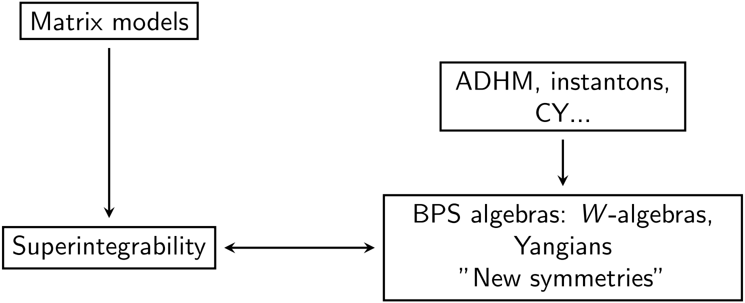
Superintegrability of β-deformed monomial matrix models and Uglov polynomials

Victor Mishnyakov, Ivan Myakutin

Generalities



Superintegrability of matrix models

Matrix models:

$$\left\langle f(X) \right\rangle = \int DX \, e^{V(X)} \, f(X)$$

Superintegrability of matrix models

Matrix models:

$$\left\langle f\left(\mathrm{Tr}\,X^k\right)\right\rangle = \int DX\, e^{\mathrm{Tr}\,V(X)}\, f\left(\mathrm{Tr}\,X^k\right)$$

Superintegrability of matrix models

Matrix models:

$$\left\langle f\left(\operatorname{Tr} X^k\right)\right\rangle = \int DX\, e^{\operatorname{Tr} V(X)} f\left(\operatorname{Tr} X^k\right)$$

As an eigenvalue integral (for the Hermitian case)

$$\left\langle f\left(x\right)\right\rangle = \int \prod_{i=1}^{N} dx_i\, \Delta^2(x) \exp\left(\sum_{i=1}^{N} V(x_i)\right) f\left(x\right)$$

$$\Delta(x) = \prod_{i<j=1}^{N} (x_i - x_j)$$

Superintegrability of matrix models

Sometimes expectation values can be computed exactly [A.Mironov, A.Morozov (2022)]:

$$\Big\langle \text{Character} \Big\rangle \sim \text{Character}$$

Superintegrability of matrix models

Sometimes expectation values can be computed exactly [A.Mironov, A.Morozov (2022)]:

$$\left\langle f(x) \right\rangle \sim f(*)$$

Superintegrability monomial potentials

Among many - monomial matrix models are exactly solvable:

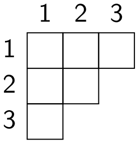
$$\int DX \exp\left(-\operatorname{Tr}\frac{X^s}{s}\right)\ldots$$

Superintegrability monomial potentials

Simplest case - Gaussian matrix model, $\operatorname{Tr} V(X) = -\frac{1}{2}\operatorname{Tr} X^2$:

$$\Big\langle \underbrace{S_R(\operatorname{Tr} X^k)}_{\text{Schur functions}} \Big\rangle = \left(\prod_{(i,j)\in R} (N+j-i)\right) S_R\left\{p_k = \delta_{k,2}\right\} =$$

$$= \underbrace{\frac{S_R\left\{p_k = N\right\}}{S_R\left\{p_k = \delta_{k,1}\right\}} S_R\left\{p_k = \delta_{k,2}\right\}}_{\text{Schur functions}}$$



$$\rightarrow \quad N^2(N+1)^2(N+2)(N-1)(N-2)$$

Superintegrability of monomial potentials

Non-Gaussian models:

$$\underset{?}{\int} DX \exp\left(-\operatorname{Tr}\frac{X^s}{s}\right)\dots$$

Superintegrability of monomial potentials

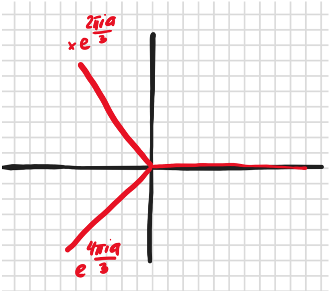
Non-Gaussian models:

$$\int_{C_{s,a}^{\otimes N}} \prod_{i=1}^{N} dx_i \Delta^2(x) \exp\left(-\sum_{i=1}^{N} \frac{x_i^s}{s}\right) \dots$$

$$C_{s,a} = \sum_{b=0}^{s-1} e^{\frac{2\pi iab}{s}} \times \left(0, e^{\frac{2\pi ib}{s}} \infty\right)$$

