Generalization of the Bargmann-Wigner approach to constructing relativistic fields

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Unitary Irreps of $ISO^{\uparrow}(1,3)$ (or its covering $ISL(2,\mathbb{C})$) $\longleftrightarrow$ relativistic particles

Classification of unitary irreps of Poincaré group $ISO^{\uparrow}(1,3)$, and its covering $ISL(2,\mathbb{C})$, was given by $\left[\begin{array}{c} E.Wigner\,(1939, 1947), \\ V.Bargmann, E.Wigner\,(1948) \end{array}\right]$.
Note that unitary irreps of the noncompact group $ISL(2,\mathbb{C})$ are infinite dimensional.

Irreducible representations of covering of the Poincaré group $ISL(2,\mathbb{C})$ are defined (as induced irreps) in the infinite dimensional space of the Wigner-Bargmann wave functions. These functions explicitly do not carry an information about the relativistic equations for the relativistic fields (e.g., Dirac equations for spin $1/2$ fields, or Rarita-Schwinger equations for spin $3/2$ fields).

The aim of our works is to deduce relativistic equations for local fields from the first principles.

The transformation of the Wigner-Bargmann wave functions into the local relativistic fields (corresponding to the unitary irreps of the Poincaré group) are carried out by the special Wigner operators.

The name Wigner operators was introduced by S. Weinberg (1967).

$$\text{WB wave functions } \phi_{\bar{\alpha}}(k) \overset{\textit{Wigner operators}}{\longleftrightarrow} \text{ relativistic fields } \psi_{\bar{\alpha}}(k)$$

All information about relativistic equations for local fields is contained in the Wigner operators. Moreover these operators give the general solutions of the relativistic equations.

In this report we define and study generalized Wigner operators $A_{(k)}$ for massive and massless irreps of $ISL(2,\mathbb{C})$.

As an example, we show how the relativistic equations for local massive/massless relativistic fields are dictated by the form of Wigner operators.

Induced Representations

Let G be a finite group of order n with elements $(g_1, g_2, \ldots, g_n)$. The group G always has n-dimensional real representation $T^{(R)}$: $G \to S_n$. Indeed, left (right) action of a given element $g_i \in G$ on vector $(g_1, g_2, \ldots, g_n)$ gives:

$$g_i \cdot (g_1, \, g_2, \ldots, g_n) = (g_i \cdot g_1, \, g_i \cdot g_2, \ldots, g_i \cdot g_n) = (g_{k_1}, \, g_{k_2}, \ldots, g_{k_n}) \; ,$$

where $(k_1, ..., k_n)$ is a permutation of $(1, 2, ..., n)$, which is given by $(n \times n)$ - matrix $||T^{(R)}_{jk}(g_i)||$:

$$g_i \cdot g_j = g_k \, T^{(R)}_{kj}(g_i) \; . \tag{1}$$

Permutation matrices $||T^{(R)}_{jk}(g_i)||$ contains in each row and each column exactly one element 1 and other elements are 0. Representation $T^{(R)}$ is left regular representations of G.

The definition of $T^{(R)}$ can be modified for infinite groups G. It is done in terms of algebra of functions on G in view of difficulties in defining infinite-dimensional analogs of (1).

Regular representations $T^{(R)}$ for finite group G can be generalized in the following way. Let H be subgroup in G and $\mathrm{ord}(G) = n$, $\mathrm{ord}(H) = m$.
Consider quotient set G/H of left cosets of H in G. We have $\mathrm{ord}(G/H) = n/m = k$.
Let $\mathrm{k}_\alpha|_{\alpha=1,\dots,k} \in G$ be representatives in each coset, and arrange them into a row $(\mathrm{k}_1, \mathrm{k}_2, \dots, \mathrm{k}_k)$. Then the left action of $g \in G$ on $(\mathrm{k}_1, \mathrm{k}_2, \dots, \mathrm{k}_k)$ gives permutation of its components modulo their multiplication on the right by elements of H:

$$g \cdot (\mathrm{k}_1, \dots, \mathrm{k}_k) = (g \cdot \mathrm{k}_1, \dots, g \cdot \mathrm{k}_k) = \Big(\mathrm{k}_{\alpha_1} \cdot h_{(g,\mathrm{k}_1)}, \dots, \mathrm{k}_{\alpha_k} \cdot h_{(g,\mathrm{k}_k)}\Big) \;, \qquad (2)$$

where $(\alpha_1, \dots, \alpha_k)$ is some permutation of $(1, 2, \dots, k)$ and $h_{(g,\mathrm{k}_\alpha)} \in H$. Thus, left action (2) can be written in terms of $(k \times k)$ matrix $||T_{\alpha\beta}(h_{(g,\mathrm{k}_\beta)})||$:

$$g \cdot \mathrm{k}_\beta = \mathrm{k}_\alpha \, T_{\alpha\beta}(h_{(g,\mathrm{k}_\beta)}) \;. \qquad (3)$$

Each row and each column of $||T_{\alpha\beta}(h_{(g,\mathrm{k}_\beta)})||$ contain exactly one non-zero element $h_{(g,\mathrm{k}_\beta)} \in H$. Assume now that Δ is some d-dimensional representation of H, which maps any $h \in H$ into matrix $\Delta(h) \in GL(d, \mathbb{K})$. Then, in accordance with (3), each $g \in G$ is mapped to $(k \times k)$ block matrix $||T_{\alpha\beta}\big(\Delta(h_{(g,\mathrm{k}_\beta)})\big)||$ with $(d \times d)$ blocks.
Mapping T of group G to group $GL(k \cdot d, \mathbb{K})$:

$$g \;\; \to \;\; ||T_{\alpha\beta}\big(\Delta(h_{(g,\mathrm{k}_\beta)})\big)|| \;, \qquad \forall g \in G \;, \qquad (4)$$

defines $(k \cdot d)$-dimensional representation of G, induced from representation of subgroup H.

Let G be Lie group and H its Lie subgroup. Consider coset space G/H and choose a representative $k \in G$ for each coset from G/H. The set of all representatives is a subset K in G, which is in one-to-one correspondence with G/H.
The left action of group G in coset space G/H is written as

$$g \cdot k = \mathrm{k}(g,k) \cdot h(g,k) \ , \qquad \forall k \in K \ , \ \ \forall g \in G \ , \tag{5}$$

where $\mathrm{k}(g,k) \in K$ and $h(g,k) \in H$. The group property of action of G in G/H translates to the properties:

$$\mathrm{k}(g_2, \, \mathrm{k}(g_1,k)) = \mathrm{k}(g_2 \cdot g_1, \, k) \ , \tag{6}$$

$$h(g_2, \, \mathrm{k}(g_1,k)) \cdot h(g_1, \, k) = h(g_2 \cdot g_1, \, k) \ . \tag{7}$$

These are obtained by comparing the r.h.s. of two equivalent expressions

$$g_2 \cdot g_1 \cdot k = g_2 \cdot \mathrm{k}(g_1,k) \cdot h(g_1, \, k) = \mathrm{k}(g_2, \, \mathrm{k}(g_1,k)) \cdot h(g_2, \, \mathrm{k}(g_1,k)) \cdot h(g_1, \, k) \ ,$$

$$g_2 \cdot g_1 \cdot k = \mathrm{k}(g_2 \cdot g_1, \, k) \cdot h(g_2 \cdot g_1, \, k) \ .$$

Relation (6) coincides with the standard formula of G-action. The second eq. (7) does not imply that mapping $h(.,k)$: $G \to H$ (with fixed k) is homomorphism, so it cannot be directly used for constructing representation of G.
It is remarkable, however, that formulas (6) and (7) enable one to make use of the function $h(g,k)$: $G \times K \to H$ for constructing a representation of G out of representation of its subgroup H, which is called induced representation. Let us study this construction in Lie group context (induced representations in the case of finite groups were discussed above.

