Semi-classicsl quantization for equations with singularities.

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Outline

1. Geometric asymptotics for equations with smooth coefficients (Maslov theory)
   - Spectral problems
   - Cauchy problems
2. Equations with singularities
   - Spectral problems for Schrödinger operator with δ-potential
     - Operator with δ-potential on the surface of revolution
     - Surface of revolution with conic point
   - Cauchy problem for Schrödinger equation with delta-potential
     - Reflection of Lagrangian manifolds
     - Reflection of vector bundles
   - Strictly hyperbolic systems with discontinuous coefficients

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Spectral problem

Spectral problem for the Schrödinger operator.
Let $x \in \mathbb{R}^n$,

$$\hat{H} = H(x, -ih\frac{\partial}{\partial x})$$

$$H(x,p) : \mathbb{R}^{2n} \to \mathbb{R}-$$

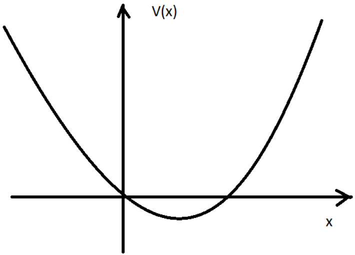
smooth function. Problem: asymptotics of the spectrum as $h \to 0$.

1D example

Let $n = 1$,

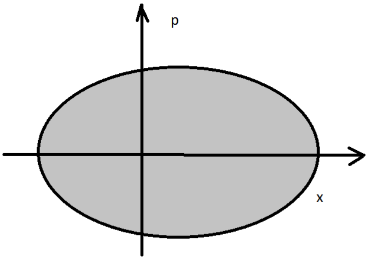
$$\hat{H} = -\frac{h^2}{2}\frac{d^2}{dx^2} + V(x),$$

$V(x) \to +\infty, \quad |x| \to \infty.$


V(x)
x

Λ — curve on the phase plane.

$$\frac{1}{2}p^2 + V(x) = E.$$



Theorem

Let E be solution of the Bohr — Sommerfeld equation

$$\frac{1}{2\pi h}\int_{\Lambda} pdx + \frac{1}{2} = m \in \mathbb{Z}.$$

Then there exists an eigenvalue λ of $\hat{H}$:

$$\lambda = E + o(h).$$

Maslov theory for smooth Hamiltonians

Maslov theory for smooth Hamiltonians.
$\hat{H} = H(x - ih\frac{\partial}{\partial x})$.
Let Λ be compact Lagrangian manifold, invariant with respect to the classical Hamiltonian system in $\mathbb{R}^{2n}$ with the Hamilton function $H(x, p)$.

Theorem (V.P. Maslov)

Let Λ satisfy quantization condition

$$\frac{1}{2\pi h}[\theta] + \frac{1}{4}[\mu] \in H^1(\Lambda, \mathbb{Z})$$

and let $\hat{H}$ be self-adjoint. Then there exists a point λ of the spectrum, such that

$$\lambda = H|_\Lambda + O(h^2).$$

$\theta = \sum_j p_j dx_j$.

$[\mu]$ — Maslov class.

Gauss map $P \in \Lambda \to T_P\Lambda$, $G : \Lambda \to L(n)$, $L(n)$ — Lagrangian Grassmanian, $L(n) = U(n)/O(n)$.
$\det^2 : L(n) \to U(1)$.

$[\mu] = G^*(\det^2)^*[\frac{dz}{2\pi iz}]$

$$\frac{1}{2\pi h}\int_\gamma \theta + \frac{1}{4}\mu(\gamma) = m \in \mathbb{Z}.$$

μ — Maslov index. $\pi : \mathbb{R}^{2n}_{(x,p)} \to \mathbb{R}^n_x$ — natural projection, Σ — cycle of singularities of π.

$$\mu(\gamma) = \gamma \circ \Sigma.$$

Example: integrable Hamiltonian system.
Λ — Liouville tori, I — action variables. Quantization conditions

$$\frac{1}{h}I_j + \frac{1}{4}\mu_j = m_j \in \mathbb{Z}.$$

$$\lambda = H(I(m)) + O(h^2).$$

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Cauchy problem for *h*-pseudodifferential evolutionary equation

$$ih\frac{\partial u}{\partial t} = H(x, -ih\frac{\partial}{\partial x})u, \quad x \in \mathbb{R}^n, h \to +0,$$

