb{R})$ conformal Ward identities.

• Finite version. Using $SL(2,\mathbb{R}) : z \to w(z)$ yields

$$\mathcal{V}_{j\tilde{j}}(\rho, \mathbf{z}) = \Big(\frac{\partial z}{\partial w}\Big)^{j_1}\Big|_{w=w_1} \cdots \Big(\frac{\partial z}{\partial w}\Big)^{j_n}\Big|_{w=w_n} \mathcal{V}_{j\tilde{j}}(\rho, \mathbf{w}) + O\big(e^{\rho\left(\sum_{i=1}^{n} j_i - 1\right)}\big)$$

Asymptotic cap states

Requiring conformal symmetry of the AdS vertex function only at large-ρ one finds asymptotic equations

$$W_j[x,0](J_0 - j)\,|a\rangle = 0 + O(e^{\rho(j-1)})$$

$$\langle a'|\,(J_0 + j)W_j[0,x] = 0 + \mathcal{O}(e^{\rho(j-1)})$$

These are called quasi-Ishibashi cap states. Note that $|a\rangle$ and $\langle a'|$ are generally two different vectors, $\langle a'| \neq |a\rangle^{\dagger}$.

- For infinite-dimensional modules $\mathcal{D}_j^-$:

$$|a\rangle = |j,j\rangle + (J_0 - j)^{-1}\,\hat{R}\,|j,j\rangle \quad \text{and} \quad \langle a'| = \langle j,j|\,\check{R}\,(J_0 + j)^{-1}$$

  where $\hat{R}, \check{R} \in \mathcal{U}\,(sl(2,\mathbb{R}))$ are some constant elements.

- For finite-dimensional modules $\mathcal{D}_j$ the operator $(J_0 + j)$ has a kernel described by LW vectors (recall that $(\mathcal{D}_j)^* \approx \mathcal{D}_j$) so that in this case the general solution can be represented as

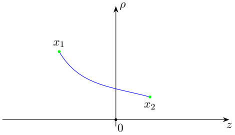
$$|a\rangle = |j,j\rangle + (J_0 - j)^{-1}\,\hat{L}\,|j,j\rangle \quad \text{and} \quad \langle a'| = \langle j,-j| + \langle j,-j|\,\check{L}\,(J_0 + j)^{-1}$$

  with some new constant $\hat{L}, \check{L} \in \mathcal{U}\,(sl(2,\mathbb{R}))$.

- It immediately follows that the LW/HW vectors from (Besken et al 2016) solve the asymptotic equations

$$|j\rangle\rangle = |j,j\rangle \in \mathcal{D}_j\,, \qquad j \in \mathbb{N}_0/2$$

2-point functions



- 2-point AdS vertex function

$$\mathcal{V}_{j_1 j_2}(\rho, \mathbf{z}) = \langle \tilde{a}_1 | \, I_{j_1 j_2} \, | \tilde{a}_2 \rangle = \sum_{m_1, m_2} [I_{j_1 j_2}]^{m_1}{}_{m_2} \, \langle \tilde{a}_1 | j_1, m_1 \rangle \, \langle j_2, m_2 | \tilde{a}_2 \rangle$$

where the the 2-valent intertwiner ($j_3 = 0$) is given by $[I_{j_1 j_2}]^{m_1}{}_{m_2} \sim \delta_{j_1 j_2} \delta^{m_1}{}_{m_2}$. The final expression (see also Castro et all 2018):

$$\mathcal{V}_{j_1 j_2}(\rho, \mathbf{z}) \sim \; q_{12}^{2j_1} \; {}_2F_1 \left(-j_1, -j_1; -2j_1 \,|\, -\frac{4}{q_{12}^2} \right) \Big|_{q_{12} = z_{12} e^{\rho} \to \infty} \;\; \longrightarrow \;\; q_{12}^{2j_1} \sim e^{2\rho j_1} z_{12}^{2j_1} \equiv e^{2\rho j_1} \frac{1}{z_{12}^{2h_1}}$$