Let there be a representation of $H \subset G$ in space $\mathcal{V}$, i.e., we have $v \to h \cdot v \in \mathcal{V}$ $(\forall h \in H)$. The induced representation T of group G, which we are going to define, acts in space of functions $v(k)$ $(k \in K)$ taking values in $\mathcal{V}$. In other words, the functions we are talking about are mappings: $K = G/H \to \mathcal{V}$. The definition of T is

$$T(g) \cdot v(k) = \left[h(g^{-1}, k)\right]^{-1} \cdot v(\mathrm{k}(g^{-1}, k)) \equiv v_g(k) \; , \quad \forall g \in G \; . \tag{8}$$

Proposition. The mapping T given in (8) is a homomorphism.
Proof. We write

$$\begin{aligned} T(g_1) \cdot T(g_2) \cdot v(k) = T(g_1) \cdot v_{g_2}(k) = \left[h(g_1^{-1}, k)\right]^{-1} \cdot v_{g_2}(\mathrm{k}(g_1^{-1}, k)) = \\ = \left[h(g_1^{-1}, k)\right]^{-1} \cdot \left[h(g_2^{-1}, \mathrm{k}(g_1^{-1}, k))\right]^{-1} \cdot v\left(\mathrm{k}(g_2^{-1}, \mathrm{k}(g_1^{-1}, k))\right) \; . \end{aligned} \tag{9}$$

On the other hand, we have

$$T(g_1 \cdot g_2) \cdot v(k) = \left[h\left((g_1 \cdot g_2)^{-1}, k\right)\right]^{-1} \cdot v(\mathrm{k}((g_1 \cdot g_2)^{-1}, k)) \; . \tag{10}$$

The two expressions (9) and (10) coincide due to (6) and (7), so that we have $T(g_2) \cdot T(g_1) = T(g_2 \cdot g_1)$. □

Thus, Eq. (8) indeed defines a representation of G, which is representation induced from representation of subgroup $H \subset G$. We note that in (8) we use (see (5))

$$h(g, k) = \mathrm{k}^{-1}(g, k) \cdot g \cdot k \in H \; .$$

Algebra $\mathfrak{iso}(1,3)$ and its Casimirs

To construct the unitary irreps of 4-dimensional Poincaré group $ISO^{\uparrow}(1,3)$ (or its covering $\mathrm{ISpin}^{\uparrow}(1,3) \simeq ISL(2,\mathbb{C})$) we need to consider the coset space $\mathbb{R}^{1,3} \simeq ISL(2,\mathbb{C})/SL(2,\mathbb{C})$ and then reduce subgroup $SL(2,\mathbb{C})$ to the stability subgroup $H \simeq G_q \subset SL(2,\mathbb{C})$, which keeps $q \in \mathbb{R}^{1,3}$ unchanged. For massive case $SL(2,\mathbb{C})/SU(2) \sim$ all point $k \in \mathbb{R}^{1,3}$ such that $k^2 = \mathrm{m}^2$.

We construct the unitary irreps of 4-dimensional Poincaré group $ISO^{\uparrow}(1,3)$, or its covering $\mathrm{ISpin}^{\uparrow}(1,3)$, by considering the corresponding irreps of the Lie algebra $\mathfrak{iso}(1,3) = i\mathrm{spin}(1,3)$ with generators $\hat{P}_n$ and $\hat{M}^{mk}$:

$$[\hat{P}_n,\, \hat{P}_m] = 0\ , \quad [\hat{P}_n,\, \hat{M}_{mk}] = i\,(\eta_{kn}\hat{P}_m - \eta_{mn}\hat{P}_k)\ ,$$

$$[\hat{M}_{nm},\, \hat{M}_{k\ell}] = i(\eta_{nk}\hat{M}_{m\ell} - \eta_{mk}\hat{M}_{n\ell} + \eta_{m\ell}\hat{M}_{nk} - \eta_{n\ell}\hat{M}_{mk})\ ,$$

where $||\eta_{mk}|| = \mathrm{diag}(+1,-1,-1,-1)$ – metric in $\mathbb{R}^{1,3}$.

The Lie algebra $\mathfrak{iso}(1, d-1)$ has $[(d+1)/2]$ Casimir operators, where $[A]$ denote the integer part of A. Thus, the Lie algebra $\mathfrak{iso}(1,3)$ of the d=4 Poincaré group has 2 Casimir operators

$$C_2 = \hat{P}^n \hat{P}_n \; , \quad C_4 = \hat{W}^n \hat{W}_n$$

where $\hat{W}_n = \frac{1}{2}\, \varepsilon_{nmkr}\, \hat{M}^{mk}\, \hat{P}^r$ (ε_{mnkr} – completely skew-symmetric tensor) are components of Pauli-Lubansky vector. These components satisfy

$$\hat{W}_n \hat{P}^n = 0 \, , \quad [\hat{W}_k, \, \hat{P}_n] = 0 \, , \quad [\hat{W}_m, \, \hat{W}_n] = i \, \varepsilon_{mnkr} \hat{W}^k \, \hat{P}^r \, .$$

The values of the Casimir operators $C_2, \; C_4$ characterize the irreps of $\mathfrak{iso}(1,3)$.

 

Classification of the $ISL(2,\mathbb{C})$-irreps:
[E.P.Wigner (1939);
V.Bargmann, E.P.Wigner(1948)]

1. Massive irreps. On the space of states of massive irreps the Casimir operators are proportional to the unite operator I:

$$\hat{P}^n \hat{P}_n = \mathrm{m}^2 \, I \quad (\mathrm{m}^2 > 0), \qquad \hat{W}^n \hat{W}_n = -\mathrm{m}^2 j(j+1) \, I \; ,$$

where I – unit operator, the real number $\mathrm{m} > 0$ is called mass and the real number $j \in \mathbb{Z}_{\geq 0}/2$ is called spin.

2. Massless irreps. The Casimir operators of $iso(1,3)$ are

$$\hat{P}^n \hat{P}_n = \mathrm{m}^2 = 0 \; , \quad \hat{W}^2 = \hat{W}^n \hat{W}_n = -\mu^2 \, I \; .$$

In this case we have two subcases: A. $\mu^2 = 0$ and B. $\mu^2 \neq 0$.

In massless case A, when $\mu^2 = 0$, we obtain

$$\hat{W}^2 = 0, \quad \hat{P}^2 = 0, \quad \hat{P}_n \hat{W}^n = 0 \quad \stackrel{\mathbb{R}^{1,3}}{\Longrightarrow} \quad \hat{W}_n = \hat{\Lambda} \cdot \hat{P}_n \ ,$$

where element $\hat{\Lambda} \in iso(1,3)$ is central and called helicity operator. Its eigenvalues are $\Lambda = 0, \pm 1/2, \pm 1, \pm 3/2, \dots$.
We call these representations as helicity representations.
For example, photon is a particle with two possible states characterized by helicities $\Lambda = \pm 1$.

In massless case B, when $\mu^2 \neq 0$, we have

$$\hat{W}^2 = -\mu^2 \, I \, , \quad \hat{P}^2 = 0, \quad \hat{P}_n \hat{W}^n = 0 \, .$$

These irreps are called the infinite spin (or continues spin) representations.

The covering group $ISL(2,\mathbb{C})$ of the Poincare group $ISO^{\uparrow}(1,3)$ is the set of all pairs (A,X), where $A \in SL(2,\mathbb{C})$, and X belongs to the space $\mathbf{H}$ of Hermitian (2×2) matrices

$$X \equiv (x\,\sigma) = x_0\,\sigma^0 + x_1\,\sigma^1 + x_2\,\sigma^2 + x_3\,\sigma^3 = \begin{pmatrix} x_0+x_3 & x_1-ix_2 \\ x_1+ix_2 & x_0-x_3 \end{pmatrix} \in \mathbf{H}\,.$$

Here $x_m \in \mathbb{R}$, the set of matrices: $\sigma^m = (\sigma^0 = I_2, \sigma^1, \sigma^2, \sigma^3)$, where $\sigma^k|_{k=1,2,3}$ are Pauli matrices, form the basis in $\mathbf{H}$ which is isomorphic to Minkowski space $\mathbf{H} \simeq \mathbb{R}^{\mathbf{1,3}}$. We also need the dual set of σ-matrices:

$$\tilde{\sigma}^m = (\sigma^0, -\sigma^1, -\sigma^2, -\sigma^3) \ .$$

The multiplication in the group $ISL(2,\mathbb{C})$ is given by the formula

$$(A,Y)\cdot(A',X) = (A\cdot A',\ \underline{A\cdot X\cdot A^\dagger} + Y)\ ,$$

with the $SL(2,\mathbb{C})$ group action in the Minkowski space $\mathbf{H} \simeq \mathbb{R}^{1,3}$

$$X \ \to\ X' = A\cdot X\cdot A^\dagger \ \in \mathbf{H}\,.$$

Indeed, if $X = (x\sigma) \in \mathbf{H}$, then X' is also belongs to $\mathbf{H}$ and $X' = (x'\sigma)$.

