D. N. Voskresensky, JINR, NRNU MEPhI

Vacuum stability and phase transitions

DIAS-BLTP Winter School 2024

- Aim: to introduce theory of scalar field and study phenomena of vacuum stability and phase transitions carrying out analogy with phenomenological models of phase transitions.

Content:

Theory of one-component scalar field

Principles: Lorentz invariance, locality of interaction → *c*=const, symmetry of vacuum (ground state), Occam razor principle, principle of least action:

Action is scalar relatively Lorentz transformations $S_{\text{tot}} = S_{\text{part}} + S_{\text{int}} + S_{\text{field}}$.

Interval is Lorentz scalar $s_{12} = \sqrt{c^2 t_{12}^2 - x_{12}^2 - y_{12}^2 - z_{12}^2}\,,$

➡ For free particle $S_{\text{part}} = \alpha \int_a^b ds = \alpha c \int_{t_1}^{t_2} dt \sqrt{1 - v^2/c^2}\,, \qquad \alpha = -mc$

Scalar particle interacting with a scalar external field

Interaction term S_{int} cannot be such as it was in non-rel. mechanics, $\delta S = \int U dt$

Scalar one-component field is determined by single real variable $\phi(x^\mu)$.

Examples: scalar σ meson field, pseudo-scalar pion field (responsible for attraction between baryons), Higgs field, still unknown cosmological fields, etc.

Let ϕ interacts with a scalar charge density of external source $\rho_s(x^\mu)$, μ=0,1,2,3.

Simplest scalar $$S_{\rm int}^{\phi} = \alpha_0 \int \rho_s(x)\phi(x) d^4x$$

Fields interact in one world point –locality principle.

α_0 is coupling constant. Dimensionalities of new quantities ρ, ϕ are not yet fixed.

Take $\alpha_0 = -1/(c\hbar^3)$ Planck constant is introduced to get further relation to QFT.

One often uses units $\hbar = c = 1$

Action & eq. of motion for scalar one-component self-interacting field

Simplicity +locality +symmetry of vacuum $S_\phi = S_\phi[\partial_\mu \phi, \phi]$

Covariant notations $\partial^\mu \phi = \partial \phi / \partial x_\mu$

As in non-relativistic mechanics Lagrangian L may depend only on q and $\dot{q}$

$$S_\phi = \int d^4x \mathcal{L}/c = \frac{1}{c} \int d^4x \left[\hbar^2 \frac{\partial_\mu \phi \partial^\mu \phi}{2} - V(\phi(x^\mu)) \right] \frac{c}{\hbar^3},$$

½ for convenience—units of new quantity ϕ ,

$L = \int d^3x \mathcal{L}$, $V(\phi)$ is a potential energy, cannot depend on x_μ explicitly.

By variation of $S_{\rm part} + S_\phi + S^\varphi_{\rm int}$ over ϕ ($S_{\rm part}$ does not depend on ϕ) we recover

$$\partial_\nu \partial^\nu \phi + \frac{\partial V(\phi)}{\partial \phi} = -\rho_s \,, \qquad (*)$$

the term in the r.h.s. appears if there is interaction of the scalar field with the source of the scalar charge of particles of another kind

Expanding $V(\phi)$ in the series of ϕ we find

$$V(\phi) = \rho_{sc}\phi + \frac{c^2\mu^2\phi^2}{2} + \frac{\lambda_3\phi^3}{3} + \frac{\lambda_4\phi^4}{4} + \ldots$$

$\rho_{sc},\ \mu^2,\ \lambda_3,\ \lambda_4,$ are constants, ρ_{sc} can be interpreted as an interaction of the scalar field with a constant scalar density, and included in ρ_s.

Energy density $$E=\dot{\phi}\frac{\partial\mathcal{L}}{\partial\dot{\phi}}-\mathcal{L}=\frac{\dot{\phi}^2}{2}+\frac{(\nabla\phi)^2}{2}+V+\rho_s\phi$$ (*a)

For ρ_s, *V*=0 vacuum - minimum of energy state- is ϕ=0: $\frac{\dot{\phi}^2}{2}+\frac{(\nabla\phi)^2}{2}$ =0.

Introduce operators

$$\hat{k}^\mu=(\hat{\omega},\hat{\vec{k}})=(\frac{i\partial_t}{c},\frac{1}{i}\nabla),\quad \hat{p}^\mu=(\hat{\epsilon},\hat{\vec{p}})=(\frac{i\hbar\partial_t}{c},\frac{\hbar}{i}\nabla)$$

From (*) →

$$\hbar^2\left(\frac{\hat{\omega}^2}{c^2}-\hat{\vec{k}}^2\right)\phi-\frac{\partial V(\phi)}{\partial\phi}=0\,,$$ (**)

For ρ_s =*0*, ρ_{sc} =0: $$\hat{p}^\mu\hat{p}_\mu\phi=\frac{\partial V(\phi)}{\partial\phi}=\mu^2c^2\phi+O(\phi^2)$$

reminds (non-linear) Klein-Gordon equation where μ has a sense of a mass.

Dimensionality of coupling constants and renormalizable theories

In convenient units $\hbar = c = 1$ action *S* is dimensionless, Lagrange density ~$1/l^4$

$$\partial_\nu \sim 1/l, \quad \phi \sim 1/l. \qquad \mu \sim 1/l,\ \lambda_3 \sim 1/l,\ \lambda_4 \text{ is dimensionless},\ \lambda_{4+n} \sim l^n.$$

Locality: to reach $r \ll 1/\mu$, particle needs $E \gg \mu$. In perturbation series of V (ϕ) parameter of expansion should be dimensionless.

➡ λ_{4+n} enters as $\lambda_{4+n}E^n$ and $E \to \infty$ corresponds to the local interaction.

➡ Series should be cut at term $\lambda_4\phi^4$--**necessary condition of renormalizability.**

Also expansion cannot be stopped on term $\rho_{sc}\phi$ or on $\lambda_3\phi^3/3$ since vacuum instability, e.g. either for $\lambda_3>0$ or $\lambda_3<0$.

➡ appropriate are only theories with $V(\phi) = \rho_{sc}\phi + \mu^2\phi^2/2 + \lambda_3\phi^3/3 + \lambda_4\phi^4/4$

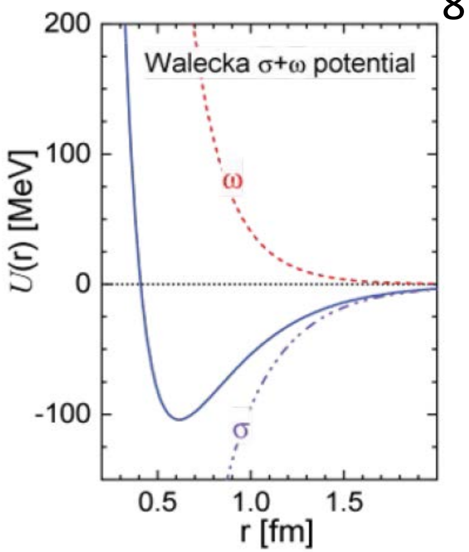
$\lambda_4>0$ for stability. In phenomenological theories one may use higher-order terms.

Interaction of point-like baryon with classical scalar field

Let source of static external scalar charge is $\rho_s = Q_s \delta(\vec{r})$

For small ϕ eq. (**) yields $\Delta\phi - \mu^2\phi = Q_s\delta(\vec{r})$

Solution: $V_{Qq} = q_s\phi = -\frac{q_s Q_s}{4\pi r} e^{-\mu r}$



Potential of NN interaction

Scalar charges of one-sign attract, of opposite sign, repulse.

Nucleons of one-sign baryon charge attract each other by exchange of scalar σ mesons at intermediate distances $r \sim 1/m_\sigma$, m_σ is σ mass ~600 MeV and by exchange of π at $r \sim 1/m_\pi$, m_π is π mass 140 MeV. Replacing *qQ* to *mM* one may describes Newton gravity: charges of one-sign (masses) attract each other.

By exchange of Ω mesons, m_ω=783 MeV, **baryon charges of one-sign repulse each other**, similarly to exchange of electrons by photons in electrodynamics. Classical scalar and vector fields as responses on sources of scalar and baryon charges can be treated as condensates of virtual scalar and vector field quanta. Coulomb field can be treated as a condensate of virtual photons.

Waves

Let $V = \mu^2\phi^2/2$ and **$\mu^2>0$.** $\mathcal{L} = \frac{c}{\hbar^3}\left\{\frac{\hbar^2\partial_\mu\phi\partial^\mu\phi - \mu^2c^2\phi^2}{2}\right\}.$ (*b)

Eq. of motion: $\left(\hat{\omega}^2 - \hat{\vec{k}}^2 - \mu^2\right)\phi = 0\,.$ (***)

This equation coincides with Klein-Gordon equation in QFT for scalar particle (for one-component field spin 2s+1=1).