- Up to the constant, the 2-point AdS vertex function is the bulk-to-bulk propagator in AdS_2 (Fronsdal 1974) on the $\rho = const$ hyperplane

$$G_h(x_1, x_2) = e^{-h\sigma(x_1, x_2)} \; {}_2F_1 \left(h, \frac{1}{2}; h + \frac{1}{2} \,\Big|\, e^{-2\sigma(x_1, x_2)} \right), \quad e^{\sigma(x_1, x_2)} = \frac{|q_{12}|\sqrt{4 + q_{12}^2} + 2 + q_{12}^2}{2}$$

where $j_1 = j_2 = -h$ and $\sigma(x_1, x_2)$ is the geodesic length between points x_1 and x_2.

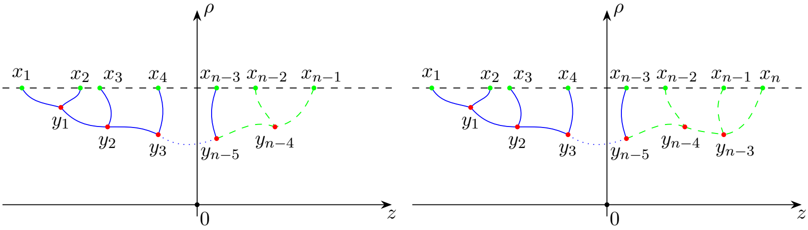
Known results and perspectives

- 2-point AdS vertex and CFT functions (Castro et all 2018, Bhatta et all 2016, Besken et all 2016)
- 3-point and 4-point CFT functions (Bhatta et all 2016, Besken et all 2016)
- 5-point CFT functions (Bhatta et all 2016 for K.A., Belavin 2015)

No exact expressions for higher-point AdS vertex functions (no results) and their near-the-boundary asymptotics (CFT conformal blocks (Rosenhaus 2018))

Recursion for asymptotic AdS vertex functions

Near-the-boundary analysis, $\rho \to \infty$:



The recursion relation:

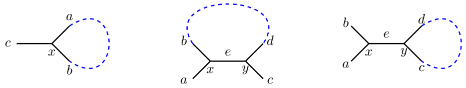
$$\mathcal{V}^{(n)}_{j_1\cdots j_n\,\tilde{j}_1\cdots\tilde{j}_{n-3}} \approx \sum_{k_n} \gamma_{n,k_n,j_n}\, \mathcal{V}^{(n-1)}_{j_1\cdots j_{n-2}(\tilde{j}_{n-3}-k_n)\,\tilde{j}_1\cdots\tilde{j}_{n-4}}\,, \qquad n = 5, 6, \ldots$$

- E.g. Appell F_2 vs Gauss ${}_2F_1$ (splitting identity)
- OPEs ordering in CFT
- Coordinates z_i must organize into cross-ratios χ_j

Wilsonian networks with loops

Toroidal Wilson networks in the thermal AdS

• Let us build torus blocks from the Wilson networks described in the plane topology case by gluing together any two extra edges modulo $2\pi\tau$ and then identifying the corresponding irreps:



• Consider a toroidal 1-point Wilson network:

(1) The toroidal gravitational connection $A = e^{-\rho J_0}\left[J_1 + \frac{1}{4}J_{-1}\right] dz e^{\rho J_0} + J_0 d\rho$.

(2) Identify any two endpoints: $z_1 = -2\pi\tau$ and $z_2 = 0$ lie on the thermal cycle.

(3) Identify $\mathcal{R}_a \cong \mathcal{R}_b$, choose $|a\rangle = |b\rangle = |j_a, m\rangle$ and sum up over all basis states in $\mathcal{R}_a$.