For basis $\sigma^m \in \mathbf{H}$ and coordinates $x_m \in \mathbb{R}$ we respectively obtain

$$\boxed{A \cdot \sigma^m \cdot A^\dagger = \sigma^k \, \Lambda_k{}^m(A)}\,, \qquad x'_k = \Lambda_k{}^m(A)\, x_m\,,$$

where the (4×4) matrix $||\Lambda^m{}_k(A)|| \in SO^\uparrow(1,3)$ is the image of $A \in SL(2,\mathbb{C})$ under the homomorphism $SL(2,\mathbb{C}) \to SO^\uparrow(1,3)$. By using component notation $X_{\alpha\dot\beta} = x_k \; \sigma^k_{\alpha\dot\beta} \quad (\alpha, \dot\beta = 1,2)$ we write transformation of the basis $\sigma^m \in \mathbf{H}$ as

$$A_\xi{}^\alpha \, A^*_{\dot\gamma}{}^{\dot\beta} \; \sigma^m_{\alpha\dot\beta} = \sigma^k_{\xi\dot\gamma} \; \Lambda_k{}^m(A)\,,$$

We apply doted and undoted indices below.

Let us fix some test momentum q: $q_m q^m = \mathrm{m}^2$, $q_0 > 0$, $\mathrm{m} \geq 0$.

Definition. The *stability subgroup* (*little group*) $G_q \subset SL(2, \mathbb{C})$ of $q \in \mathbb{R}^{1,3}$ is the set of matrices $A \in SL(2, \mathbb{C})$ satisfying condition

$$\boxed{A \cdot (q\sigma) \cdot A^\dagger = (q\sigma)} \quad \Leftrightarrow \quad A_\alpha{}^\gamma \, (q^n \sigma_n)_{\gamma\dot{\alpha}} \, (A^*)_{\dot{\gamma}}{}^{\dot{\alpha}} = (q^n \sigma_n)_{\alpha\dot{\gamma}} \; .$$

- For massive case $\mathrm{m} > 0$, one can take test momentum

$$q = (\mathrm{m}, 0, 0, 0) \; \rightarrow \; (q\sigma) = \mathrm{m} I_2 \quad \Rightarrow \quad A \cdot A^\dagger = I_2 \; ,$$

and $G_q \simeq SU(2)$.

- For massless case $\mathrm{m} = 0$, we take

$$q_m = (E, 0, 0, E) \quad \rightarrow \quad (q\sigma) = \mathrm{diag}(2E, 0)$$

and obtain $G_q \simeq ISO(2)$ for $\mu^2 \neq 0$, or $G_q \simeq SO(2)$ for $\mu^2 = 0$.

The unitary irrep $\mathcal{U}$ of the group $ISL(2,\mathbb{C})$ acts in the spaces of Wigner-Bargmann (WB) wave functions $\phi_{\bar{\alpha}}(k)$ as follows
[E.P.Wigner (1939,1947); V.Bargmann, E.P.Wigner(1948)]

$$[\mathcal{U}(A, x_m\sigma^m)\cdot\phi]_{\bar{\alpha}}(k) \equiv \phi'_{\bar{\alpha}}(k) = e^{ix^m k_m}\sum_{\bar{\beta}} T_{\bar{\alpha}\bar{\beta}}(h_{A,\Lambda^{-1}\cdot k})\,\phi_{\bar{\beta}}(\Lambda^{-1}\cdot k)\,. \tag{11}$$

Here $(A, x_m\sigma^m) \in ISL(2,\mathbb{C})$, 4-vector $k = (k_0, k_1, k_2, k_3)$ denotes the momentum of a particle with mass $\mathrm{m} \geq 0$: $k_n k^n = \mathrm{m}^2$, $\bar{\alpha}$ is a multi-index, matrix $\Lambda \in SO^{\uparrow}(1,3)$ is the image of $A \in SL(2,\mathbb{C})$:

$$A\cdot\sigma^m\cdot A^{\dagger} = \sigma^k\;\Lambda_k{}^m(A)\,,$$

and we use notation $\phi_{\bar{\alpha}}(k)$ for the vector in the space of the unitary irreducible representation T of elements

$$h_{A,\Lambda^{-1}\cdot k} = A^{-1}_{(k)}\cdot A\cdot A_{(\Lambda^{-1}\cdot k)}$$

of stability subgroup $G_q \subset SL(2,\mathbb{C})$. One can directly check that for (11) we have $\mathcal{U}(A,X)\cdot\mathcal{U}(B,Y) = \mathcal{U}\big((A,X)\cdot(B,Y)\big)$.

The crucial point is that $h_{A,\Lambda^{-1}\cdot k} \in G_q$ depends on k. Thus $\phi_{\bar{\alpha}}(k)$ could not be a local field !!!

The stability subgroup element has the factorized form

$$h_{A,\Lambda^{-1}\cdot k} = A_{(k)}^{-1} \cdot A \cdot A_{(\Lambda^{-1}\cdot k)}$$

and its substitution into Wigner represenation (11) gives (for $x = 0$)

$$[\mathcal{U}(A) \cdot \phi]_{\bar{\alpha}}(k) = \sum_{\bar{\beta}} T_{\bar{\alpha}\bar{\beta}}(A_{(k)}^{-1} \cdot A \cdot A_{(\Lambda^{-1}\cdot k)})\, \phi_{\bar{\beta}}(\Lambda^{-1} \cdot k) \tag{12}$$

which is written as

$$[\mathcal{U}(A) \cdot \psi]_{\bar{A}}(k) \equiv \psi'_{\bar{A}}(k) = \sum_{\bar{B}} T_{\bar{A}\bar{B}}(A)\, \psi_{\bar{B}}(\Lambda^{-1} \cdot k)\,. \tag{13}$$

Here we define the local field $\psi_{\bar{A}}(k)$ in the form

$$\psi_{\bar{A}}(k) := \sum_{\bar{\alpha}} T_{\bar{A},\bar{\alpha}}(A_{(k)})\, \phi_{\bar{\alpha}}(k)\,.$$

The transformation (13) is the standard $SL(2,\mathbb{C})$ transformation for local fields. Operators $T_{\bar{A},\bar{\alpha}}(A_{(k)})$ and $A_{(k)}$ are called Wigner operators.

The Wigner operators $A_{(k)}$ contain the whole information about the relativistic equations (and conditions) which reduce the space of local fields $\psi_{\bar{A}}(k)$ into the space of irreps of $ISL(2,\mathbb{C})$ group.

For the test 4-momentum $(q\sigma) = q^n \sigma_n$ and arbitrary 4-momentum $(k\sigma)$, we define operator $A_{(k)} \in SL(2,\mathbb{C})$:

$$\boxed{(k\sigma) = A_{(k)} \cdot (q\sigma) \cdot A^{\dagger}_{(k)}} \quad \Leftrightarrow \quad k_m = (\Lambda_{(k)})_m{}^n \, q_n \; , \qquad (\star)$$

where (4×4) matrix $\Lambda_{(k)} := \Lambda(A_{(k)}) \in SO^{\uparrow}(1,3)$ is the image of $A_{(k)} \in SL(2,\mathbb{C})$. In eq. $(\star)$, the matrix $A_{(k)} \in SL(2,\mathbb{C})$ is fixed up to right multiplication $A_{(k)} \to A_{(k)} \cdot U$ by an element $U \in G_q$:

$$(A_{(k)} \cdot U) \cdot (q\sigma) \cdot (A_{(k)} \cdot U)^{\dagger} = A_{(k)} \cdot (U \cdot (q\sigma) \cdot U^{\dagger}) \cdot A^{\dagger}_{(k)} = (k\sigma) \; .$$

For each k we chose a unique element $A_{(k)} \in SL(2,\mathbb{C})$ and this element numerates the left coset in $SL(2,\mathbb{C})$ with respect to the right action of the subgroup G_q on the elements $A \in SL(2,\mathbb{C})$: $A \;\; \to \;\; A \cdot U$, i.e. $A_{(k)}$ are points in the coset space $SL(2,\mathbb{C})/G_q$.