$H(x,p) : \mathbb{R}^{2n} \to \mathbb{R}$ is smooth.

$$u|_{t=0} = \varphi^0(x)e^{\frac{iS_0(x)}{h}}, \quad S_0 \in C^\infty, \varphi^0 \in C_0^\infty.$$

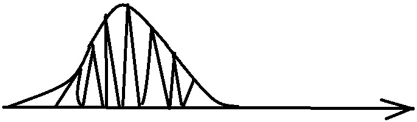

Figure: Wave packet

Solutions, corresponding to Lagrangian manifolds.

Solutions, corresponding to Lagrangian manifolds.
Rapidly oscillating wave packet - S_0 is real. Consider initial Lagrangian surface $\Lambda_0 \subset \mathbb{R}^{2n}_{(x,p)}$, $p = \frac{\partial S_0}{\partial x}$ and shift it by the flow g_t of the classical Hamiltonian system

$$\dot{x} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial x}, \quad \Lambda_t = g_t \Lambda_0.$$

Volume form $\sigma_0 = dx$ on Λ_0, $\sigma_t = g_t^* dx$ on Λ_t

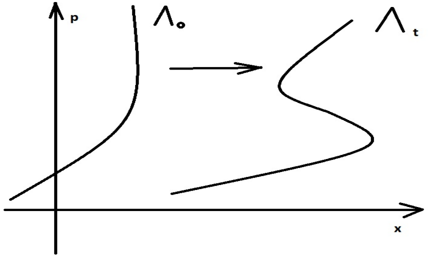

Figure: Lagrangian surface

Theorem

(V.P. Maslov, $\sim$ 1965). Under certain technical conditions the solution $u(x,t,h)$ can be represented as asymptotic series

$$u \sim K_{\Lambda_t,\sigma_t}\Big(\sum_{k=0}^{\infty} h^k \varphi_k\Big),$$

$K : C_0^\infty(\Lambda_t) \to C^\infty(\mathbb{R}^n_x)$ is the Maslov canonical operator, φ_k are smooth functions on Λ_t, $\varphi_0(\alpha) = \varphi^0(g_{-t}\alpha)$.

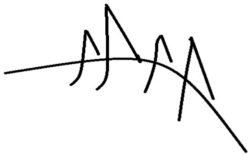

Figure: Squeezed state

Solutions, corresponding to complex vector bundles

Solutions, corresponding to complex vector bundles
Localized ("squeezed") initial state $S_0(x)$ is complex, $\Im S_0 \geq 0$, $\Im S_0 = 0$ on the smooth k-dimensional surface W_0, $d^2 \Im S_0|_{NL_0} > 0$. Consider k-dimensional isotropic surface $\Lambda_0 \subset \mathbb{R}^{2n}$: $x \in W_0, p = \frac{\partial S_0}{\partial x}$ and n-dimensional complex vector bundle ρ_0 over Λ_0 (Maslov complex germ): fiber $\rho(x,p)$ is the plane in ${}^{\mathbb{C}}T_{x,p}\mathbb{R}^{2n}$, $\xi_p = \frac{\partial^2 S_0}{\partial x^2}\xi_x$. Shifted bundle $\Lambda_t = g_t \Lambda_0$, $\rho_t = dg_t \rho_0$.

Theorem (V.P. Maslov)

Under certain technical conditions the solution $u(x, t, h)$ can be represented as asymptotic series

$$u \sim \hat{K}_{\Lambda_t, \rho_t}\left(\sum_{k=0} h^k \varphi_k\right),$$

$\hat{K} : C_0^\infty(\Lambda_t) \to C^\infty(\mathbb{R}_x^n)$ *is the Maslov canonical operator on the complex germ,* φ_k *are smooth functions on* Λ_t*,*
$\varphi_0(\alpha) = \varphi^0(g_{-t}\alpha)$.