Expand classical field ϕ in plane waves:

$$\phi = \mathrm{Re}\sum_{\vec{k}} a_{\vec{k}} e^{-i\omega t + i\vec{k}\vec{r}}\,, \quad \sum_{\vec{k}} = \int d^3x \frac{d^3k}{(2\pi)^3}\,,$$

$a_{\vec{k}} = a(\vec{k})$ are coefficients of expansion. Solution of eq. (***) :

$$k^\mu k_\mu = \omega_\pm^2(\vec{k}^2) - \vec{k}^2 = \mu^2\,, \qquad \omega_\pm = \pm\sqrt{\mu^2 + \vec{k}^2}.$$

Now quantity μ>0 can be interpreted as mass of scalar particle.

Solution with $\omega_-(\vec{p}^2)$ <0 can be associated with antiparticle with $-\omega_- = \omega_+$,

$$\phi = \mathrm{Re} \sum_{\vec{k}} \left(a_{\vec{k}} e^{-i\omega_+ t + i\vec{k}\vec{r}} + b_{\vec{k}} e^{-i\omega_- t + i\vec{k}\vec{r}} \right) = \sum_{\vec{k}} \left(a_{\vec{k}} e^{-i\omega_+ t + i\vec{k}\vec{r}} + [a_{\vec{k}} e^{-i\omega_+ t + i\vec{k}\vec{r}}]^* \right),$$

in the sum we replaced $\vec{k} \to -\vec{k}$ and used that ϕ is real quantity

$\phi_+ = \sum_{\vec{k}} a_{\vec{k}} e^{-ikx}$ wave can be associated with particles

$\phi_- = \sum_{\vec{k}} a_{\vec{k}} e^{-ikx}$ with antiparticles, $\phi = \phi_+ + \phi_-^*$.

After integration of energy density over volume:

$$\mathcal{E} = \sum_{\vec{k}} V_3 \omega_+^2(\vec{k}^2) 2 a_{\vec{k}} a_{\vec{k}}^* = \sum_{\vec{k}} \omega_+(\vec{k}^2) N_{\vec{k},+}, \quad V_3 = \int d^3x \quad \text{cf. eqs. (*a), (*b).}$$

$N_{\vec{k},+}$ can be interpreted as number of modes (particles) in each state, classical treatment requires condition $N_{\vec{k},+} \gg 1$. **Vacuum – *N*$_k$*=0, ϕ*=0.**

Generalization to QFT-operators in 2-d quantization obeying commutator relations.

Using canonical variables $Q_{\vec{k}} = \sqrt{V_3}(a_{\vec{k}}(t) + a^*_{\vec{k}}(t))$ and $P_{\vec{k}} = \dot{Q}_{\vec{k}}$, where $a_{\vec{k}}(t) = a_{\vec{k}} e^{-i\omega_+ t}$,

$$\mathcal{H} = \sum_{\vec{k}} \frac{P^2_{\vec{k}} + \omega^2_+(\vec{k}^2) Q^2_{\vec{k}}}{2} \, .$$

Non-interacting scalar field is described by infinite sum of harmonic oscillators, at mass coefficient equal unity. Recall that motion of oscillating particle in nonrelativistic mechanics is described by the plane wave.

If we included the self-interaction $H_{\rm int} = \lambda_3 \phi^3/3 + \lambda_4 \phi^4/4$, unharmonic oscillators.

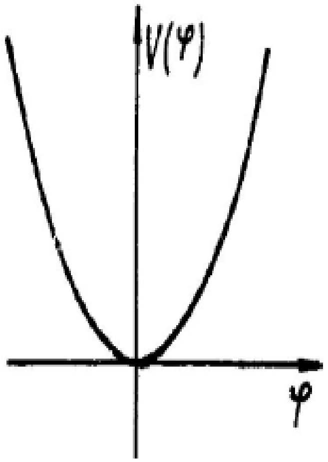
Generalization to QFT–commutator relation for operators in second quantization→ sum of quantum oscillators and zero vibration contribution to energy.

Model $\lambda_4\phi^4$

$$V = \mu^2\phi^2/2 + \lambda_4\phi^4/4.$$

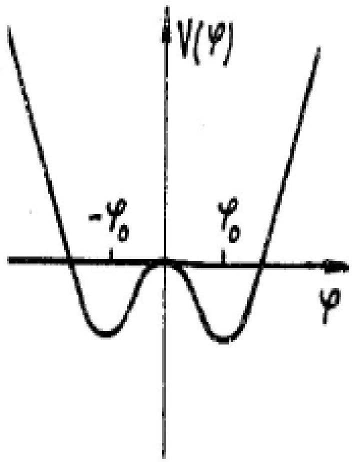
Symmetry $\phi \to -\phi$

Unharmonic oscillator



Potential $V(\phi)$ for the case $\mu^2 > 0$.

Two humped potential well



Potential $V(\phi)$ for the case $\mu^2 < 0$.

$$E = \dot{\phi}\frac{\partial \mathcal{L}}{\partial \dot{\phi}} - \mathcal{L} = \frac{\dot{\phi}^2}{2} + \frac{(\nabla\phi)^2}{2} + \frac{\mu^2\phi^2}{2} + \frac{\lambda_4\phi^4}{4}$$

Spontaneous $\phi \to -\phi$ *vacuum* symmetry breaking. Field in vacuum

Consider case **$\mu^2<0$.** ➡ $\partial_\mu \partial^\mu \phi + \mu^2 \phi + \lambda_4 \phi^3 = 0$ (****)

solutions $\phi = \pm\phi_0$, where $\phi_0 = +\sqrt{-\mu^2/\lambda_4} > 0$

If initially system (e.g. Universe) was in state ϕ=0 in a fluctuation it may reach one of minima and lives there for ever. Probability of tunneling from one minimum to other $W(B,C) = e^{-2\int_{x_B}^{x_C} |p| dx/\hbar}$, ϕ plays role of *x.* Kin. energy $\frac{\dot{\phi}^2}{2} V_3$ ➡

Mass M=V_3, *U* =VV_3, |*p*|= (|2*M*(ε -*U*)|)$^{1/2}$ ➡ exponent enters infinite volume V_3.

The energy in the ground state $\vec{p}^2 = 0$ is as follows,

$$\mathcal{E} = -V_3 \mu^4/(4\lambda_4) < 0\,, \qquad (*****)$$

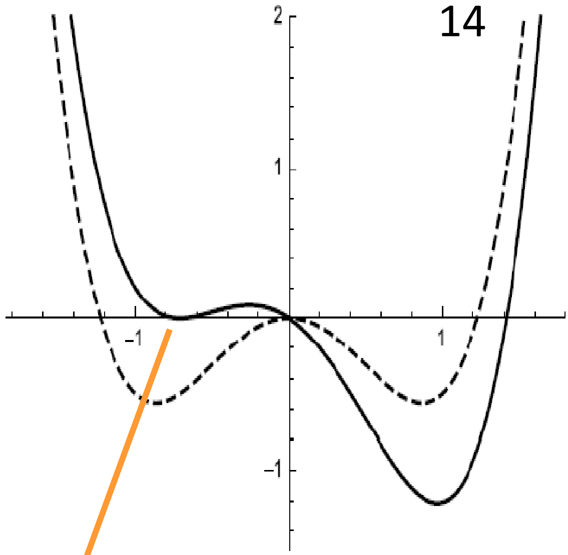
this energy of scalar field is infinite for $V_3 \to \infty$.

Ignoring gravity, this divergence does not hurt (removed by renormalization).

Expand $\phi = \phi_0 + \phi'$. Eq.(****) yields $\partial^\mu \partial_\mu \phi' - 2\mu^2 \phi' + 3\lambda_4 \phi_0 \phi'^2 + \lambda_4 \phi'^3 = 0$

It describes stable waves/ particles with mass $\sqrt{-2\mu^2} > 0.$

Symmetric *(dash)* & asymmetric potentials *(solid line)*



dash-symmetric potential

Let now $V(\phi) = \rho_{sc}\phi + \mu^2\phi^2/2 + \lambda_3\phi^3/3 + \lambda_4\phi^4/4$

Now we consider case $\rho_{sc} \neq 0$. We have

$$\partial_\mu \partial^\mu \phi + \mu^2 \phi + \rho_{sc} + \lambda_4 \phi^3 = 0$$

Assume system (e.g. Multiverse) was initially in metastable phase . Gain & loss:

$$(\epsilon_{\rm st} - \epsilon_{\rm meta}) V_{\rm seed}(R) + \sigma S_{\rm seed}(R)$$

$\epsilon_{\rm st}$ is energy density of stable phase, $\epsilon_{\rm meta}$ of metastable phase, σ surface tension,

V_{seed} is volume of seed, S_{seed}-surface. ⇨ For $R>R_{cr}$ seeds grow, energy is gained similarly to what occurs at I-order liquid-vapor phase transition (e.g., at boiling of water).

Experimnt: Cf. Zenesini et al., False vacuum decay via bubble formation in ferromagnetic superfluids, Nature 22.01.2024.