Recall that the elements $h_{A,\,k}$ of the stability subgroup $G_q \subset SL(2,\mathbb{C})$, which appeared in the E.Wigner irreps (11), are defined as

$$A^{-1}_{(\Lambda\cdot k)} \cdot A \cdot A_{(k)} = h_{A,\,k} \;\in\; G_q \qquad \Rightarrow \qquad A \cdot A_{(k)} = A_{(\Lambda\cdot k)} \cdot h_{A,\,k}\;, \qquad (14)$$

The second relation is justified as

$$(A \cdot A_{(k)})\,(q\sigma)\,(A \cdot A_{(k)})^{\dagger} = A\,(k\sigma)\,A^{\dagger} = (\Lambda \cdot k\,,\sigma) = A_{(\Lambda\cdot k)}\,(q\sigma)\,A^{\dagger}_{(\Lambda\cdot k)}\;.$$

The first relation is visualized as

$$\begin{tikzcd} & k \arrow[d, "A"] \\ q \arrow[ur, "A_{(k)}"] & \Lambda\cdot k \arrow[l, "A^{-1}_{(\Lambda\cdot k)}"] \end{tikzcd}$$

Note, that (in view of second eq. in (14)) the matrices $(A_{(k)})_a{}^{\alpha}$ have two kind of indices: the left $SL(2,\mathbb{C})$-type incoming index a and the right G_q-type outgoing index α.

First we consider the massive case.

In the massive case the test momentum satisfies $(q)^2 = \mathrm{m}^2 > 0$, and the stability subgroup G_q, which is defined as a set of elements A

$$\boxed{(q\sigma) = A \cdot (q\sigma) \cdot A^\dagger},$$

is isomorphic to $SU(2)$, since for $q = (\mathrm{m}, 0, 0, 0)$ we have $(q\sigma) = \mathrm{m} \cdot I_2$.

The Wigner operators $A_{(k)} \in SL(2, \mathbb{C})$

$$\boxed{(k\sigma) = A_{(k)} \cdot (q\sigma) \cdot A^\dagger_{(k)}} \quad \Leftrightarrow \quad k_m = (\Lambda_{(k)})_m{}^n \, q_n \ ,$$

parametrizes points in the coset space

$$A_{(k)} \ \in \ SL(2, \mathbb{C})/SU(2) \ ,$$

and matrices $(A_{(k)})_\alpha{}^a$ have two kind of indices: the left $SL(2, \mathbb{C})$-type index α and the right $SU(2)$-type index a.

In the massive case, the Wigner formula for unitary spin j irreps $\mathcal{U}$ of $ISL(2,\mathbb{C})$ is defined by the following action on the WB wave function $\phi_{\bar{\alpha}}(k)$ of the element $(A,a) \in ISL(2,\mathbb{C})$:

$$[\mathcal{U}(A,a) \cdot \phi]_{\bar{\alpha}}(k) \equiv \phi'_{\bar{\alpha}}(k) = e^{ia^m k_m}\; T^{(j)}_{\bar{\alpha}\bar{\beta}}(h_{A,\Lambda^{-1}\cdot k})\; \phi_{\bar{\beta}}(\Lambda^{-1} \cdot k)\;.$$

Here $T^{(j)}$ – finite-dimensional irrep of the stability subgroup $SU(2)$; we use the concise notation for WB wave function $\phi_{\bar{\alpha}}(k) \equiv \phi_{(\alpha_1 \dots \alpha_{2j})}(k)$. The element (dependent on k)

$$h_{A,\Lambda^{-1}\cdot k} = A^{-1}_{(k)} \cdot A \cdot A_{(\Lambda^{-1}\cdot k)} \;\; \in \;\; SU(2)\;, \tag{15}$$

belongs to the stability subgroup $SU(2) \subset SL(2,\mathbb{C})$ and the matrix $\Lambda \in SO^{\uparrow}(1,3)$ is related to $A \in SL(2,\mathbb{C})$ in standard way.
We note that, since $h_{A,\Lambda^{-1}\cdot k} \in SU(2)$, Eq. (15) is written in the equivalent form

$$h_{A,\Lambda^{-1}\cdot k} = h^{\dagger -1}_{A,\Lambda^{-1}\cdot k} = A^{\dagger}_{(k)} \cdot A^{\dagger -1} \cdot A^{\dagger -1}_{(\Lambda^{-1}\cdot k)}\;. \tag{16}$$

As we mentioned above, the element $h_{A,\Lambda^{-1}\cdot k}$ depends on k and, in the coordinate representation, the BW wave function $\phi_{\bar{\alpha}}(k)$ can not be a local field!!! How do we solve this problem in the massive case?

In the representation $T^{(j)}$, the element $h \in SU(2)$ is the matrix, which can be written in the factorized form ($p + r = 2j$)

$$T^{(j)}_{\bar{\beta}\bar{\alpha}}(h) = \left(h^{\otimes(p+r)}\right)_{\bar{\beta}\bar{\alpha}} = \left[h_{\beta_1}^{\ \alpha_1} \cdots h_{\beta_p}^{\ \alpha_p} \cdot h_{\beta_{p+1}}^{\ \alpha_{p+1}} \cdots h_{\beta_{p+r}}^{\ \alpha_{p+r}}\right] =$$
$$= \left[h_{\beta_1}^{\ \alpha_1} \cdots h_{\beta_p}^{\ \alpha_p} \cdot \left(h^{\dagger-1}\right)_{\beta_{p+1}}^{\ \alpha_{p+1}} \cdots \left(h^{\dagger-1}\right)_{\beta_{p+r}}^{\ \alpha_{p+r}}\right],$$

where r factors are chosen as $h \ \rightarrow \ h^{\dagger-1}$, since $h \in SU(2)$.
Then, we use the factorized forms (15), (16) and write the matrix $T^{(j)}_{\bar{\beta}\bar{\alpha}}\left(h_{A,\Lambda^{-1}\cdot k}\right)$ in the factorized form

$$T^{(j)}\left(h_{A,\Lambda^{-1}\cdot k}\right) = \left(A^{-1}_{(k)} \cdot A \cdot A_{(\Lambda^{-1}\cdot k)}\right)^{\otimes p} \otimes \left(A^{\dagger}_{(k)} \cdot A^{\dagger-1} \cdot A^{\dagger-1}_{(\Lambda^{-1}\cdot k)}\right)^{\otimes r} =$$
$$= \left(A^{-1\ \otimes p}_{(k)} \otimes A^{\dagger\ \otimes r}_{(k)}\right) \cdot \left(A^{\otimes p} \otimes A^{\dagger-1\ \otimes r}\right) \cdot \left(A^{\otimes p}_{(\Lambda^{-1}\cdot k)} \otimes A^{\dagger-1\ \otimes r}_{(\Lambda^{-1}\cdot k)}\right)$$

and for the Wigner formula we have $\phi'(k) = T^{(j)}\left(h_{A,\Lambda^{-1}\cdot k}\right) \phi(\Lambda^{-1}\cdot k) =$

$$= \left(A^{\otimes p}_{(k)} \otimes A^{\dagger-1\ \otimes r}_{(k)}\right)^{-1} \cdot \left(A^{\otimes p} \otimes A^{\dagger-1\ \otimes r}\right) \cdot \underbrace{\left(A^{\otimes p}_{(\Lambda^{-1}\cdot k)} \otimes A^{\dagger-1\ \otimes r}_{(\Lambda^{-1}\cdot k)}\right) \phi(\Lambda^{-1}\cdot k)} =$$
$$= \left(A^{\otimes p}_{(k)} \otimes A^{\dagger-1\ \otimes r}_{(k)}\right)^{-1} \cdot \left(A^{\otimes p} \otimes A^{\dagger-1\ \otimes r}\right) \quad \cdot \quad \psi^{(r)}(\Lambda^{-1}\cdot k)$$