Superintegrability of monomial potentials

Non-Gaussian models:



Contours in the cubic model

Superintegrability of monomial potentials

Non-Gaussian models:

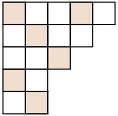
$$\int_{C_{s,a}^{\otimes N}} \prod_{i=1}^{N} dx_i \Delta^2(x) \exp\left(-\sum_{i=1}^{N} \frac{x_i^s}{s}\right) \dots$$

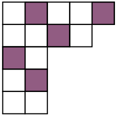
$$\left\langle S_R(x) \right\rangle_{s,a} = \left(\prod_{(i,j)\in R} [[N+j-i]]_{s,a} [[N+j-i]]_{s,0} \right) S_R \left\{ p_k = \delta_{k,s} \right\}$$

where $[[n]]_{s,a} = n$ if $n \mod s = a$ and 1 otherwise. Also, $N = 0, a \mod s$ (for concreteness choose 0)
[C.Cordova, B. Heidenreich, A. Popolitov, S. Shakirov (2018)]

Superintegrability of monomial potentials

Non-Gaussian models:

$[[N+j-i]]_{3,0}$:  $\rightarrow N^3(N+3)(N-3)^2$

$[[N+j-i]]_{3,1}$:  $\rightarrow (N+1)^2(N+4)(N-2)^2$

β-deformation

The Gaussian model is known to survive a deformation of the measure:

$$\Delta^2(x) \quad \to \quad \Delta^{2\beta}(x)$$

This corresponds to:

$$S_R \to J_R \text{ - Jack polynomials}$$

and

$$\left\langle J_R(x) \right\rangle = \quad \underbrace{\frac{J_R\left\{N\right\}}{J_R\left\{\delta_{k,1}\right\}}}_{\prod\limits_{(i,j)\in R}(\beta N + j - \beta i)} \quad J_R\left\{\delta_{k,2}\right\}$$

β-deformation

β-deformation is consistent among many subjects :

- "Temperature" in statistical physics, interpolation between $\beta = 1/2, 1, 2$
- Central charge in CFT $c = 1 - 6\left(\sqrt{\beta} - \frac{1}{\sqrt{\beta}}\right)^2$
- AGT-related equiviariant parameters in Nekrasov functions: $\beta = -\frac{\epsilon_1}{\epsilon_2}$
- Non trivial coupling $g = \beta(\beta - 1)$ in Calogero model. Jack polynomials - eigenfunctions.
- more ...

β-deformation

Is there a β-deformation of Non-Gaussian superintegrability?

β-deformation

Is there a β-deformation of Non-Gaussian superintegrability?

$$\left\langle S_R(x) \right\rangle_{s,a} = \left(\prod_{(i,j)\in R} [[N+j-i]]_{s,a}[[N+j-i]]_{s,0} \right) S_R\left\{p_k = \delta_{k,s}\right\}$$

β-deformation

Naive Guess does not work!

$$\Delta^2 \to \Delta^{2\beta}$$

$$\left\langle J_R(x) \right\rangle_{s,a} = \text{ complete mess}$$

β-deformation

The problem is we are doing an *uneducated guess* but need to do an *educated* guess

We need some symmetry argument. The symmetry appears to be not the *internal* for the matrix model, but the symmetry of it's *moduli space*

β-deformation

We need to cook up a " deformed" partition function

β-deformation

We need to cook up a " deformed" partition function

In QFT terms:

$$Z(T_{O_i}) = \int D\Phi \exp\left(-S[\Phi] + \sum_i \int T_{O_i} O_i[\Phi(x)]\right)$$