We studied vacuum of one-component scalar field.
Now study vacuum state for two-component field

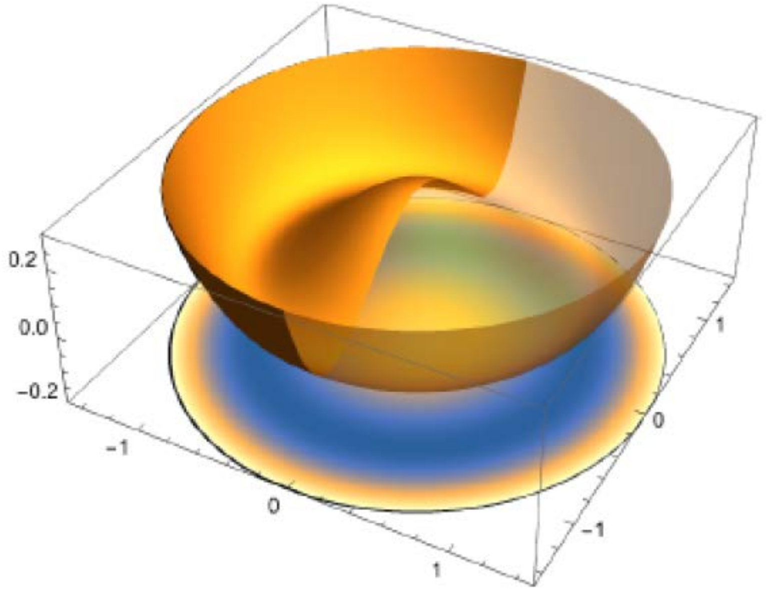
Self-interacting complex scalar field. Action and equation of motion

Consider two scalar fields $\phi_1,\ \phi_2,$ as $\phi = (\phi_1 + i\phi_2)/\sqrt{2}$ and $\phi^* = (\phi_1 - i\phi_2)/\sqrt{2}$.

$$S_\phi = \int d^4x \mathcal{L} = \int d^4x \left[\partial_\mu\phi\partial^\mu\phi^* - V(|\phi(x^\mu)|^2) - \rho_s(\phi + \phi^*)\right] .$$

By variation of $S_{\rm part} + S_\phi + S_{\rm int}^\phi$ over ϕ^* ($S_{\rm part}$ does not depend on ϕ, ϕ^*) we recover

$$\partial_\nu\partial^\nu\phi + \frac{\partial V(\phi,\phi^*)}{\partial\phi^*} = -\rho_s\ , \qquad (*)$$



In case of ***symmetric potential*** *(ρ_{sc}=0):*

$$V(|\phi|^2) = \mu^2|\phi|^2 + \frac{\lambda_4|\phi|^4}{2} .$$

In Fig. $\mu^2 = (d^2V/(d\phi d\phi^*))_{\phi=0}$ <0

In such model we shall deal with 2-order phase transition

Energy density: $E = \dot\phi \frac{\partial \mathcal{L}}{\partial \dot\phi} + \dot\phi^* \frac{\partial \mathcal{L}}{\partial \dot\phi^*} - \mathcal{L} = |\dot\phi|^2 + |\nabla\phi|^2 + \mu^2|\phi|^2 + \frac{\lambda_4|\phi|^4}{2} + \rho_s(\phi + \phi^*)\,.$

Eq. (*) is $\left(\frac{\hat\omega^2}{c^2} - \hat k^2\right)\phi - \frac{\partial V(|\phi|^2)}{\partial \phi^*} = 0$ We recovered dependence on *c*.

Multiplying this eq. by ϕ^* and complex conjugated eq. by ϕ,

recover conservation of flux/current $\partial^\mu(\phi^*\partial_\mu\phi - \phi\partial_\mu\phi^*) = 0\,.$

$j_\mu = C(\phi^*\partial_\mu\phi - \phi\partial_\mu\phi^*)$ *C*=const, $\rho = C(\phi^*\dot\phi - \phi\dot\phi^*)$

Interpretation needs a care (enters not $|\psi|^2$ as it is in nonrelativistic QM)

Waves

Let $V(|\phi|^2) = \mu^2|\phi|^2$ for $\mu^2 > 0$. ➡ $\left(\hat{\omega}^2 - \hat{\vec{k}}^2 - \mu^2\right)\phi = 0\,.$

$$\phi = \sum_{\vec{k}} \left(a_{\vec{k}} e^{-i\omega_+ t + i\vec{k}\vec{r}} + b_{\vec{k}} e^{-i\omega_- t + i\vec{k}\vec{r}}\right) = \sum_{\vec{k}} \left(a_{\vec{k}} e^{-i\omega_+ t + i\vec{k}\vec{r}} + [b_{\vec{k}} e^{-i\omega_+ t + i\vec{k}\vec{r}}]^*\right),$$

we replaced $\vec{k} \to -\vec{k}$

$a_{\vec{k}}$ and $b_{\vec{k}}$ are complex coefficients.

$\phi_+ = \sum_{\vec{k}} a_{\vec{k}} e^{-ikx}$ can be associated with the particles

$\phi_- = \sum_{\vec{k}} b_{\vec{k}} e^{-ikx}$ with antiparticles. $\phi = \phi_+ + \phi_-^*$.

Introducing *C=-ie*, *e* is particle charge, $\sqrt{2\omega_+}a_{\vec{k}} = \tilde{a}_{\vec{k}}$, $\sqrt{2\omega_+}b_{\vec{k}} = \tilde{b}_{\vec{k}}$. ➡

charge number density, $\rho/e = \sum_{\vec{k}} (|\tilde{a}_{\vec{k}}|^2 - |\tilde{b}_{\vec{k}}|^2)\,,$ ➡ clear interpretation.

In Klein-Gordon eq. 2 degrees of freedom: one for *s*=0 particle, other antiparticle

with the frequency $-\omega_- = \omega_+$. Complex field ϕ describes, e.g., pseudoscalar π^+ and π^- mesons.

Energy:

$$\mathcal{E} = \sum V_3 \omega_+^2(\vec{k}^2) 2(a_{\vec{k}} a_{\vec{k}}^* + b_{\vec{k}}^* b_{\vec{k}}) \, .$$

$$\mathcal{E} = \sum_{\vec{k}} \hbar \omega_+(\vec{k}^{\,2})(N_{\vec{k},+} + N_{\vec{k},-}) \, ,$$

$$N_{\vec{k},+} = \frac{V_3 \omega_+(\vec{k}^{\,2}) c^2}{2\pi\hbar} a_{\vec{k}} a_{\vec{k}}^* \, , \quad N_{\vec{k},-} = \frac{V_3 \omega_+(\vec{k}^{\,2}) c^2}{2\pi\hbar} b_{\vec{k}} b_{\vec{k}}^* \, ,$$

$N_{\vec{k},+}$ and $N_{\vec{k},-}$ can be treated as numbers of particles and antiparticles classical treatment requires that all $N_{\vec{k}} \gg 1.$

Introducing canonical variables $Q_{\vec{k}}^{(1)} = \sqrt{V_3}(a_{\vec{k}}(t) + a_{\vec{k}}^*(t))$ and $P_{\vec{k}}^{(1)} = \dot{Q}_{\vec{k}}^{(1)}$

$Q_{\vec{k}}^{(2)} = \sqrt{V_3}(b_{\vec{k}}(t) + b_{\vec{k}}^*(t))$ and $P_{\vec{k}}^{(2)} = \dot{Q}_{\vec{k}}^{(2)}$

$$\mathcal{H} = \sum_{\vec{k}} \frac{P_{\vec{k}}^{(1)\,2} + \omega_+^2(\vec{k}^2)(Q_{\vec{k}}^{(1)})^2 + P_{\vec{k}}^{(2)\,2} + \omega_+^2(\vec{k}^2)(Q_{\vec{k}}^{(2)})^2}{2} \, .$$

Generalization to QFT – commutator relation for operators in second quantization.

Goldstone model (No interaction with electromagnetic field)

$$\mathcal{L} = \partial_\mu \phi \partial^\mu \phi^* - \mu^2 |\phi|^2 - \lambda |\phi|^4 / 2 \,. \qquad \mu^2 < 0.$$

$$\partial_\mu \partial^\mu \phi + \mu^2 \phi + \lambda |\phi|^2 \phi = 0 \,.$$

$|\phi|^2 = -\mu^2/\lambda$ This static uniform solution corresponds to minimal energy.

Lagrangian is symmetric under *U*(1) transform. $\phi \to \phi e^{i\alpha}$ for α being real and constant.

System spontaneously chooses one of α states, e.g. α=0, $\phi = (|\mu^2|/\lambda)^{1/2} = \phi_0$

Present two-component field as $\phi = \phi_0 (1 + \rho'(x)) e^{i\chi'(x)}.$

In terms of new real fields $\rho'(x)$ and $\chi'(x)$

$$\mathcal{L} = \phi_0^2 \partial_\mu \rho' \partial^\mu \rho' + \phi_0^2 \partial_\mu \chi' \partial^\mu \chi' + 2\mu^2 \phi_0^2 \rho'^2 + \ldots$$

$$-\partial_\mu \partial^\mu \rho' + 2\mu^2 \rho' \simeq 0 \,, \quad \partial_\mu \partial^\mu \chi' = 0 \,.$$

Solutions are waves with dispersion laws: $\omega^2_{\rho'} = \vec{q}^{\,2} + \boxed{2|\mu^2|}$, $\omega^2_{\chi'} = \vec{q}^{\,2}$.