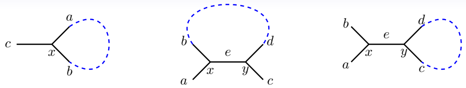
$$\overset{\circ}{V}_{a|c}(\tau, \mathbf{z}) = \sum_m \Big(\langle j_a, m|\, W_a[2\pi\tau, 0]\, I_{a,a,c}\, |j_a, m\rangle\Big)\, W_c[0, z]|c\rangle$$
$$\equiv \mathrm{Tr}_a\Big(W_a[2\pi\tau]\, I_{a;a,c}\Big) W_c[0, z]|c\rangle$$

(a) If $\mathcal{D}_c = 1$ (i.e. $j_c = 0$), then $W_c = 1_c$ and $I_{a;a,0} = 1_a$ so that we find the Wilson loop operator,

$$\overset{\circ}{V}_{a|0}(\tau) = \mathrm{Tr}_a\Big(W_a[2\pi\tau]\Big)$$

It is known to be a character of the representation $\mathcal{D}_a$ (Witten 1988).

(b) For non-trivial $\mathcal{D}_c$ this yields 1-point global torus block (Kraus et al 2017, K.A., Belavin 2020).



The irreps labelled by a, b, c, d are associated with endpoints ordered as z_1, z_2, z_3, z_4.

Wilson network in the t-channel (OPE)

(1) Identify irreps $\mathcal{D}_d \cong \mathcal{D}_c$ and respective endpoints $z_4 = -2\pi\tau$ and $z_3 = 0$.

(2) Choose $|d\rangle = |c\rangle = |j_c, m\rangle$ and sum up over all m to produce a trace over $\mathcal{D}_c$.

$$\overset{\circ}{V}_{(\mathrm{t})\, c,e|a,b}\,(\tau, \mathbf{z}) = \mathrm{Tr}_c\Big(W_c[0, 2\pi\tau]\, I_{c;c,e}\Big) I_{e;a,b} W_a[0, z_1] W_b[0, z_2]\, |a\rangle \otimes |b\rangle$$

Wilson network in the s-channel (necklace)

(1) Fix endpoints as $z_4 = -2\pi\tau$ and $z_2 = 0$.

(2) Identify irreps $\mathcal{D}_d \cong \mathcal{D}_b$ and then sum up over states $|d\rangle = |b\rangle = |j_b, m\rangle$.

$$\overset{\circ}{V}_{(\mathrm{s})\, b,e|a,c}\,(\tau, \mathbf{z}) = \mathrm{Tr}_b\Big(W_b[0, 2\pi\tau]\, I_{b;c,e}\, I_{e;a,b}\Big) W_a[0, z_1]\, W_c[0, z_3]\, |a\rangle \otimes |c\rangle$$

These two AdS vertex functions calculate 2-point global torus blocks in respectively t-channel and s-channel (K.A., Belavin 2020, K.A., Mandrygin 2023, K.A., Khiteev, in progress).

Perspectives:

- HKLL reconstruction for n-point Wilsonian networks (K.A., Khiteev, Kanoda, in progress)
- Wilsonian networks on spaces with defects (2-point Wilson line around BTZ Castro et al 2018 and in BF JT gravity Blommaert, Mertens, Verschelde 2018)
- Large-c CFT on Riemann surfaces of genus g (torus $g = 1$ CFT: K.A., Belavin 2016, 2020, Kraus et al 2017, K.A., Mandrygin 2023, K.A., Khiteev, in progress)
- $1/c$ corrections: general Virasoro conformal blocks via Wilson networks (Besken, Hegde, Kraus, D'Hoker 2017-2019 for quasi-Ishibashi cap states of fin-dim $sl(2,\mathbb{R})$ modules)
- Extended CFT – W_N conformal algebras: adding higher spin fields in lower dimensions (Ammon et al 2013, de Boer et al 2015, Hegde et al 2015, Belavin et all 2022)
- HS gravity in higher dimensions within the unfolded formulation