Simplest case:

$$S_0 = (p_0, x - x_0) + \frac{1}{2}(x - x_0, Q_0(x - x_0))), \quad p_0 \in \mathbb{R}^n, Q^t = Q, \Im Q > 0.$$

W_0 is the point x_0, $\rho_0 : \xi_p = Q_0 \xi_x$.

$$u(x, t, h) \sim e^{\frac{iS(x,t)}{h}} \sum_{k=0}^{\infty} (h^k \varphi_k(x, t)).$$

$$S = q(t) + (P(t), x - X(t)) + \frac{1}{2}(x - X(t), Q(t)(x - X(t))),$$

$$\dot{X} = \frac{\partial H}{\partial p}, \quad \dot{P} = -\frac{\partial H}{\partial x},$$

Q can be expressed explicitly in terms of solutions of the linearized system.

Problem

What happens if coefficients of initial equation contain singularities?

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1D example

Let $n = 1$,

$$\hat{H} = -\frac{h^2}{2}\frac{d^2}{dx^2} + V(x) + \alpha\delta(x - x_0).$$

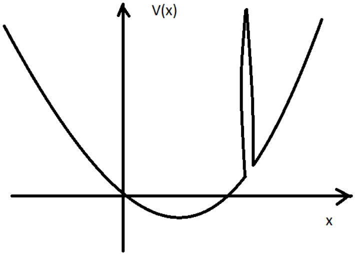
Formal definition:

$$\hat{H}_0 = -\frac{h^2}{2}\frac{d^2}{dx^2} + V(x), \quad x \in \mathbb{R}\backslash x_0.$$

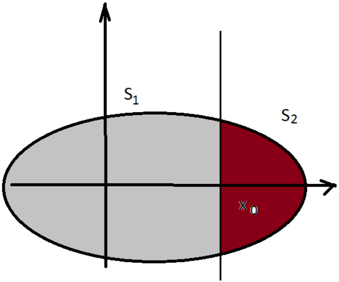
Boundary conditions

$$\psi(x_0 + 0) = \psi(x_0 - 0),$$

$$\psi'(x_0 + 0) - \psi'(x_0 - 0) = \frac{2\alpha}{h^2}\psi(x_0).$$


V(x)
x

$$\frac{1}{2}p^2 + V(x) = E.$$



Theorem

Let E be solution of the equation

$$\cos\left(\frac{1}{2h}(S_1+S_2)\right)=$$

$$=\frac{\alpha}{hp(x_0)}\left(\sin\left(\frac{1}{2h}(S_1+S_2)\right)-\cos\left(\frac{1}{2h}(S_1-S_2)\right)\right).$$

Then there exists an eigenvalue λ of $\hat{H}$:

$$\lambda = E + o(h).$$

Limit cases

$\frac{\alpha}{h} \to 0$,

$$\frac{S_1 + S_2}{2\pi h} + \frac{1}{2} = m \in \mathbb{Z},$$

$\frac{\alpha}{h} \to \infty$,

$$\frac{S_1}{2\pi h} + \frac{1}{4} = m_1 \in \mathbb{Z}, \quad \frac{S_2}{2\pi h} + \frac{3}{4} = m_2 \in \mathbb{Z}.$$

M — Riemannian manifold, $\dim M \leq 3$,

$$\hat{H} = -\frac{h^2}{2}\Delta + \alpha\delta_P$$

Definition of the operator with delta-potential δ_P (Berezin, Faddeev). 2 properties

- $\hat{H}$ is self-adjoint;
- If $\psi(P) = 0$, then $\hat{H}\psi = -\frac{h^2}{2}\Delta\psi$.

Formal definition. $\hat{H}_0 = -\frac{h^2}{2}\Delta|_{\psi\in H^2(M),\psi(P)=0}$.
$\hat{H}$ is a self-adjoint extension of $\hat{H}_0$.

Explicit description of the domain.
For $\psi \in D(\hat{H})$ we have a decomposition

$$\psi = aF(x) + b + o(1),$$

$$F = -\frac{1}{4\pi d(x, P)}, \quad \mathrm{dim} M = 3, \quad F = \frac{1}{2\pi} \log d(x, P), \quad \mathrm{dim} M = 2.$$

Boundary condition

$$a = \frac{2\alpha}{h^2} b.$$

Symmetric manifold

Let M be 2D surface of revolution or 3D spherically symmetric manifold, $M \cong S^2$ or $M \cong S^3$.