One massive and 1 massless (Goldstone) neutral excitations.

Goldstone theorem: at spontaneous *U*(1) symmetry breaking there appears a massless mode. In the given example it is mode of "scalar light".

For nonrelativistic condensed matter, e.g. for He-II $\phi(x_\mu) = \psi(t, \vec{r})e^{-imt}$

$m > 0$ is the effective mass of He atoms and $\psi(t, \vec{r})$ is a smooth function

gappless excitations in neutral superfluids such as ^{4}He !

$\omega = vq$, $v < c$.

Self-interacting pions

We have considered 1-component field and 2-component fields at $\mu^2>0$ and <0.

There are three types of pions: π^+, π^- and π^0 of mass $m_\pi \simeq 140$ MeV.

strong coupling constant $\lambda_4 \sim 1$ Electromagnetic interaction $e^2=1/137$

In strong interactions SU(2) global symmetry $\vec{\phi} \to e^{i\vec{\alpha}\vec{T}}\vec{\phi}, \quad \vec{\phi} = (\phi_1, \phi_2, \phi_3)$

Isospin operators $\vec{T}_i\vec{T}_k - \vec{T}_k\vec{T}_i \neq 0,$ isospin is conserved in strong interactions.

$$\mathcal{L} = \frac{1}{2}\partial_\mu\vec{\phi}\,\partial^\mu\vec{\phi} - \frac{m_\pi^2}{2}\vec{\phi}^2 - \frac{\lambda_4}{4}(\vec{\phi}^2)^2 .$$

Each meson π^+, π^-, π^0 is spinless $(2s+1=1)$ and the pion triplet has isospin 1 since $2T+1=3$. Interaction with electromagnetic field spoils isospin symmetry.

$\pi^\pm$ can be described by complex fields, e.g. π^- field $\phi = (\phi_1 - i\phi_2)/\sqrt{2}$

Similarly we could introduce actions for photons, Ω mesons, ρ mesons, gluons…

We studied vacuum of two-component scalar field.

Now come back to one-component scalar field but now in a potential well

One-component scalar field in external scalar field of another origin

Consider field ϕ, e.g. of neutral pions with **squared mass $m^2>0$**, placed in a scalar potential well U (analogy with s-wave interaction of pions and nucleons)

Effective squared mass term

$$\mathcal{L} = \frac{\partial_\mu \phi \partial^\mu \phi - (m^2 + U)\phi^2}{2} - \frac{\lambda_4 \phi^4}{4} .$$

Varying action: $\partial_\mu \partial^\mu \phi + (m^2 + U)\phi + \lambda_4 \phi^3 = 0$. cf. eq. (****) slide 13,

$$E = \dot{\phi}\frac{\partial \mathcal{L}}{\partial \dot{\phi}} - \mathcal{L} = \frac{\dot{\phi}^2}{2} + \frac{(\nabla\phi)^2}{2} + \frac{(m^2 + U)\phi^2}{2} + \frac{\lambda_4 \phi^4}{4}$$

Let λ_4=0. Then eq. of motion renders $\hat{\epsilon}^2\phi = (m^2 + U + \hat{\vec{p}}^2)\phi$

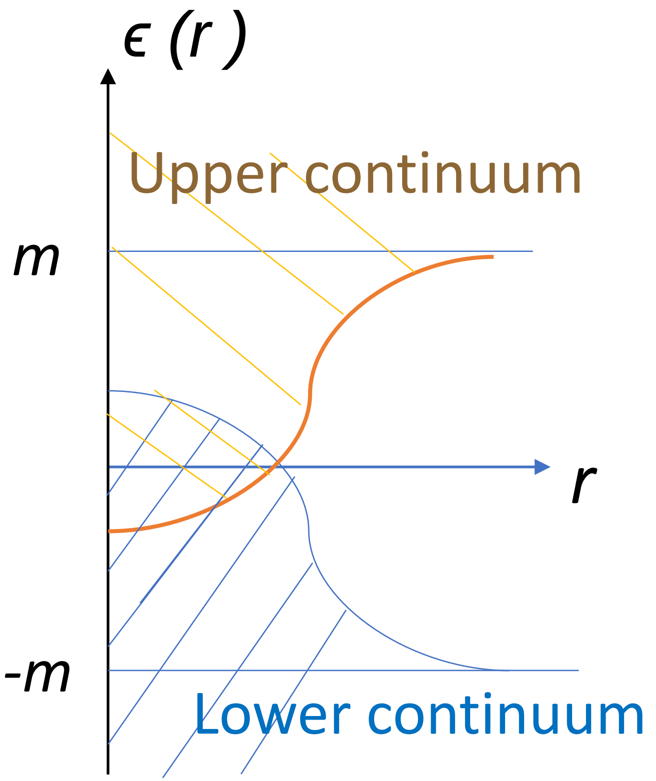
Let U is static. Solution ϕ = $\phi_{\rm st} \mathrm{Re}\, e^{-i\epsilon(\vec{r})t}$. ⟹ $\epsilon^2 \phi_{\rm st} = (m^2 + U(\vec{r}) + \hat{\vec{p}}^2)\phi_{\rm st}$.

For slowly spatially varying field *U(r)* take $\phi_{\rm st} \propto e^{i\vec{p}(\vec{r})\vec{r}}$

$$\vec{p}^2(\vec{r}) = \epsilon^2(\vec{r}) - U(\vec{r}) - m^2$$

Upper and lower continuums $\vec{p}^2 > 0$

$\vec{p}^2_\pm(\vec{r}) = 0$ --solid lines--boundaries of continuums.



For $-U>m^2$ continuums are overlapped

that corresponds to the exponential growth of the field $\phi \sim \exp(\mathrm{Im}\epsilon \cdot t)$.

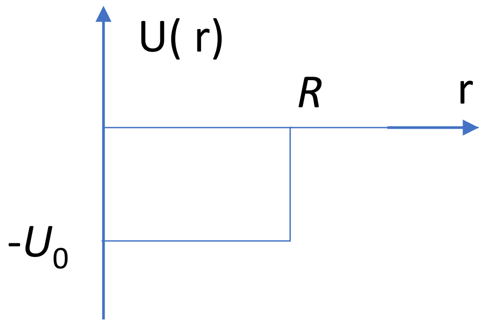
Typical time of production of ϕ condensate classical field

$$\tau \sim (-m^2 - \bar{U})^{-1/2}\theta(-m^2 - \bar{U})$$

Spherical square potential well, case $\lambda_4=0$

Now consider ϕ field in broad spherical square potential well

$$U = -U_0 = const \text{ for } r < R \text{ and } U = 0 \text{ for } r > R$$



$R \gg 1/m,\ \lambda_4 = 0$ Classical field ↔QM one-particle problem

$$\hat{\epsilon}^2\phi = (m^2 + U + \hat{\vec{p}}^2)\phi$$

$$\text{boundary conditions } |\phi(0)| < \infty, \quad \phi(r \to \infty) \to 0$$

$$\phi(R-0) = \phi(R+0) \text{ and } \phi'(R-0) = \phi'(R+0)$$

For level *l*=0:

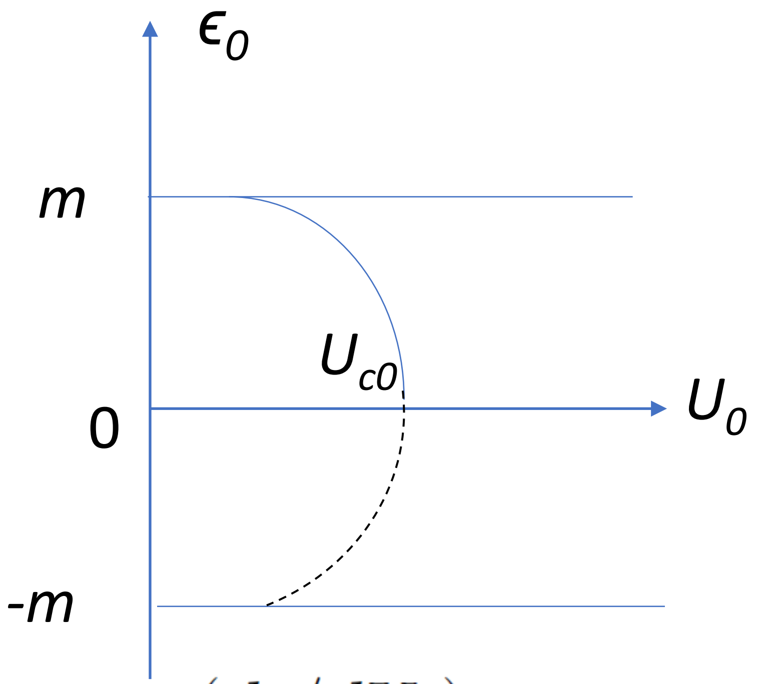
$$\phi_< = C\cos(\epsilon t)\cdot \sin(kr)/r\,, \text{ for } r < R\,,$$

$$\phi_> = C\cos(\epsilon t)\cdot e^{-\lambda r}/r \text{ for } r > R\,,$$ *C* is arbitrary constant.