Here we introduce (instead of the Wigner WFs $\phi_{(\delta_1 \dots \delta_{p+r})}(k)$) spin-tensor fields of $(\frac{p}{2}, \frac{r}{2})$-type (with r dotted and p undotted indices):

$$\psi^{(r)(\dot\beta_1 \dots \dot\beta_r)}_{\ \ (\alpha_1 \dots \alpha_p)}(k) := \left[[A_{(k)}^{\otimes p} \otimes (A_{(k)}^{\dagger -1})^{\otimes r}] \cdot \phi(k) \right]^{(\dot\beta_1 \dots \dot\beta_r)}_{(\alpha_1 \dots \alpha_p)} =$$

$$= (A_{(k)})_{\alpha_1 \dots \alpha_p}^{\ \ \delta_1 \dots \delta_p} \cdot (A_{(k)}^{-1\dagger})^{\dot\beta_{p+1} \dots \dot\beta_{p+r}; \delta_{p+1} \dots \delta_{p+r}} \phi_{(\delta_1 \dots \delta_p \delta_{p+1} \dots \delta_{p+r})}(k) \ ,$$

where we use concise notation

$$(A_{(k)})_{\alpha_1 \dots \alpha_p}^{\ \ \delta_1 \dots \delta_p} = (A_{(k)})_{\alpha_1}^{\ \ \delta_1} \cdots (A_{(k)})_{\alpha_p}^{\ \ \delta_p} \ ,$$

$$(A_{(k)}^{-1\dagger} \cdot (q\tilde\sigma))^{\dot\beta_1 \dots \dot\beta_r; \delta_1 \dots \delta_r} = (A_{(k)}^{-1\dagger} \cdot (q\tilde\sigma))^{\dot\beta_1 \delta_1} \cdots (A_{(k)}^{-1\dagger} \cdot (q\tilde\sigma))^{\dot\beta_r \delta_r}$$

and we restore the case of the arbitrary test momentum q.
The upper index (r) of the spin-tensors $\psi^{(r)}$ distinguishes these spin-tensors with respect to the number of dotted indices.

Definition. The operators $A_{(k)}^{\otimes p} \otimes \left(\frac{1}{m} A_{(k)}^{\dagger -1}(q\tilde\sigma)\right)^{\otimes r}$ which convert Wigner wave functions $\phi(k)$ into spin-tensor fields $\psi^{(r)}(k)$ of $(\frac{p}{2}, \frac{r}{2})$-type are called *the Wigner operators*.

Proposition 1. The $ISL(2,\mathbb{C})$-representation $\mathcal{U}$ is written for fields $\psi^{(r)}$ as following

$$[\mathcal{U}(A,a)\cdot\psi^{(r)}]^{(\dot\beta_1\ldots\dot\beta_r)}_{(\alpha_1\ldots\alpha_p)}(k) =$$
$$= e^{ia^m k_m}\left[A_{\alpha_1\ldots\alpha_p}^{\ \gamma_1\ldots\gamma_p}\ \left(A^{\dagger-1}\right)^{\dot\beta_1\ldots\dot\beta_r}_{\ \dot\kappa_1\ldots\dot\kappa_r}\right]\ \psi^{(r)(\dot\kappa_1\ldots\dot\kappa_r)}_{(\gamma_1\ldots\gamma_p)}(\Lambda^{-1}\cdot k)\ ,$$

where $A^{\cdots}_{\cdots}\,(A^{\dagger-1})^{\cdots}_{\cdots} = [A^{\otimes p}\otimes(A^{\dagger-1})^{\otimes r}]^{\cdots}_{\cdots}$, $A\in SL(2,\mathbb{C})$.

Thus, in the coordinate representation, the functions $\psi^{(r)(\dot\beta_1\ldots\dot\beta_r)}_{(\alpha_1\ldots\alpha_p)}(k)$ are local relativistic fields.

Proposition 2. *The wave functions $\psi^{(r)}$ satisfy the Dirac-Pauli-Fierz (DPF) equations [P.A.M.Dirac (1936), M. Fierz and W. Pauli (1939)]:*

$$k^n(\tilde{\sigma}_n)^{\dot{\gamma}_1\alpha_1}\psi^{(r)(\dot{\beta}_1\dots\dot{\beta}_r)}_{(\alpha_1\dots\alpha_p)}(k) = \mathrm{m}\,\psi^{(r+1)(\dot{\gamma}_1\dot{\beta}_1\dots\dot{\beta}_r)}_{(\alpha_2\dots\alpha_p)}(k)\,, \quad (r=0,\dots,2j-1)\,,$$
$$k^n(\sigma_n)_{\gamma_1\dot{\beta}_1}\psi^{(r)(\dot{\beta}_1\dots\dot{\beta}_r)}_{(\alpha_1\dots\alpha_p)}(k) = \mathrm{m}\,\psi^{(r-1)(\dot{\beta}_2\dots\dot{\beta}_r)}_{(\gamma_1\alpha_1\dots\alpha_p)}(k)\,, \quad (r=1,\dots,2j)\,,$$

which describe the dynamics of a massive relativistic particle with spin $j=(p+r)/2$. The compatibility conditions for the system of DPF eqs are given by the mass shell relations $(k^n k_n - m^2)\,\psi^{(r)}(k) = 0$.

Proof. Use the definitions of matrices $A_{(k)} \in SL(2,\mathbb{C})/SU(2)$:

$$(k\tilde{\sigma})\cdot A_{(k)} = A^{\dagger-1}_{(k)}\cdot(q\tilde{\sigma}) \quad \Rightarrow \quad (k\tilde{\sigma})\cdot A_{(k)} = \mathrm{m}\,A^{\dagger-1}_{(k)}\,,$$
$$(k\sigma)\cdot A^{\dagger-1}_{(k)} = A_{(k)}\cdot(q\sigma) \quad \Rightarrow \quad (k\sigma)\cdot A^{\dagger-1}_{(k)} = \mathrm{m}\,A_{(k)}\,,$$

where we use the test momentum frame $(q\tilde{\sigma}) = (q\sigma) = \mathrm{m}\,I_2$.

Examples. **1**.) For the case of spin $j = 1/2$ we obtain $(p + r) = 1$ and this case is described by two local fields $\psi^{(0)}_{\alpha}(k)$ and $\psi^{(1)\dot{\beta}}(k)$. From Proposition 2 we obtain Weyl eqs.

$$k^m (\tilde{\sigma}_m)^{\dot{\gamma}\alpha} \, \psi^{(0)}_{\alpha}(k) = \mathrm{m} \, \psi^{(1)\dot{\gamma}}(k) \, , \quad k^m (\sigma_m)_{\gamma\dot{\beta}} \, \psi^{(1)\dot{\beta}}(k) = \mathrm{m} \, \psi^{(0)}_{\gamma}(k) \, ,$$

which are combined into Dirac equation for bispinor $\Psi^{\mathrm{T}} = (\psi^{(0)}_{\alpha}(k), \psi^{(1)\dot{\beta}}(k))$ and describe the spin $\frac{1}{2}$ particles:

$$k^m \, \gamma_m \, \Psi = \mathrm{m} \, \Psi \, , \qquad \gamma_m = \begin{pmatrix} 0 & \sigma^m \\ \tilde{\sigma}^m & 0 \end{pmatrix}$$

2.) For the case of spin $j = 1$ we have $(p + r) = 2$ and this case is described by 3 local fields $\psi^{(0)}_{\alpha_1\alpha_2}(k)$, $\psi^{(1)\dot{\beta}}_{\alpha}(k)$ and $\psi^{(2)\dot{\beta}_1\dot{\beta}_2}(k)$. For the vector field $A_n = (\tilde{\sigma}_n)^{\dot{\beta}\alpha} \psi^{(1)}_{\alpha\dot{\beta}}(k)$, from Proposition 2 we deduce Proca equations.

For details see: [A.P. Isaev, M.A. Podoinitsyn, Nucl. Phys. B **929** (2018) 452, arXiv:1712.00833].

In the case of $p + r = 2j$, the system of spin-tensor wave functions $\psi^{(r)}$ which obey the Dirac-Pauli-Fierz equations describes relativistic particles with spin j.

Proposition 3. Spin-tensor wave functions $\psi^{(r)(\dot{\beta}_1 \dots \dot{\beta}_r)}_{(\alpha_1 \dots \alpha_p)}(k)$ of type $(\frac{p}{2}, \frac{r}{2})$, which obey the Dirac-Pauli-Fierz equations, automatically satisfy the equations

$$[(\hat{W}^m \hat{W}_m)\, \psi]^{(r)(\dot{\beta}_1 \dots \dot{\beta}_r)}_{(\alpha_1 \dots \alpha_p)}(k) = -\mathrm{m}^2 j(j+1)\, \psi^{(r)(\dot{\beta}_1 \dots \dot{\beta}_r)}_{(\alpha_1 \dots \alpha_p)}(k) \,,$$

where $j = (\frac{p}{2} + \frac{r}{2})$, $\hat{W}_m$ are the components of the Pauli-Lubanski vector

$$\hat{W}_m = \frac{1}{2} \varepsilon_{mnij} M^{ij} P^n = \frac{1}{2} \varepsilon_{mnij} \hat{\Sigma}^{ij} P^n \,,$$

and $\hat{W}_m \hat{W}^m$ is the Casimir operator for the group $ISL(2, \mathbb{C})$; $\hat{\Sigma}^{ij}$ — spin part of M^{ij}.