$$M \subset \mathbb{R}^3, \quad y = (f(z)\cos\varphi, f(z)\sin\varphi f(z), z)$$

or

$$M \subset \mathbb{R}^4, \quad y = (f(z)\cos\theta\cos\varphi, f(z)\cos\theta\sin\varphi, f(z)\sin\theta, z)$$

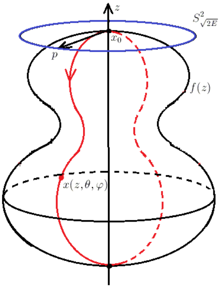
$z \in [z_1, z_2]$,
$f = \sqrt{(z - z_1)(z_2 - z)}w(z)$, w — analytic. Let δ-potential be localized in a pole.

Result: Lagrangian manifold

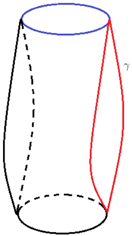
$\Lambda_0 : p \in T^*_P M, \quad |p| = 2E$, $\Lambda = \bigcup_t g_t \Lambda_0$, g_t — geodesic flow.

$\Lambda \cong T^2, \quad \dim M = 2, \quad \Lambda \cong S^2 \times S^1, \quad \dim M = 3.$

Trajectories



Lagrangian manifold



Result: eigenvalues

Theorem (Asilya Suleimanova, Tudor Ratiu, A.S.)

Let E be solution of the equation

$$\tan\left(\frac{1}{2h}\oint_\gamma (p, dx)\right) = \frac{2}{\pi}\left(\log\left(\frac{\sqrt{2E}}{h}\right) + \frac{\pi h^2}{\alpha} + c\right), \quad n = 2,$$

c is Euler constant,

$$\tan\left(\frac{1}{2h}\oint_\gamma (p, dx)\right) = \frac{2h^3}{\sqrt{2E\alpha}}, \quad n = 3.$$

Theorem (Asilya Suleimanova, Tudor Ratiu, A.S.)

Here γ is closed geodesic.
There exists an eigenvalue λ of $\hat{H}$, such that

$$\lambda = E + o(h).$$

Critical values of α.

Jump of the Maslov index
2D-case. Let

$$\frac{\alpha \log 1/h}{h^2} \to 0 \quad or \quad \frac{\alpha \log 1/h}{h^2} \to \infty.$$

Then E up to small terms satisfies

$$\frac{1}{2\pi h} \int_{\gamma} (p, dx) + \frac{1}{2} = m \in \mathbb{Z}.$$

Critical value

$$\alpha \sim \frac{h^2}{\log(1/h)}.$$

Critical values of α.

3D case.
Let $\alpha/h^3 \to 0$. Then E satisfies

$$\frac{1}{2\pi h}\int_\gamma (p, dx) + \frac{1}{2} = m \in \mathbb{Z}.$$

Let $\alpha/h^3 \to \infty$. Then E satisfies

$$\frac{1}{2\pi h}\int_\gamma (p, dx) = m \in \mathbb{Z}.$$

Critical value $\alpha \sim h^3$.

Jump of the Maslov index

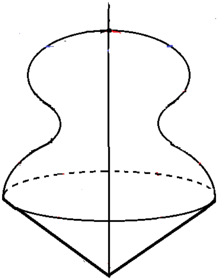
In 3D case the analog of the Maslov index jumps as α passes through the critical value. $\Lambda_0 : p \in T_P^* M, |p| = 2E,$
$F : \Lambda_0 \to \Lambda_0, \quad F(p) = -p$
General formula for big α

$$\frac{1}{2\pi h} \int_\gamma (p, dx) + \frac{1}{4}(\mu(\gamma) + (\deg F - 1)) = m \in \mathbb{Z}.$$

Surface of revolution with conic point.

$$ds^2 = dz^2 + u^2(z)d\varphi^2, \quad z \in [0, L/2]$$

1. $u(z) > 0$ if $z \in (0, L/2)$, $u(0) = u(L/2) = 0$.
2. $z = 0$ is a conic point with total angle $2\pi\beta$ ($\beta > 0$). Near the point $z = 0$ $u(z) = \beta z u_0(z)$, near the point $z = L/2$ $u(z) = (\frac{L}{2} - z)u_1(\frac{L}{2} - z)$, u_0, u_1 — analytic functions, $u_j(0) = 1$.