$$\boxed{k\cot(kR) = -\lambda\,,}$$ recall similar eq. in nonrelativistic QM

$$k = (\epsilon_n^2 - m^2 + U_0)^{1/2},\ \lambda = (m^2 - \epsilon_n^2)^{1/2}$$, *n*-quantum number

Behavior of ground state level:



Symmetry $\epsilon \to -\epsilon$

At $U_0=U_{c0}\approx m^2+O(1/R^2)$ $\quad (d\epsilon/dU_0)_{\epsilon=0,U_0=U_{c0}} = \infty.$

For $|U|<U_{c0}$ energetically favorable is solution $\phi=0$, i.e.,

if there were no particles inside the well, the level remains empty for $|U|<U_{c0}$

If N particles/antiparticles were put on ground level: energy is $N\varepsilon_0 \to 0$ for $\varepsilon_0 \to 0$

➡ **At N=0 instability of vacuum inside the overcritical potential well**

Taking into account that for ε<<m (for dangerous level), $\lambda R \sim mR >> 1$ we get approximately $\sin(kR) \simeq 0$ and $kR \sim 1$ for the ground state and thus

$$\epsilon_{0\pm} \simeq \pm\sqrt{m^2 - U_0 + \pi^2/R^2}\,.$$

The branch with $\epsilon > 0$ can be interpreted as describing particles and $\epsilon < 0$ after change $\epsilon \to -\epsilon$ describes antiparticles

Real solutions disappear for $|U| > |U_c| = m^2 + \pi^2/R^2 \simeq m^2$.

For $U_0 > U_{c0} \simeq m^2 + O(1/R^2)$ solutions correspond to $\mathrm{Im}\epsilon \neq 0$.

$$\mathrm{Im}\epsilon_{0+} = +\sqrt{|m^2 - U_0 + \pi^2/R^2|}$$

that corresponds to **exponential growth of condensate field** $\phi \sim \exp(\mathrm{Im}\epsilon \cdot t)$

$$\tau \sim (-m^2 - \bar{U})^{-1/2}\theta(-m^2 - \bar{U})$$

Case $\lambda_4 \neq 0$

For $U_0 < U_{c0}$ in absence of external particles being put inside potential well, the minimum of energy (E=0) corresponds to static classical solution ϕ =0.

Now let $U_0 > U_{c0}$. Solve Eq. of motion for **static field** (minimizing energy):

$$-\Delta\phi + (m^2 + U)\phi + \lambda_4\phi^3 = 0\,. \quad (*)$$

For $r>R$ solution of eq. (*) yields $\phi \sim e^{-m(r-R)}$ ➡

length of change of field ϕ in exterior region for any λ_4 is $l_> \sim 1/m$

length of change of field ϕ inside the well, $l_< = 1/|m_{\rm ef}|$. $m_{\rm ef}^2 = m^2 - U_0 < 0$

Assume that $R \gg l_<$ ➡ Deeply inside the system all processes

should occur as in infinite matter. Thus for $r \ll R - l_<$,

we expect $\phi = \pm\phi_0\theta(U_0 - m^2)$ for $U_0 > m^2$

introduce the variable $x = (r - R)/l_<$ and seek $\phi = \pm\phi_0\chi(x)$.

$$\chi'' + 2\chi'/(x + R/l_<) - m_{\rm ef}^2 l_<^2 \chi + m_{\rm ef}^2 l_<^2 \chi^3 = 0 \qquad (**)$$

$$m_{\rm ef}^2 = m^2 - U_0 < 0 \quad l_< = 1/|m_{\rm ef}|. \qquad \chi(-\infty) = 1.$$

At the condition $l_< \gg l_> \sim 1/m$, i.e. for $|m_{\rm ef}| \ll m$, (Landau condition of small overcriticality) as another boundary condition we may employ $\chi(x = 0) = 0$ cf. condition used in phenomenological theories (e.g. in hydrodynamics), whereas exact matching of χ and χ' at $x = 0$ yields $\chi(0) = O(1/(ml_<))$.

Solution of dimensionless eq. with dimensionless boundary conditions is changed on x~1 and we can drop $|\chi'/(x+R/l_<)| \ll |\chi''|$ (flat geometry for $R \gg l_<$)

$$\Rightarrow \qquad \chi'' + \chi - \chi^3 = 0\,, \quad \chi(-\infty) = 1\,, \quad \chi(0) = 0$$

Solution $\phi = \pm\theta(-m_{\rm ef}^2)\phi_0 {\rm th}[(r-R)|m_{\rm ef}|/\sqrt{2}]\,, \quad r<R\,, \quad \phi_0 = (|m_{\rm ef}^2|/\lambda_4)^{1/2}\,.$

Sign + or – is not important since *E* enters ϕ^2.

$$\mathcal{E} = \int E d^3x = -\frac{m_{\rm ef}^4}{4\lambda_4} V_3 \theta(-m_{\rm ef}^2) + O(R^2)\,, \quad V_3 = \frac{4\pi R^3}{3}\,.$$ Cf. (*****) slide 13.

$\mathcal{E}$ is now finite and has physical meaning even at ignorance of the gravity,

first derivative of $\mathcal{E}$ in U_0 is zero at $U = U_c$, second derivative gets a jump.

➡ We deal with **2-order phase transition** according to Ehrenfest classification.

Note, classical condensate field appeared in absence of real particles!

$\mathcal{E} \to -\infty$ for $\lambda_4 \to 0$ **Stability of vacuum is determined only by self-interaction**

Excitations: If ground state is pertubed, excitations in presence of condensate field can be described by variable shift $\phi = \phi_c + \phi'$

For $r \ll R - l_<$ in the linear approximation the equation for ϕ' becomes

$$-\ddot{\phi}' + \Delta\phi' + 2m_{\rm ef}^2\phi' + O(\phi'^2) = 0\,. \qquad (*)$$

Inside system it describes now waves related to particles with positive mass

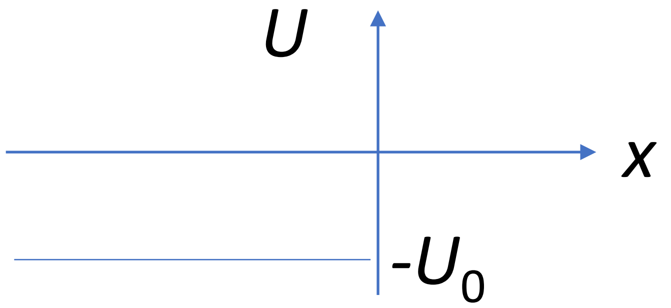
$$\mu = \sqrt{-2m_{\rm ef}^2}$$

In quantum case, ϕ' describes zero fluctuations over the ground state described by classical field $\phi = \phi_c$.

Neutral complex scalar field in a scalar potential well of other origin

Consider complex neutral scalar field not interacting with electromagnetic field.

$$\mathcal{L} = \partial_\mu \phi \partial^\mu \phi^* - \mu^2 |\phi|^2 - \lambda |\phi|^4 / 2 \,.$$



Let for $x<0$, $U=-U_0<-m$, $\mu^2 = m_{ef}^2(x<0) = m_0^2 = m^2 - U_0$,
for $x>0$, $U=0$, $m_{ef}^2(x>0) = m^2 > 0$.

➡ the same real solution as for one-component field

$$\phi = f_0 \mathrm{th}[x/(\sqrt{2} l_\phi)] \,, \quad x < 0 \,,$$

$$f_0 = \pm\sqrt{-m_0^2/\lambda}\, \theta(-m_0^2) \,, \quad l_\phi = 1/\sqrt{|m_0|}.$$

For $x>0$ we may put $\phi=0$ provided $l_\phi >> 1/m$, i.e., $|m_0| << m$, that we assume to be the case (Landau condition).

➡ $$\mathcal{E} = -\frac{m_0^4}{2\lambda} V_3 \theta(-m_0^2)$$

We studied one-component scalar field in a scalar potential well.
Now discuss thermal fluctuations of one-component scalar field

Thermal fluctuations in model $\lambda\phi^4$ with symmetry breaking

Thermal fluctuations: Now consider field ϕ *in* equilibrium state at T$\neq$ 0 (in a medium), then besides zero fluctuations there are thermal ones. The latter can be treated on classical level. **Continue to study theory with**

$$\mathcal{L} = \frac{\partial_\nu \phi \partial^\nu \phi - m^2 \phi^2}{2} - \frac{\lambda \phi^4}{4} \qquad \text{at } m^2 < 0.$$

Eq. of motion $\quad \partial_\nu \partial^\nu \phi + m^2 \phi + \lambda \phi^3 = 0$

Take $\phi = \phi_c + \phi'$, ϕ_c is classical (condensate) field, ϕ' - describes excitations

$$\boxed{\partial_\nu \partial^\nu \phi' + m^2 \phi + 3\lambda \phi_c^2 \phi'} + O(\phi'^2) = 0$$

Mass term for excitations $\quad \mu^2 = m^2 + 3\lambda \phi_c^2$. Now $\phi_c = \phi_c\,(T)$.