β-deformation

We need to cook up a " deformed" partition function

In matrix models:

$$Z(p_k) = \int DX \exp\left(\mathrm{Tr}\, V(X) + \sum_{k=1}^{\infty} \frac{p_k}{k} \mathrm{Tr}\, X^k\right) =$$

$$= \sum_R \left\langle S_R(\mathrm{Tr}\, X^k)\right\rangle S_R(p)$$

β-deformation

Symmetry acts on the p_k parameters. It's main realization is the W-representation:

$$Z(p_k) = e^{W\left[p_k, \frac{\partial}{\partial p_k}\right]} \cdot 1$$

β-deformation

It is intimately connected to superintegrability

$$W_n[g] \cdot S_R(p) = \sum_{\substack{Q: \\ |Q|=|R|+n}} \prod_{(i,j)\in Q/R} g(j-i) \Big\langle p_n S_R \Big| S_Q \Big\rangle S_Q(p)$$

$$\Big\langle S_R(x) \Big\rangle = \left(\prod_{(i,j)\in R} g(j-i) \right) S_R \left\{ p_k = \delta_{k,n} \right\}$$

β-deformation

It is intimately connected to superintegrability

$$W_n[g] \cdot S_R(p) = \sum_{\substack{Q: \\ |Q|=|R|+n}} \prod_{(i,j)\in Q/R} g(j-i) \Big\langle p_n S_R \Big| S_Q \Big\rangle S_Q(p)$$

These operators are nicely β deformed :

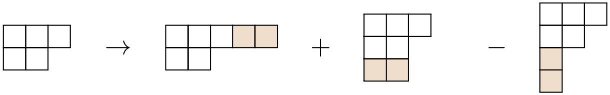
$$W_n[g] \cdot J_R(p) = \sum_{\substack{Q: \\ |Q|=|R|+n}} \prod_{(i,j)\in Q/R} g(j-\beta i) \Big\langle p_n J_R \Big| J_Q \Big\rangle J_Q(p)$$

β-deformation

In the Gaussian model:

$$W_2 S_R = \sum_{R+[2]} (N + j_\square - i_\square)(N + j_\square + 1 - i_\square) S_{R+[2]} -$$

$$- \sum_{R+[1,1]} (N + j_\square - i_\square)(N + j_\square - i_\square - 1) S_{R+[1,1]}$$



β-deformation

β-deformation of W-operators can be deduced from algebra.

For the Gaussian model it corresponds to the deformation:

$$\mathcal{W}_{1+\infty} \quad \longrightarrow \quad Y(\hat{gl}_1)$$

β-deformation

Generically it is known, at least for Schur functions, how to reverse engineer operator from function:

$$\{g(j-i), n\} \quad \longrightarrow \quad W_n[g] \in \mathcal{W}_{1+\infty}$$

We can try to do this for non-Gaussian model. What we find is quite complicated, but the important part is that it corresponds to quantum toroidal DIM algebra, with (q,t) parameters at:

$$(q,t) = (\omega_s, \omega_s)\,, \quad \omega_s = e^{\frac{2\pi i}{s}}$$

β-deformation

$$V(q) = \oint dz : \frac{e^{\varphi(qz)-\varphi(z)}}{q-1} : \quad \text{- at } q = \omega_s$$

$$\varphi(z) = \sum_{k=1}^{\infty} z^{-k} p_k + \sum_{k=1}^{\infty} z^k \frac{1}{k} \frac{\partial}{\partial p_k}$$

$$V(q) S_R = \left(\sum_{(i,j)\in R} q^{j-i} \right) S_R$$

β-deformation

For the DIM algebra the relevant polynomials are Macdonald polynomials.

At $q = t$ Macdonald polynomials always reduce to Schur! They are insensitive to the value of $q = t$.

But the operators and the deformation are!

Hence we should deform from this special point

β-deformation

Finally, the clue! One should take a deformation near the root of unity point in the (q, t) plane.

$$q = \omega_r u \,,\; t = \omega_r u^{\beta} \,, \quad u \to 1$$

β-deformation

This hints us both at the measure and the symmetric functions.

Measure

$$\Delta_{(q,t)}(x) \to \Delta^{(r)}_{\text{Uglov}}(x) = \prod_{i<j}^{N} (x_i^r - x_j^r)^{\frac{2(\beta-1)}{r}} (x_i - x_j)^2$$

Reduce to ordinary β-deformation and Jack polynomials at $r = 1$

β-deformation

This hints us both at the measure and the symmetric functions.