Spectral problem

$$-\frac{h^2}{2}\Delta\psi = \lambda\psi$$

Domain of the Laplacian.

$$F_0^+ = 1, \quad F_0^- = \log z,$$
$$F_k^{\pm} = \left(\frac{|k|}{\beta}\right)^{-1/2} z^{\pm(\frac{|k|}{\beta})} e^{ik\varphi}, \quad k \in \mathbb{Z}, 0 < |k| < \beta.$$

$$\psi = \sum_k (\alpha_k^+ F_k^+ + \alpha_k^- F_k^-) + \psi_0, \quad \psi_0 = O(z).$$

$$i(I+U)\alpha^- + (I-U)\alpha^+ = 0.$$

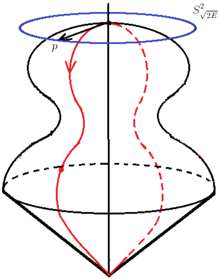
Lagrangian manifold.

$\Lambda_0 : p \in T^*_{x_1} M, \quad |p| = 2E$, x_1 — antipodal of the conic point.

$\Lambda = \bigcup_t g_t \Lambda_0$, g_t — geodesic flow.

$\Lambda \cong T^2$.

γ is closed geodesic.


$S^2_{\sqrt{2E}}$
p

Large harmonics. Fix integer l, $l \geq \beta$.

Theorem (A.S.)

Let E be solution of the equation

$$\frac{1}{2\pi h}\int_\gamma \theta = \frac{l+\beta(l+1)}{2\beta} + m, \quad m \in \mathbb{Z}, \quad m = O(\frac{1}{h}),$$

$$\theta = (p, dx).$$

Then there exist an eigenvalue $\lambda = E + o(h)$.

Small harmonics. *U* does not depend on *h*.

Theorem (A.S.)

Let E be solution of the equation

$$\frac{1}{2\pi h}\int_\gamma \theta = \frac{|k| + \beta(|k|+1)}{2\beta} + m_k \in \mathbb{Z}, \quad |k| \le \beta; \quad k \neq 0,$$

or

$$\frac{1}{2\pi h}\int_\gamma \theta + \frac{1}{2} = m_0 \in \mathbb{Z};$$

Then there exist an eigenvalue $\lambda = E + o(h)$.

- If $\beta < 1$ we have standard Bohr-Sommerfeld equation on Λ.
- Explicit formulae

$$E_k = \frac{4\pi^2 h^2}{L^2}(m_k - \frac{|k| + \beta(|k|+1)}{2\beta})^2, \quad k \neq 0,$$

$$E^{(0)} = \frac{4\pi^2 h^2}{L^2}(m_0 - \frac{1}{2})^2.$$

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$$ih\frac{\partial u}{\partial t} = -\frac{h^2}{2}\Delta u + V(x)u + q(x)\delta_M u, \quad x \in \mathbb{R}^n,$$

$$u|_{t=0} = \varphi^0 e^{\frac{iS_0}{h}}$$

M is a smooth oriented hypersurface, S_0 is real. Boundary conditions on M:

$$u_-|_M = u_+|_M, \quad h\frac{\partial u}{\partial m_-}|_M - h\frac{\partial u}{\partial m_+}|_M = qu|_M$$

Expanded phase space $\mathbb{R}^{2n+2}_{(x,t,p,p_0)}$. Isotropic surface Λ_0: $t = 0, p = \frac{\partial S_0}{\partial x}, H = 0$, $H = p_0 - \frac{1}{2}|p|^2 - V(x)$, Lagrangian manifold $\Lambda^+ = \bigcup_s g_s \Lambda_0$.

Hypersurface $\hat{M} \subset \mathbb{R}^{2n+2}$, $x \in M$. $N^+ = \Lambda \bigcap \hat{M}$. For $x \in M$ let p_τ denote the projection of p to $T_x M$, p_n – normal component. Map $Q : \hat{M} \to \hat{M}$, $Q(x, t, p_\tau, p_n, p_0) = (x, t, p_\tau, -p_n, p_0)$, $N^- = Q(N^+)$. Reflected Lagrangian manifold $\Lambda^- = \bigcup_s g_s N^-$.