For $T\neq0$ setting $\phi=\phi_c+\phi'$ ➡

$$\langle(-\mathcal{L})\rangle_T=\frac{m^2\phi_c^2}{2}+\frac{\lambda\phi_c^4}{4}$$
$$-\frac{\langle\partial_\mu\phi'\partial^\mu\phi'\rangle_T-(m^2+3\lambda\phi_c^2)\langle\phi'^2\rangle_T}{2}+O(\phi'^4)\,. \quad (**)$$

Averaging is now done over Gibbs state, within Grand canonical treatment.

Terms in 1-line depend only on ϕ_c. 2-line yields the free -energy density, F', of noninteracting Bose gas with positive squared mass $\mu^2=m^2+3\lambda\phi_c^2\,.$

For $T\gg\mu\sim|m|$ the quantity F' is simply calculated yielding the result, cf. e.g. LL5,

$$F'=-\frac{\pi^2T^4}{90}+\frac{\mu^2T^2}{24}\left(1+O(\mu/T)\right).\quad F[\phi_c]=-\frac{|m^2|-\lambda T^2/4}{2}\phi_c^2+\frac{\lambda\phi_c^4}{4}\,. \quad (***)$$

Minimization over ϕ_c yields $\phi_{c,m}^2=\frac{|m^2|-\lambda T^2/4}{\lambda}\theta(|m^2|-\lambda T^2/4)\,.$

➡ The value of the critical temperature $T_c=2|m|/\sqrt{\lambda}\,.$

condition $T\gg\mu,|m|$ is fulfilled for λ<<1.

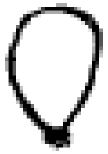
$$F[\phi_{cm}] = -\frac{(|m^2| - \lambda T^2/4)^2}{4\lambda}\,,$$ near T_c: $$F[\phi_{cm}] = -\frac{(T-T_c)^2 T_c^2}{\lambda}\theta(T_c - T)\,.$$

Second-order phase transition: $dF'[\phi_{cm}]/dT)_{T_c} = 0$ and $(d^2F'[\phi_{cm}]/dT^2)_{T_c} \neq 0$,

restoration of symmetry for $T>T_c$ occurs owing to long-range fluctuations of field ϕ.

The contribution to the free-energy density associated with fluctuations in (**)

also can be found by computing the closed boson line tadpole diagram:

$$iG^R(x=0) = \langle\phi'^2\rangle = \int \frac{d^4p}{(2\pi)^4}\frac{i}{p^2-\mu^2+i0}e^{ipx}|_{x=0}$$ = 

Within the Matsubara diagram technique we should perform replacements

$\int \frac{d^4p}{(2\pi)^4} \to iT\sum_{n=-\infty}^{\infty}\frac{d^3p}{(2\pi)^3}$ and $\omega \to \omega_n = 2\pi i n T$. $$\sum_{n=-\infty}^{\infty}(y^2+n^2)^{-1} = \pi y^{-1}\mathrm{cth}(\pi y)\,,$$

$$\langle\phi'^2\rangle_T = \int\frac{d^3p}{(2\pi)^3}\frac{1}{2\sqrt{\vec{p}^2+\mu^2}}\mathrm{cth}\frac{\sqrt{\vec{p}^2+\mu^2}}{2T}$$. For $T>>\mu$ convergent T-dependent term:

$\langle\phi'^2\rangle_T \simeq T^2/12\,.$ Setting it in (**) we again recover (***).

We studied fluctuations of one-component scalar field.
Now study charged scalar field in electric potential well.

Interaction of particles & scalar field with electromagnetic field

Vector-potential $A^\mu = (\Phi, \vec{A})$ Action $S[A,x]=S_{part}+S^A{}_{int}+S_A$

$$S^A_{\rm int} = \frac{e}{c}\int A_\mu(x)dx^\mu = \frac{1}{c^2}\int \rho_e d^4x(dx^\mu/dt)A_\mu(x) = \frac{1}{c^2}\int j^\mu(x)A_\mu(x)d^4x\,. \quad j^\mu = -c\frac{\partial \mathcal{L}^A_{\rm int}}{\partial A_\mu}\,,$$

$A^\mu = A'^\mu + \partial^\mu f\,,\ \delta(S_{\rm part} + S^A_{\rm int}) = 0$ for arbitrary *f(x)*→ **gauge invariance,**

➡ $\partial_\mu j^\mu = 0\,.$ Einstein relation $(p^\mu - (e/c)A^\mu)(p_\mu - (e/c)A_\mu) = m^2c^4\,.$

$$S_A = -\frac{1}{16\pi c}\int d^4x F_{\mu\nu}F^{\mu\nu}\,.\quad F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$$

Excitations: $A^\mu = A^\mu_{\rm cl} + A'^\mu\,,$

$$\partial_\nu F^{\nu\mu}_{\rm cl} = \frac{4\pi}{c}j^\mu_{\rm cl}\,,\qquad \left(\hat{\omega}^2 - \hat{\vec{k}}^2\right)A'^\mu = 0,\quad \partial_\nu A'^\nu = 0.$$

Photons are massless, spin 1, 2*s*+1=3.

For complex self-interacting scalar field interacting with gauge field A_μ :

$$\mathcal{L} = (\partial_\mu + ieA_\mu)\phi(\partial^\mu - ieA^\mu)\phi^* - m^2|\phi|^2 - \lambda|\phi|^4/2 - F_{\mu\nu}F^{\mu\nu}/(16\pi)\,.$$

4-vector is added to 4-vector. Gauge invariance is kept.

Charged scalar field placed in external static electric field *V=eΦ*

$$\mathcal{L}_{V\phi} = (\partial_t + iV)\phi(\partial_t - iV)\phi^* - |\nabla\phi|^2 - m^2|\phi|^2 - \lambda_4|\phi|^4/2$$

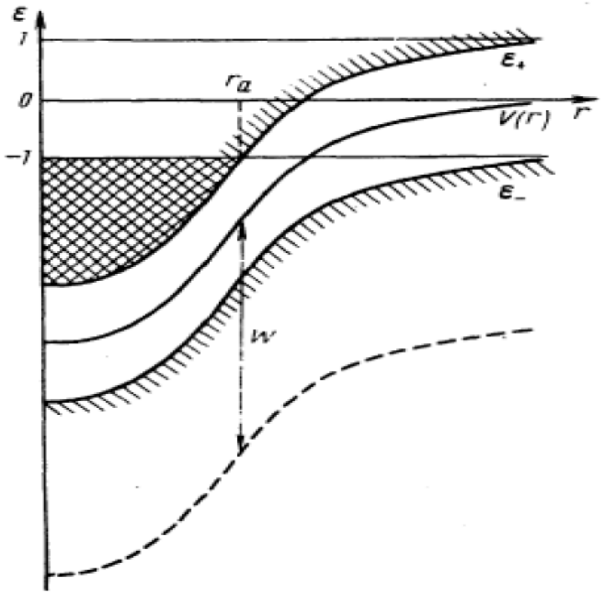
Let $m^2>0$, e.g. $\pi^\pm$. After replacement $\phi \to e^{-i\epsilon t}\phi_{\rm st}$ equation of motion renders

$$\Delta\phi_{\rm st} + [(\epsilon - V)^2 - m^2]\phi_{\rm st} - \lambda_4|\phi_{\rm st}|^2\phi_{\rm st} = 0 \qquad (*)$$

For $\lambda_4=0$: Classical field ↔QM one-particle problem

$$\Delta\phi + [(\epsilon - V)^2 - m^2]\phi = 0\,,$$

$\epsilon \to -\epsilon$ is equivalent to replacement $V \to -V$, or $e \to -e$



Boundaries of upper and lower continuums in field *V(r) now:*

$$c^2\vec{p}^2(r) = (\epsilon_\pm - V)^2 - m^2 = 0$$

For $V<V_c \approx -2m$ single-particle level in broad potential well *$R>>1/m$* crosses boundary *ε=-m,* **π^-s occupy dangerous level**, π^+-s with $-\epsilon$ tunnel to infinity.

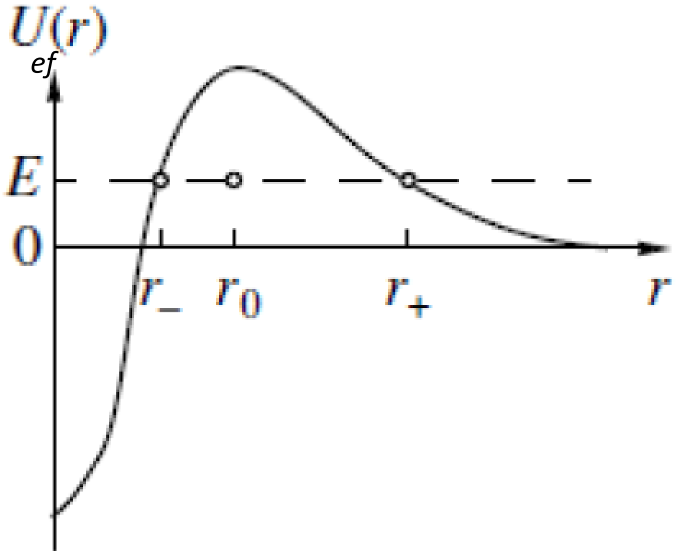
After doing replacements $E = \frac{\epsilon^2 - m^2}{2m}\,, \quad U_{\rm ef} = -\frac{V^2 - 2\epsilon V}{2m}$

get **Schroedinger form** of equation of motion $\Delta\phi + 2m(E - U_{\rm ef})\phi = 0\,.$

Notice relativistic attraction V^2 term in $U_{\rm ef}$ even for repulsive V.