The matrices $A_{(k)}$ numerate points of the coset space $SL(2,\mathbb{C})/SU(2)$. The left action of the group $SL(2,\mathbb{C})$ on $SL(2,\mathbb{C})/SU(2)$ is

$$A \cdot A_{(k)} = A_{(\Lambda \cdot k)} \cdot U_{A,k} \; , \qquad A \in SL(2,\mathbb{C}) \; , \;\; \Lambda \in SO^{\uparrow}(1,3) \; ,$$

where $A\sigma^k A^\dagger = \Lambda^k_m \sigma^m$ and $U_{A,k} \in SU(2)$. Under this left action of $A \in SL(2,\mathbb{C})$, two columns of the matrix $A_{(k)}$ are transformed as two Weyl spinors. Therefore, it is convenient to represent the matrix $A_{(k)}$ in the following way:

$$(A_{(k)})_\alpha{}^\beta = \frac{1}{(\mu^\rho \, \lambda_\rho)^{1/2}} \begin{pmatrix} \mu_1 & \lambda_1 \\ \mu_2 & \lambda_2 \end{pmatrix} \;\Rightarrow\; (A^{\dagger -1}_{(k)})^{\dot\alpha}{}_{\dot\beta} = \frac{1}{(\overline{\mu}^{\dot\rho} \, \overline{\lambda}_{\dot\rho})^{1/2}} \begin{pmatrix} \overline{\lambda}_{\dot 2} & -\overline{\mu}_{\dot 2} \\ -\overline{\lambda}_{\dot 1} & \overline{\mu}_{\dot 1} \end{pmatrix} ,$$

This representation is convenient to write explicitly the polarization vectors for higher spin fields.

Massless irreps. Stability subgroup.

For massless irreps we choose the test vector $\overset{\circ}{p} \in \mathbb{R}^{1,3}$ as follows

$$||\overset{\circ}{p}_\nu|| = (\overset{\circ}{p}_0, \overset{\circ}{p}_1, \overset{\circ}{p}_2, \overset{\circ}{p}_3) = (E, 0, 0, E) \tag{17}$$

By definition, the finite-dimensional Wigner operators are the matrices $A_{(p)} \in SL(2, \mathbb{C})$ that transform the test momentum $\overset{\circ}{p}$ into an arbitrary momentum p

$$A_{(p)}(\overset{\circ}{p}\,\sigma)A^\dagger_{(p)} = (p\,\sigma)\,, \tag{18}$$

where $(p\,\sigma) := p_\mu \sigma^\mu$. The stability subgroup $G_{\overset{\circ}{p}}$ of $\overset{\circ}{p}$ is formed by the set of matrices $h \in SL(2, \mathbb{C})$ that preserve $\overset{\circ}{p}$:

$$h(\overset{\circ}{p}\,\sigma)h^\dagger = (\overset{\circ}{p}\,\sigma)\,, \tag{19}$$

Equation (19) has the following general solution

$$h = \begin{pmatrix} e^{\frac{i}{2}\theta} & e^{-\frac{i}{2}\theta}\,\mathbf{b} \\ 0 & e^{-\frac{i}{2}\theta} \end{pmatrix} = \begin{pmatrix} 1 & \mathbf{b} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^{\frac{i}{2}\theta} & 0 \\ 0 & e^{-\frac{i}{2}\theta} \end{pmatrix}\,, \tag{20}$$

where $\theta \in [0, 2\pi]$ and $\mathbf{b} = b_1 + ib_2$.

The matrices (20) form the $ISO(2)$ group, i.e. $G_{\overset{\circ}{p}} \cong ISO(2)$ and stability group is not compact. The generators of $ISO(2)$:

$$\hat{R} = -\frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \hat{T}_1 = \begin{pmatrix} 0 & i \\ 0 & 0 \end{pmatrix}, \quad \hat{T}_2 = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}, \tag{21}$$

satisfy the defining relations for the real algebra $\mathfrak{iso}(2)$

$$[\hat{T}_1, \hat{T}_2] = 0, \quad [\hat{R}, \hat{T}_a] = i\varepsilon_{ad}\hat{T}_d \ . \tag{22}$$

An element $h(\theta, \vec{b})$ of the $ISO(2)$ group can be written as a product

$$h(\theta, \vec{b}) = T(\vec{b}) \cdot R(\theta) \equiv e^{-ib_a\hat{T}_a} e^{-i\theta\hat{R}} \ , \tag{23}$$

where $T(\vec{b})$ is the element of the translation subgroup of $ISO(2)$ and $R(\theta)$ is the element of $SO(2) \subset ISO(2)$. Here $\vec{b} = (b_1, b_2) \in \mathbb{R}^2$, and $\theta \in [0, 2\pi)$ is the angular variable.

The unitary irreps of $ISO(2)$ group is given by the relation

$$\Phi'(\varphi) \;=\; \left[\mathcal{U}(h(\theta,\vec{b}))\,\Phi\right](\varphi) \;=\; e^{-i\,\vec{b}\cdot\vec{t}_\varphi}\Phi(\varphi-\theta)\,. \qquad (24)$$

It is also convenient to use another discrete basis $|\,n\,\rangle$, $n\in\mathbb{Z}$, in the space of the unitary irrep of $ISO(2)$ group. In this basis, the generator $\hat{R}$ is diagonal while the generators of T_a are not diagonal

$$\hat{R}|\,n\,\rangle = n|\,n\,\rangle\,, \qquad T_\pm|\,n\,\rangle = \rho|\,n\pm 1\,\rangle\,. \qquad (25)$$

The set of vectors $|\,n\,\rangle$ are orthogonal and complete $\langle\,n\,|\,m\,\rangle = \delta_{nm}$, $\sum_{n=-\infty}^{\infty}|\,n\,\rangle\langle\,n\,| = 1$. The function $\langle\,\varphi\,|\,n\,\rangle$ relating basis vectors $|\,n\,\rangle$ and $|\,\varphi\,\rangle$ is $\langle\,\varphi\,|\,n\,\rangle = \frac{1}{\sqrt{2\pi}}\,e^{in\varphi}$. Thus the wave function $\Phi(\varphi)$ is expanded as a Fourier series ($\Phi_n := \langle\,n\,|\,\Phi\,\rangle/\sqrt{2\pi}$)

$$\Phi(\varphi) = \langle\,\varphi\,|\,\Phi\,\rangle = \sum_{n=-\infty}^{\infty}\langle\,\varphi\,|\,n\,\rangle\langle\,n\,|\,\Phi\,\rangle = \sum_{n=-\infty}^{\infty}\Phi_n e^{in\varphi}\,. \qquad (26)$$

In the massless case, Wigner induced unitary representations of the group $SL(2,\mathbb{C})$ realized on the Wigner functions $\Phi(p,\varphi)$ are constructed according to (24) have the form:

$$\begin{aligned}\Phi'(p,\varphi) \; &:= \; [U(A)\Phi](p,\varphi) = \sum_{\varphi'} \mathcal{D}_{\varphi\varphi'}(\theta_{A,\Lambda^{-1}p},\vec{b}_{A,\Lambda^{-1}p})\,\Phi(\Lambda^{-1}p,\varphi')\,, \\ &= \; e^{-i\vec{b}_{A,\Lambda^{-1}p}\cdot\vec{t}_{\varphi}}\,\Phi(\Lambda^{-1}p,\varphi-\theta_{A,\Lambda^{-1}p})\,.\end{aligned} \tag{27}$$

The transformations $\mathcal{D}_{\varphi\varphi'}(\theta_{A,\Lambda^{-1}p},\vec{b}_{A,\Lambda^{-1}p})$ of the Wigner functions depend on the momentum variable p_μ. We assume that the Lorentz-covariant field $\Psi(p,y)$, which describes massless particles, is constructed from WF $\Phi(p,\varphi)$ via integral transformation

$$\Psi(p,\eta) \; = \; \int_0^{2\pi} d\varphi\,\mathcal{A}(p;\eta,\varphi)\,\Phi(p,\varphi)\,, \tag{28}$$

where η is the set of auxiliary variables. We consider two cases: $\eta=(\eta_0,\eta_1,\eta_2,\eta_3)\in\mathbb{R}^{1,3}$, or $\eta=(u,\bar{u})$ – couple of 2 spinors. The kernel $\mathcal{A}(p;\eta,\varphi)$ plays the role of the Wigner operator, which is an infinite-dimensional analogue of $A_{(p)}$ from (18).
In the kernel $\mathcal{A}(p;\eta,\varphi)$ the variables η and φ plays the role of the $SL(2,\mathbb{C})$-type and $ISO(2)$-type continues indices.