Volume form. On Λ_0 we have $\sigma_0 = dx$, construct invariant form on Λ^+: $\sigma^+(\alpha, s) = g_s^*\sigma_0 \wedge ds$. On N^+ consider $i_{p_n}\sigma^+$, map it to N^- and construct invariant form σ^-.

Consider formal series

$$u = K_{\Lambda^+}(\sum_{k=0}^{\infty} h^k \varphi_k^+) + K_{\Lambda^-}(\sum_{k=0}^{\infty} h^k \varphi_k^-)$$

on the negative side of M,

$$u = K_{\Lambda^+}(\sum_{k=0}^{\infty} h^k \varphi_k^*)$$

on the positive side.

$$\varphi_0^*|_{N^+} = \frac{2ip_n}{2ip_n + q}\varphi_0^+|_{N^+}, \quad \varphi_0^-|_{N^-} = \frac{-q}{q + 2ip_n}\varphi_0^+|_{N^+}$$

Theorem (Olga Shchegortsova, A.S.)

This series is asymptotic for the solution of the Cauchy problem for $t \in [0, T]$.

Remark

$$\tau = \frac{2ip_n}{2ip_n + q}, \quad r = \frac{-q}{q + 2ip_n}$$

are the analogs of the coefficients of transmission and reflection.

Complex Lagrangian planes correspond to quadratic forms — matrices $Q^{\pm}$: $\rho : p = Qx$. Rules of reflection:

$$Q^{-}|_{T_M} = Q^{+}|_{T_M} + 2p_n^{+}b,$$

$$< p^{-}, Q^{-}p^{-} > = < p^{+}, Q^{+}p^{+} > + 2p_n^{+}\partial_m(V),$$

$$< p^{-}, Q^{-}r_i > = < p^{+}, Q^{+}r_i >,$$

b is the second fundamental form of M.

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Hyperbolic systems

$$(i\frac{\partial}{\partial t})^m u = A(t, x, i\frac{\partial}{\partial t}, -i\frac{\partial}{\partial x})u,$$
$$x \in \mathbb{R}^n, \quad u \in \mathbb{C}^l, \quad A(t, x, p_0, p) - l \times l \quad matrix.$$

We assume that A is discontinuous on an orientable hypersurface $M^{n-1} \subset \mathbb{R}^n_x$ and smooth outside M, $A = A^{\pm}(t, x, p_0, p)$ at the positive (negative) side of M. Hyperbolicity in Petrovsky sense: equation

$$\det(p_0^m - A_m^{\pm}) = 0$$

has ml real roots $p_0 = H_k^{\pm}(t, x, p)$ and $|H_j - H_k| \geq C|p|$. Initial conditions

$$u|_{t=0} = \varphi^0(x)e^{\frac{iS_0(x)}{h}}, \quad (\frac{\partial}{\partial t})^j u|_{t=0} = 0, \quad j = 1, \ldots m-1$$

Example: wave equation ($m = 2, l = 1$)

$$\frac{\partial^2 u}{\partial t^2} = c^2(x,t)\Delta u$$

$H_k = \pm c|p|$

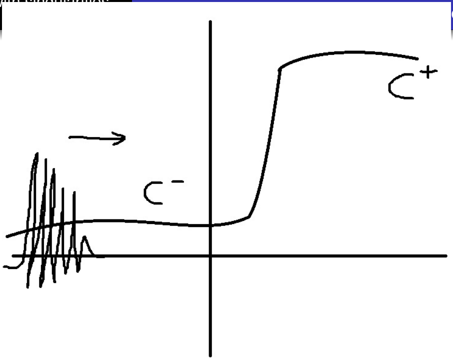

Figure: Scattering

New effects

1. Many reflected and transmitted waves.

2. Total reflection. Transmitted wave can dissapear.

Figure: Total reflection

Lagrangian surfaces, corresponding to incident waves
$\Lambda_k^0 \subset \mathbb{R}^{2n+2}$, $p = \frac{\partial S_0}{\partial x}$, $t = 0$, $p_0 = H_k^-(t, x, p)$,
Hamiltonian systems

$$\dot{x} = \frac{\partial H_k^-}{\partial p}, \quad \dot{p} = -\frac{\partial H_k^-}{\partial x}, \quad \dot{t} = 1, \quad \dot{p}_0 = -\frac{\partial H_k^-}{\partial t},$$

$\Lambda_k = \cup_s g_\pm^s \Lambda_k^0$

Surface $\hat{M} \subset \mathbb{R}^{2n+2}$: $x \in M$, t, p_0, p — arbitrary (the lifting of M to the phase space), $N^2 = \Lambda_1 \bigcap \hat{M}$.