Effective potential U_{ef} (r) for V<-2*m*, (for $|V|>|V_c\,|$), *E>0, is* presented in Fig.

.



On this language **tunneling from lower to upper continuum in electric field is equivalently described as decay of empty quasistationary state**

Is it possible to observe $\pi^+\pi^-$ production for *V<-2m*?

Construct plain capacitor at $d \gtrsim 10^3$ cm $\quad |\nabla A_0| = |\vec{E}| \sim 10^5$ V/cm,

and we get $V<-2m_\pi$! However probability of production of pion pairs per time

$W \sim e^{-2\mathrm{Im}s}$ → **0** since $\quad \mathrm{Im}S = \int_{x_1}^{x_2} |p| dx = \frac{\pi}{2}\frac{E_0}{E}\,, \qquad E_0 = \frac{m^2c^3}{e\hbar}\,,$

$$\text{for pions } E_0 \simeq 7 \cdot 10^{20} \text{ V/cm.}$$

Note in passing that condition $\delta V<-2m$ is required for applicability of Sauter-Schwinger production of pairs in uniform electric field.

Charged bosons in Coulomb field. Case $\lambda_4 = 0$. Falling to centrum

Let $V=-Ze^2/r$. With the help of the replacement $\phi(\vec{r}) = R(r) Y_{lm}$

$$\Rightarrow \quad \Delta R + \frac{2m}{\hbar^2}\left[E + \frac{(Ze^2)^2}{2mc^2r^2} - \frac{l(l+1)}{2mr^2} + \frac{\epsilon Ze^2}{mc^2 r}\right] R = 0\,, \qquad E = \frac{\epsilon^2 - m^2c^4}{2mc^2}$$

$$U_{\rm ef}(r) = \boxed{-\frac{(Ze^2)^2}{2mc^2r^2}} + \frac{l(l+1)}{2mr^2} - \frac{\epsilon Ze^2}{mc^2 r}$$

In nonrel.mechanics falling to centrum for $U = -\alpha/r^2$ at $\alpha > \vec{L}^2/(2m)$

Semiclassical approximation $l(l+1) \to (l+1/2)^2$ $\Rightarrow$ For $Z > Z_{cr} = \hbar c/(2e^2)$

particle in ground state falls to center. $Z_{cr} = 68.5$

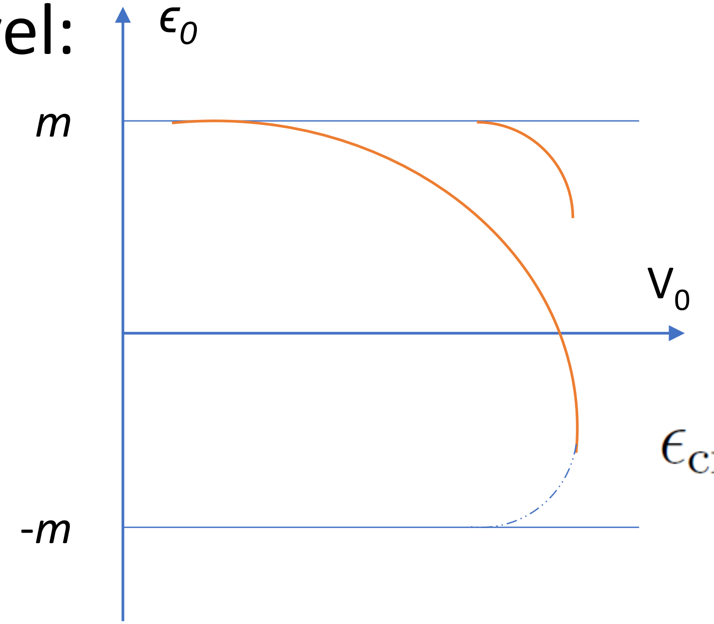
Puzzle (?) What to do with half of Mendeleev table?

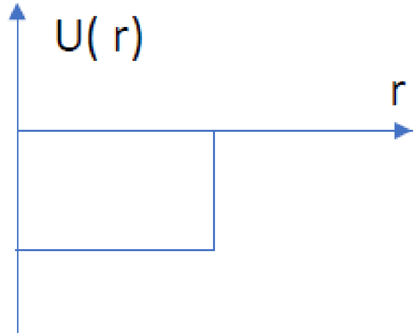
For a symmetric nucleus $A \simeq 2Z$ we estimate $R > a_{1B} = \hbar c/(m_\pi Z e^2)$ (Bohr radius for pion) already for $Z > 40$. Then the lowest pion orbit enters the nucleus and approximation of the point-like nucleus becomes invalid.

A peculiarity: ground-state level crosses boundary ε=-*m* only provided $-V(r\to\infty) > C/r^2$, $C = const > 0$. Otherwise there is bound state for antiparticle (in repulsive potential !--relativistic effect) .

For broad square well $V = -V_0 < 0$ for $r < R$, $V = 0$ for $r > R \gg 1/m$.

Ground-state level:





$$\epsilon_{\rm cr} \simeq -m(1 - O(1/(m^2R^2))).$$

In crit. point sum of particle and antiparticles energies reaches 0 and level becomes occupied. $E = \epsilon_{0+}\rho_{\rm ch}(\epsilon_{0+}) - \mathcal{L}(\epsilon_{0+}) + (-\epsilon_{0-})\rho_{\rm ch}(-\epsilon_{0-}) - \mathcal{L}(-\epsilon_{0-})$

However for $V_0 > V_{0c} \approx 2m$ single particle problem should be reconsidered with taking into account of $\lambda_4 > 0$ or including electromagnetic coupling.

Case $\lambda_4 \neq 0$. Weak binding of antiparticle is destroyed by multi-particle interactions $\lambda_4 \neq 0$, π^+ go off to infinity, π^- form condensate, cf. slide 37 above.

For broad square well similarly to the case of one-component field solving eq. (*) now for $\varepsilon_{0c} \approx -m$ we have

$$\Rightarrow \quad \phi = \pm e^{-i\epsilon_{0c}t}\theta(-m_{\rm ef}^2)\phi_0 \mathrm{th}[(r-R)|m_{\rm ef}|/\sqrt{2}]\,, \quad r<R\,, \quad \phi_0 = (|m_{\rm ef}^2|/\lambda_4)^{1/2}\,. \quad (**)$$

$$m_{\rm ef}^2 = -V_0^2 - 2\epsilon_{0c}V_0 < 0, \quad \varepsilon_{0c} \approx -m,\ V_{0c} \approx 2m\ .$$

For simplicity assume that there is positively charged substrate compensating the charge of π^- condensate. Then,

$$\mathcal{E} = -\int \mathcal{L} d^3x = -\frac{m_{\rm ef}^4}{2\lambda_4} V_3 \theta(-m_{\rm ef}^2) + O(R^2) \qquad \text{cf. slide 32}$$

In the critical point $(d\mathcal{E}/dV_0)_{V_{0c}} = 0$, $(d^2\mathcal{E}/dV_0^2)_{V_{0c}} \neq 0$.

Thus we again deal with **2-order phase transition.**

Idea of supercharged superheavy nuclei-nuclearites-stars

baryon density in the heavy nucleus is approximately constant

$n_0 \simeq 0.16\,\mathrm{fm}^{-3} \simeq 0.5 m_\pi^3,\ V_3 = 4\pi R^3/3,\ A = N + Z,$

The positive Coulomb energy grows with the charge of the nucleus

Z as $0.6 Z^2 e^2/R \sim Z^{5/3}$, where $R \simeq Z^{1/3}/m_\pi$ for $N \sim Z$.

for $Z \gtrsim 1/(5e^3)$ Coulomb energy would exceed nucleus binding energy -16 *A* MeV

➡ **to exist large size systems should be electrically neutral!**

Consider nucleus with $A \gtrsim 10^3$ at $N \simeq Z,\ n \simeq n_0$

$$\mathcal{E} = -\mathcal{E}_{\rm bind} + \mathcal{E}_{\rm sur} + \int d^3x\,\{|\nabla\phi|^2 + [m_b^2 - (\epsilon_b - V)^2]|\phi|^2 + \lambda_4|\phi|^4/2 - (\nabla V)^2/(8\pi e^2) - n_p V\} \quad (*)$$

First 2 terms follow from nuclear droplet model,

equilibrium condition for the reaction $n \leftrightarrow p + b^-$ renders $\mu_n = \mu_p$ that for $N = Z$, i.e. for $\mu_n = \mu_p$, requires $\epsilon_b = 0$, where μ_n, μ_p and ϵ_b are chemical potentials of the constituents. The lowest energy level of the boson in the deep potential well reaches zero for $V \le V_c \simeq -m_b$. Simplifying put $\lambda_4=0$.