Let the relativistic field $\Psi(p,\eta)$ given in (28) be transformed under the action of the Lorentz group in the standard way:

$$\Psi'(p,\eta) \;=\; [U(A)\Psi](p,\eta) \;=\; \Psi(\Lambda^{-1}p,\Lambda^{-1}\eta)\,, \qquad (29)$$

where the matrices $A \in SL(2,\mathbb{C})$ and $\Lambda \in SO^{\uparrow}(1,3)$ are related by standard way.

Knowing the explicit form of the unitary Lorentz transformation (27) of WF $\Phi(p,\varphi)$ and the corresponding transformation (29) of the field $\Psi(p,\eta)$, we find the equations that determine the kernel $\mathcal{A}(p,\eta,\varphi)$ of the Wigner operator.

As a result we obtain the following equation for the kernel $\mathcal{A}(p,\eta,\varphi)$:

$$\mathcal{A}(\Lambda^{-1}p,\Lambda^{-1}\eta,\varphi) \;=\; e^{-i\vec{b}_{A,\Lambda^{-1}p}\vec{t}_{\varphi+\theta_{A,\Lambda^{-1}p}}}\mathcal{A}(p,\eta,\varphi+\theta_{A,\Lambda^{-1}p})\,, \qquad (30)$$

which is the ”massless” analog of the equation

$$A\,A_{(k)} = A_{(\Lambda\cdot k)}\,h_{A,k}\;,$$

1. Non-singular solution

The expression for the kernel $\mathcal{A}(p, \eta, \varphi)$ is given by

$$\mathcal{A}(p,\eta,\varphi) = e^{\,i\mu\,\eta\cdot\varepsilon_{(1)}(\varphi)/(\eta\cdot p)}\, f(\eta\cdot\eta, \eta\cdot p)\,, \tag{31}$$

where we introduced the mass dimensional constant

$$\mu := E\,\rho\,, \tag{32}$$

$f((\eta)^2,\ \eta\cdot p)$ is an arbitrary function, and we also introduced two additional 4-vectors

$$\overset{\circ}{\varepsilon}_{(1)}(\varphi) := \overset{\circ}{\varepsilon}_{(1)}\cos\varphi - \overset{\circ}{\varepsilon}_{(2)}\sin\varphi\,, \qquad \overset{\circ}{\varepsilon}_{(2)}(\varphi) := \overset{\circ}{\varepsilon}_{(1)}\sin\varphi + \overset{\circ}{\varepsilon}_{(2)}\cos\varphi\,, \tag{33}$$

that are $SO(2)$-transformations of

$$(\overset{\circ}{\varepsilon}_{(1)})_\nu = (0,1,0,0)\,, \qquad (\overset{\circ}{\varepsilon}_{(2)})_\nu = (0,0,1,0)\,. \tag{34}$$

As a result, one obtains the relativistic field

$$\Psi(p,\eta) = \int_0^{2\pi} d\varphi\, e^{i\mu\,\eta\cdot\varepsilon_{(1)}(\varphi)/(\eta\cdot p)}\, f(\eta\cdot\eta, \eta\cdot p)\,\Phi(p,\varphi)\,. \tag{35}$$

2. Singular solution

$$\mathcal{A}(p,\eta,\varphi) \ = \ \delta(\eta\cdot p)\,\delta(\eta\cdot\varepsilon_{(2)}(\varphi))\,e^{i\mu\,\eta\cdot\varepsilon/(\eta\cdot\varepsilon_{(1)}(\varphi))}\,f(\eta\cdot\varepsilon_{(1)}(\varphi))\,, \qquad (36)$$

where we have introduced the vector

$$\varepsilon = \Lambda(A_{(p)})\,\overset{\circ}{\varepsilon}\,. \qquad (37)$$

The vector (37) is light-like and transverse to the vectors $\varepsilon_{(1)}(\varphi)$, $\varepsilon_{(2)}(\varphi)$:

$$\varepsilon\cdot\varepsilon = 0\,, \qquad \varepsilon\cdot\varepsilon_{(1)}(\varphi) = \varepsilon\cdot\varepsilon_{(2)}(\varphi) = 0\,. \qquad (38)$$

Moreover, it obeys the condition $\varepsilon\cdot p = 1$.
Expression (36) coincides with the generalized Wigner operator found in [P. Schuster, N. Toro, JHEP 09 (2013) 104, arXiv:1302.1198 [hep-th]; JHEP **09** (2013) 105, arXiv:1302.1577 [hep-th]].

From solutions for the Wigner operators $\mathcal{A}(p,\eta,\varphi)$, we deduce the equation of motions of the field $\Psi(p,\eta)$, which are

$$(\eta \cdot p)\,\Psi(p,\eta) = 0\,. \tag{39}$$

$$\Big[i\sqrt{-(\eta\cdot\eta)}\,(p\cdot\frac{\partial}{\partial\eta}) + \mu\Big]\,\Psi(p,\eta) = 0\,. \tag{40}$$

Additional condition $f(\eta\cdot\varepsilon_{(1)}(\varphi)) = \delta(\eta\cdot\varepsilon_{(1)}(\varphi) - 1)$, fixing the function $f(\eta\cdot\varepsilon_{(1)}(\varphi))$, leads to the equation

$$\big[(\eta\cdot\eta) + 1\big]\,\Psi(p,\eta) = 0 \tag{41}$$

As a result, the equation (40) becomes:

$$\Big[i\,(p\cdot\frac{\partial}{\partial\eta}) + \mu\Big]\,\Psi(p,\eta) = 0\,. \tag{42}$$

Together with the massless condition $p^2\,\Psi(p,\eta) = 0$, the equations (39), (41), (42) are the Bargmann-Wigner equations for infinite spin fields depending on an additional vector variable η. From these eqs. we have $\hat{W}^2\Psi(p,\eta) = -\mu^2\Psi(p,\eta)$.

For another choice of the auxiliary variable η, the local field $\Psi(p, u, \bar{u})$ is found in the form

$$\Psi(p,u,\bar{u}) \;=\; \int\limits_0^{2\pi} d\varphi\, \mathcal{A}(p,u,\bar{u},\varphi)\, \Phi(p,\varphi)\,, \tag{43}$$

where $\Phi(p, \varphi)$ is the Bargman-Wigner wave function and the generalized Wigner operator $\mathcal{A}(p; u, \bar{u}, \varphi)$ maps a BW wave function of φ into a function depending on u^{α}, $\bar{u}^{\dot{\alpha}}$. In this case we have equations

$$(\pi^{\alpha} u_{\alpha} - \sqrt{\mu})\, \Psi(\pi,\bar{\pi},u,\bar{u}) \;=\; 0\,, \tag{44}$$

$$(\bar{u}_{\dot{\alpha}} \bar{\pi}^{\dot{\alpha}} - \sqrt{\mu})\, \Psi(\pi,\bar{\pi},u,\bar{u}) \;=\; 0\,. \tag{45}$$

$$\left(\pi^{\beta} \frac{\partial}{\partial u^{\beta}} + i\sqrt{\mu}\right) \Psi(\pi,\bar{\pi},u,\bar{u}) \;=\; 0\,, \tag{46}$$

$$\left(\bar{\pi}^{\dot{\beta}} \frac{\partial}{\partial \bar{u}^{\dot{\beta}}} + i\sqrt{\mu}\right) \Psi(\pi,\bar{\pi},u,\bar{u}) \;=\; 0\,. \tag{47}$$