Uglov polynomials:

$$\lim_{\omega_r} M_R(p_k) = U_R^r(p_k)$$

For example:

$$U_{[2]}^{(2)}(p_k) = \frac{\beta p_1^2 + p_2}{\beta + 1} \qquad U_{[2]}^{(1)}(p_k) = \frac{p_1^2 + \beta p_2}{\beta + 1}$$

Reduce to ordinary β-deformation and Jack polynomials at $r = 1$

β-deformation

Just as Jack polynomials, Uglov polynomials are nice objects:

- Eigenfunctions for $\mathfrak{gl}_r$ spin-Calogero system [D.Uglov (1998)]
- Label highest weight vectors in superVirasoro algebra (at $r = 2$) [M.Bershtein, A.Vargulevich (2022)]
- SCFT [V.Belavin, A.Zhakenov (2020)]
- Vectors in representation of $Y(\hat{gl}_r)$ [N.Tselousov, D.Galakhov, A.Morozov (2024)]
- Uglov limit itself is related to ADHM on $\mathbb{C}^2/\mathbb{Z}_r$ [T. Kimura (2012)].

β-deformation of the monomial matrix model

Finally. Combine everything and put $r = s$:

$$\left\langle U_R^{(s)}(x) \right\rangle_{s,a}^{\beta} = \int_{C_{s,a}^{\otimes N}} \prod_{i=1}^{N} dx_i \, \Delta_{\text{Uglov}}^{(s)}(x) \exp\left(-\sum_{i=1}^{N} \frac{x_i^s}{s} \right) U_R^{(s)}(x) =$$

$$= \left(\prod_{(i,j)\in R} [[\beta N + j - \beta i]]_{s,a} [[\beta N + (j-1) - \beta(i-1)]]_{s,0} \right) U_R^{(s)} \{\delta_{k,s}\}$$

Discussion and questions?

- Looking at peculiar symmetries of matrix models we are able to deduce the correct basis for a non-trivial extension of superintegrability
- The type of special functions are tied not only to the measure, but also to the potential
- Can this result be deduced from superintegrability in some (q, t)-model?
- For the Gaussian model we have $r = 2$ and $r = 1$, why is that?
- Somehow we still land at some very special functions
- Limit is related to instanton counting on $\mathbb{C}^2/\mathbb{Z}_r$ orbifold. Is there a well-defined limit and special functions related to the $\mathbb{C}^2/\Gamma_{r,v}$ -orbifold ?

Extra slides

The Gaussian model

For $V(X) = \frac{X^2}{2}$ we actually have two options $r = s = 2$ and $r = 1$.
Both are superintegrable.

Example of superintegrability

$$\left\langle \mathrm{Schur}_R\left(\mathrm{Tr}\,X^k\right)\right\rangle = \frac{\mathrm{Schur}_R\left\{N\right\}}{\mathrm{Schur}_R\left\{\delta_{k,1}\right\}}\,\mathrm{Schur}_R\left\{\delta_{k,2}\right\}$$

Where Schur_R are Schur functions, for example

$$\mathrm{Schur}_{[2]}(p) = \frac{p_2}{2} + \frac{p_1^2}{2}\,,\ p_k = \mathrm{Tr}\,X^k$$

The r.h.s are Schur functions evaluated at special points

$$\frac{\mathrm{Schur}_R\left\{p_k = N\right\}}{\mathrm{Schur}_R\left\{p_k = \delta_{k,1}\right\}} = \prod_{(i,j)\in R}\left(N + j - i\right)$$

Schur polynomials

- Generating function:

$$\exp\left(\sum_{k=1}^{\infty}\frac{z^k p_k}{k}\right)=\sum_{k=1}^{\infty}s_k(p)z^k$$

- Determinant form

$$S_R(p)=\det_{i,j}s_{\lambda_i-i+j}(p)$$

- As symmetric functions:

$$p_k=\sum_{i=1}^{N}x_i^k$$

- As characters fo $GL(N)$:

$$S_R(p_k=\operatorname{Tr}X^k)=\operatorname{Tr}\rho_R(X)$$

Jack polynomials

$$J_{[2]} = \frac{\beta p_1^2 + p_2}{\beta + 1}$$

$$J_{[1,1]} = \frac{1}{2}\left(p_1^2 - p_2\right)$$

$$J_{[3]} = \frac{\beta^2 p_1^3 + 3\beta p_2 p_1 + 2p_3}{(\beta + 1)(\beta + 2)}$$

$$J_{[2,1]} \frac{\beta p_1^3 - \beta p_2 p_1 + p_2 p_1 - p_3}{2\beta + 1}$$