We assume that on the surface N^2, for some $\delta > 0$, $\frac{\partial H_1^-}{\partial p_n} \geq \delta$. ($p_n$ — normal to M component of the vector p).

1. Reflecting roots

$$H_k^-(t, x, p_0, p_\tau, \varkappa) = H_1^-(t, x, p_0, p_\tau, p_n), \quad \frac{\partial H_k^-}{\partial p_n} < 0$$

or

2. Transmitting roots

$$H_k^+(t, x, p_0, p_\tau, \varkappa) = H_1^-(t, x, p_0, p_\tau, p_n), \quad \frac{\partial H_k^+}{\partial p_n} > 0$$

Lemma

(A.I. Allilueva, A.S.) There exists at least one either reflecting or transmitting root

Consider also complex roots; in the first case we choose $\Im\varkappa < 0$, in the second — $\Im\varkappa > 0$.

Lemma

(A.I. Allilueva, A.S.) $\#$ (complex reflecting roots)+$\#$ (complex transmitting roots)=ml.

Proof is based on the study of intersections of a certain line in $\mathbb{R}P^n$ with the Petrovsky surface

$$\Gamma : \det(p_0^m - A_m^{\pm}) = 0$$

Theorem

(I.G. Petrovskii, 1945) $\Gamma = \bigcup_1^{ml/2} \Gamma_j$, *if ml is even,*
$\Gamma = \bigcup_1^{[ml/2]} \Gamma_j \bigcup \Gamma_0$, *if ml is odd.*
$\Gamma_j \cong S^{n-1}, \quad \Gamma_0 \cong \mathbb{R}P^{n-1}$.

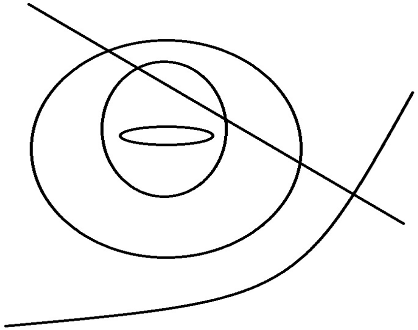

Figure: Petrovsky surface

Reflected and transmitted Lagrangian surfaces

Mappings $Q_k^{\pm} : \hat{M} \to \hat{M}$:

$Q^{\pm}(t, x, p_0, p_\tau, p_n) = (t, x, p_0, p_\tau, \varkappa(t, x, p))$,

$N_k^{\pm} = Q_k^{\pm}(N^2)$. We shift $N_k^{\pm}$ along the trajectories of the Hamiltonian systems with Hamiltonians $H_k^{\pm}$.

$\Lambda_k^{\pm} = \bigcup_{s \in \mathbb{R}} g_{s,k}^{\pm} N^{\pm}$.

Theorem

(A.I. Allilueva, A.S.) During certain time interval

$$u \sim \sum_k K_{\Lambda_k}\Big(\sum_{j=0}^{\infty} h^j \varphi_{j,k}\Big) + \sum_k K_{\Lambda_k^-}\Big(\sum_{j=0}^{\infty} h^j \varphi_{j,k}^-\Big),$$

on the negative part of M,

$$u \sim \sum_k K_{\Lambda_k^+}\Big(\sum_{j=0}^{\infty} h^j \varphi_{j,k}^+\Big)$$

on the positive part of M.

Reflection of vector bundles

Reflection of vector bundles
Rules of reflection

The fibers are positive complex Lagrangian planes – quadratic forms on $T_P\mathbb{R}^n$. On T_PM it is shifted by $p_n b$, where b is the second fundamental form of M, on the pair (m, ξ) — by the value $p_n^{\pm}\partial_\xi(c^{\pm})$, on the pair (m, m) – by $(p_n^{\pm})^2\partial_m(c^{\pm})$.

THANK YOU
FOR YOUR ATTENTION!