Again from these eqs we deduce

$$\hat{W}^2\, \Psi(p,u,\bar{u}) = -\mu^2\, \Psi(p,u,\bar{u})\,, \tag{48}$$

Helicity representations; $\mu, \rho \to 0$

For discrete basis, we use Fourier expansion of $\Phi(p, \varphi)$:

$$\Phi(p,\varphi) = \sum_{n=-\infty}^{\infty} \Phi_n(p) e^{in\varphi} . \tag{49}$$

and the induced Wigner unitary representation of $SL(2,\mathbb{C})$ becomes

$$[U(A)\Phi]_n(p) = \sum_{m=-\infty}^{\infty} \mathcal{D}_{nm}(\theta_{A,\Lambda^{-1}p}, \vec{b}_{A,\Lambda^{-1}p})\, \Phi_m(\Lambda^{-1}p) , \tag{50}$$

where $\mathcal{D}_{nm}$ is the matrix of the little group element h in the discrete basis:

$$\mathcal{D}_{nm}(\theta,\vec{b}) = (-ie^{i\beta})^{m-n}\, e^{-im\theta} J_{(m-n)}(b\rho) , \tag{51}$$

the $\beta, b \in \mathbb{R}$ are the polar coordinates of $\vec{b} = b\,(\cos\beta, \sin\beta)$ and $J_{(n)}(x)$ are the Bessel functions of integer order. In the case of $\rho \to 0$ (equivalent to $\mu \to 0$) we have $J_{n-m}(0) = \delta_{nm}$ and the matrix element (51) is written as

$$\mathcal{D}_{nm}(\theta,\vec{b}) = \delta_{nm} e^{-in\theta} , \tag{52}$$

I.e. the matrix $\mathcal{D}(\theta,\vec{b})$ becomes diagonal and the transformation (50) is written as

$$[U(A)\Phi]_n(p) = e^{-in\theta}\,\Phi_n(\Lambda^{-1}p)\,, \tag{53}$$

where θ depends on p.

The one-to-one correspondence of the relativistic fields $\Psi(p,\eta)$ and the Wigner wave functions is given by the integral transform

$$\Psi(p,\eta) \;=\; \int_0^{2\pi} d\varphi\,\mathcal{A}(p,\eta,\varphi)\,\Phi(p,\varphi)\,, \tag{54}$$

where $\mathcal{A}(p,\eta,\varphi)$ – the generalized Wigner operator. In the discrete basis, the WF $\Phi_n(p)$ and the local fields $\Psi(p,\eta)$ are related as

$$\Psi(p,\eta) = \sum_{n=-\infty}^{\infty} \mathcal{A}(p,\eta,n)\Phi_n(p)\,, \tag{55}$$

where the kernel $\mathcal{A}(p,\eta,n)$ of the generalized Wigner operator is the Fourier component of $\mathcal{A}(p,\eta,\varphi)$:

$$\mathcal{A}(p,\eta,n) = \int_0^{2\pi} d\varphi\,\mathcal{A}(p,\eta,\varphi)\,e^{in\varphi}\,. \tag{56}$$

Solving eqs for the kernel $\mathcal{A}(p,\eta,n)$, we find the explicit form of the generalized Wigner operator of helicity states for an arbitrary 4-momentum:

$$\mathcal{A}(p,\eta,n) = \begin{cases} \delta(\eta\cdot p)\,(\varepsilon_{(+)}\cdot\eta)^n & \text{with } n>0\,, \\ \delta(\eta\cdot p)\,(\varepsilon_{(-)}\cdot\eta)^{-n} & \text{with } n<0\,, \end{cases} \tag{57}$$

where the 4-polarization vectors $\varepsilon_{(\pm)}$ are used.
We use the kernel of the generalized Wigner operator (57) and define a relativistic field in the form

$$\Psi_n(p,\eta) \;=\; \delta(\eta\cdot p)\,F_n(p,\eta)\,, \tag{58}$$

where

$$F_n(p,\eta) \;=\; F_n^{(+)}(p,\eta) \;+\; F_n^{(-)}(p,\eta)\,, \qquad F_n^{(\pm)}(p,\eta) = (\varepsilon_{(\pm)}\cdot\eta)^n\,\Phi_{\pm n}(p)\,. \tag{59}$$

The component fields $F_n^{(+)}(p,\eta)$ and $F_n^{(-)}(p,\eta)$ describe states with positive and negative helicities $\lambda = n$ and $\lambda = -n$.

The explicit form of (59) reproduces the eqs for the fields $F_n(p,\eta)$:

$$p^2 F_n(p,\eta) = 0\,, \tag{60}$$

$$\left(p \cdot \frac{\partial}{\partial\eta}\right) F_n(p,\eta) = 0\,, \tag{61}$$

$$\left(\frac{\partial}{\partial\eta} \cdot \frac{\partial}{\partial\eta}\right) F_n(p,\eta) = 0\,, \tag{62}$$

$$\left(\eta \cdot \frac{\partial}{\partial\eta}\right) F_n(p,\eta) = n\, F_n(p,\eta)\,. \tag{63}$$

The last equation determines the degree of homogeneity for the field $F_n(p,\eta)$ in the variables η^μ. In addition, the presence in the definition of $\Psi_n(p,\eta)$ of the field $F_n(p,\eta)$ together with the δ-function $\delta(\eta\cdot p)$ leads to the following equivalence relation:

$$F_n(p,\eta) \sim F_n(p,\eta) + (p\cdot\eta)\,\epsilon_{n-1}(p,\eta)\,, \tag{64}$$

where the functions $\epsilon_{n-1}(p,\eta)$ satisfy equations (78) – (81).

Relation (64) is essentially a gauge transformation with parameters $\epsilon_{n-1}(p,\eta)$ and, therefore, the field $F_n(p,\eta)$ is a gauge field.
The standard tensor description of gauge fields is obtained after expansion of the field $F_n(p,\eta)$ as the polynomial in η :

$$F_n(p,\eta) = \eta^{\mu_1}\dots\eta^{\mu_n}\, f_{\mu_1\dots\mu_n}(p) \tag{65}$$

and transferring to the coordinate representation. The corresponding coordinate tensor field $f_{\mu_1\dots\mu_n}(x)$ is automatically totally symmetric $f_{\mu_1\dots\mu_n}(x) = f_{(\mu_1\dots\mu_n)}(x)$ and, thanks to (60)-(62), obeys the equations

$$\Box f_{\mu_1\dots\mu_n}(x) = 0\,, \qquad \partial^{\mu_1} f_{\mu_1\dots\mu_n}(x) = 0\,, \qquad \eta^{\mu_1\mu_2} f_{\mu_1\mu_2\dots\mu_n}(x) = 0\,. \tag{66}$$

In addition, the equivalence relation (64) means that the fields $f_{\mu_1\dots\mu_n}(x)$ are defined up to the gauge transformations:

$$\delta f_{\mu_1\mu_2\dots\mu_n}(x) = \partial_{(\mu_1}\epsilon_{\mu_2\dots\mu_n)}(x)\,. \tag{67}$$

Equations (66) and gauge symmetry (67) are standard conditions that define free massless higher spin fields.

In the case of zero helicity, the use of additional variables η^μ is not required. In this case, the relativistic field coincides with the Wigner wave function $\Psi_0(p) = \Phi_0(p)$, which is not a gauge field, and obeys only the Klein-Gordon equation in the momentum representation: $p^2\Psi_0(p) = 0$.

Summary and outlook

1.) In this report, on the basis of unitary representations of the covering group $ISL(2,\mathbb{C})$ of the Poincaré group, we have constructed explicit solutions of the wave equations for free massive particles of arbitrary spin j (the Dirac-Pauli-Fierz equations).

2.) The most interesting examples corresponding to spins $j = 1/2, 1, 3/2$ and $j = 2$ were discussed in detail in [A.P.I., M.A.Podoinitsyn, Nucl. Phys. B929 (2018) 452].

3.) We obtained that the massless case can also be considered in a similar manner. Just as in the massive case, the spin-tensor wave functions of free massless particles with arbitrary helicity are constructed (with the help of Wigner operators) from the vectors of spaces of the unitary massless Wigner representations for the covering group $ISL(2,\mathbb{C})$ of the Poincaré group.

4.) It will be interesting to apply this approach to the consideration of the unitary irreps for the Poincare superalgebra and supergroup.