Minimizing (*) get: $\Delta\phi + [(\epsilon_b - V)^2 - m_b^2]\phi = 0\,, \quad \Delta V = 4\pi e^2[2(V - \epsilon_b)|\phi|^2 + n_p]\,.$

Inside the large-size system solutions $|V| = m_b$ and $|\phi|^2 = n_p/(2m_b)$

$\Rightarrow$ $\mathcal{E} = -\mathcal{E}_{\rm bind} + m_b A/2$ For $|V|=Ze^2/R > m_b$, for $Z>1/e^3$, π^- level reaches $\epsilon = 0$,

π^- Bose condensate is formed inside nucleus. No tunneling is required.

Thus, if in the Naure existed light charged bosons with the mass $m_b \le 31$ MeV, there would exist superheavy nuclei, nuclearites and nuclei–stars with the baryon density $n \sim n_0$, $N \sim Z$, stable due to the condensation of the light charged bosons. Sufficiently heavy nuclei-stars would form the black holes.

We studied charged scalar field in electric potential.
Now study essence of Higgs-Meissner effect, and vortices at rotation.

Higgs effect (Interaction of scalar and electromagnetic field)

Interaction of complex scalar field ϕ with own electromagnetic field A_μ :

$$\mathcal{L} = (\partial_\mu + ieA_\mu)\phi(\partial^\mu - ieA^\mu)\phi^* - m^2|\phi|^2 - \lambda|\phi|^4/2 - F_{\mu\nu}F^{\mu\nu}/(16\pi) .$$

Energy density of own (no source term) electromagnetic field $\mathcal{E} = (\vec{E}^{\,2} + \vec{H}^{\,2})/(8\pi)$.

Thus vacuum can be taken with A_μ=0 and for m^2>0, ϕ=0.

Now let m^2<0. Ground state: A_μ=0, $\phi = \phi_0 e^{i\alpha}$, with $\phi_0 = \sqrt{|m^2|/\lambda}$, α is real constant.

After spontaneous symmetry breaking occurred we may put in vacuum state α=0,

we present $\phi = \phi_0(1 + \rho'(x))e^{i\chi'(x)}$ and $A_\mu = A'_\mu$.

In quadratic approximation over new fields we have

$$\mathcal{L} = \phi_0^2|\partial_\mu\rho' + ieA'_\mu + i\partial_\mu\chi'|^2 + 2m^2\phi_0^2\rho'^2 - F'_{\mu\nu}F'^{\mu\nu}/(16\pi) ,$$

$$F'_{\mu\nu} = \partial_\mu A'_\nu - \partial_\nu A'_\mu .$$

Do trick: introduce new gauge field $\widetilde{A}_\mu = A'_\mu + i\partial_\mu \chi'/e$

In new variables $\mathcal{L} = \phi_0^2 \partial_\mu \rho' \partial^\mu \rho' + \phi_0^2 e^2 \widetilde{A}_\mu \widetilde{A}^\mu + 2m^2 \phi_0^2 \rho'^2 - \widetilde{F}_{\mu\nu} \widetilde{F}^{\mu\nu}/(16\pi)$,

$\widetilde{F}_{\mu\nu} = \partial_\mu \widetilde{A}_\nu - \partial_\nu \widetilde{A}_\mu$. Goldstone mode is ``eaten'' by gauge transformation.

Variation of action over new fields yields equation of motion: $\omega^2 = k^2 + \mu^2$ for **neutral field ρ´** with mass $\mu = \sqrt{2|m^2|}$ and $\partial_\nu \widetilde{F}_{\mu\nu} = 8\pi e^2 \phi_0^2 \widetilde{A}_\mu$. $\omega^2 = k^2 + e^2 \phi_0^2$ describing **massive photons,** $m_{ef,\gamma} = |e\phi_0|$

In absence of spontaneous symmetry breaking (for $m^2>0$) we had 2 degrees of freedom for charged complex field and 2 polarizations for massless photon. In presence of spontaneous symmetry breaking (for $m^2<0$) –also 4 modes, but one describes real massive scalar field and other 3, massive photon with 3 polarizations –**Higgs effect**

Recall presence of gapped excitations in superconductors !

Superfluidity and superconductivity of complex scalar field

Lagrangian and equations of motion:

$$\mathcal{L} = D_\mu\phi D^{*\mu}\phi^* - m_{\rm sc}^2|\phi|^2 - \lambda|\phi|^4/2 - F_{\mu\nu}F^{\mu\nu}/(16\pi) - J_\mu^{\rm ext}A^\mu$$

$\phi = (\phi_1 - i\phi_2)/\sqrt{2}$ is the spin-zero complex field of a negatively charged boson (e.g., of π^- meson), ϕ_1 and ϕ_2 are real components, A_μ is electromagnetic potential, $m > 0$ is the mass of the boson, U is the external scalar potential $m_{\rm sc}^2 = m^2 + U$, $J_\mu^{\rm ext}A^\mu$ appears, if there are external 4-current sources of electromagnetic field

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\,, \quad D_\mu\phi = (\partial_\mu + ieA_\mu - i\mu_\pi\delta_{\mu 0})\phi\,,$$

For a piece of charge-neutral nuclear matter would be $\mu_{\pi^-} = \mu_e = \mu_n - \mu_p \neq 0$

$$D_\mu D^\mu\phi + m_{\rm sc}^2\phi + \lambda|\phi|^2\phi = 0\,,$$

$$\partial_\mu F^{\mu\nu} = -4\pi\delta\mathcal{L}_\phi/\delta A_\nu = 4\pi J^\nu \qquad \partial_\nu J^\nu = 0$$

$$J^\nu = -ie\phi D^{*\nu}\phi^* + \text{c.c.} + J^\nu_{\rm ext}\,, \qquad \text{c.c. denotes complex conjugation}$$

Let us deal with static uniform magnetic field and static scalar potential well such as $m_{\rm ef}^2 = m_{\rm sc}^2 = m_0^2 < 0$ in half space $x<0$, $m_{\rm ef}^2 = m^2 > 0$ for $x > 0$

Condensate energy is minimal for static field ϕ. ➡

$$(\nabla - ie\vec{A})^2\phi - m_{\rm ef}^2\phi - \lambda|\phi|^2\phi = 0\,, \qquad (*)$$

$$\Delta\vec{A} = -4\pi\vec{J}\,,\ \vec{J} = ie(\phi\nabla\phi^* - \phi^*\nabla\phi) - 2e^2\vec{A}|\phi|^2 + \vec{J}_{\rm ext},\ \ {\rm div}\vec{A} = 0 \qquad (**)$$

Let external current produces uniform constant magnetic field *H parallel z*.

In Lorenz gauge $\vec{A}_{ext} = (0, Hx, 0),\ H = const.$

Appropriate thermodynamical potential is *Gibbs energy:*

$G = F - \vec{M}\vec{H} - \vec{H}^2/(8\pi)\,,\ \vec{M} = (\vec{h} - \vec{H})/(4\pi)$--magnetic moment, $\vec{h} = {\rm curl}\vec{A}.$

$$G = |(\nabla - ie\vec{A})\phi|^2 + m_{\rm ef}^2|\phi|^2 + \frac{\lambda}{2}|\phi|^4 + \frac{(\vec{h} - \vec{H})^2}{8\pi}\,,$$

being for our convenience shifted on constant value $\vec{H}^2/(8\pi)$

Response of charged complex scalar field on uniform magnetic field

Meissner-Higgs effect:

Typical lengths: l_ϕ of change of condensate field at *x*<0, l_h -of change $A_y(x)$ at *x*<0,

and Larmor radius $R_H = p_\perp/(eH) \sim 1/(R_H eH)$ ➡ $R_H = 1/\sqrt{eH}$

For $l_\phi \gg l_> \sim 1/m$ we may put $\phi(x=0)=0$ and put $\phi(x>0)=0.$

Boundary condition for *A*: $A'_y(x\to 0) = H,\ |A_y(x)| < \infty.$

Properties in magnetic field are determined by relations between length scales.

For $1/l_\phi \gg eHl_h$ in Eq. $(\nabla - ie\vec{A})^2\phi - m_{\rm ef}^2\phi - \lambda|\phi|^2\phi = 0$ we may drop $A_y(x)$

➡ $\phi = f_0 \mathrm{th}[x/(\sqrt{2}l_\phi)]$, $x<0$, $f_0 = \pm\sqrt{-m_0^2/\lambda}\,\theta(-m_0^2)$, $l_\phi = 1/\sqrt{|m_0|}.$

From eq. (**) we get $A''_y(x) - 8\pi e^2|\phi|^2 A_y(x) = 0$, $x \le 0$.

For $l_h = 1/\sqrt{8\pi e^2 f_0^2} \gg l_\phi$ we may put $|\phi|^2 = f_0^2$ ➡ $A_y(x\le 0) = Hl_h e^{x/l_h}$

$m_\gamma = 1/l_h$ has sense of effective photon mass in superconducting region *x*<0

Quantity $\kappa = l_h/l_\phi$ is named Ginzburg-Landau parameter.

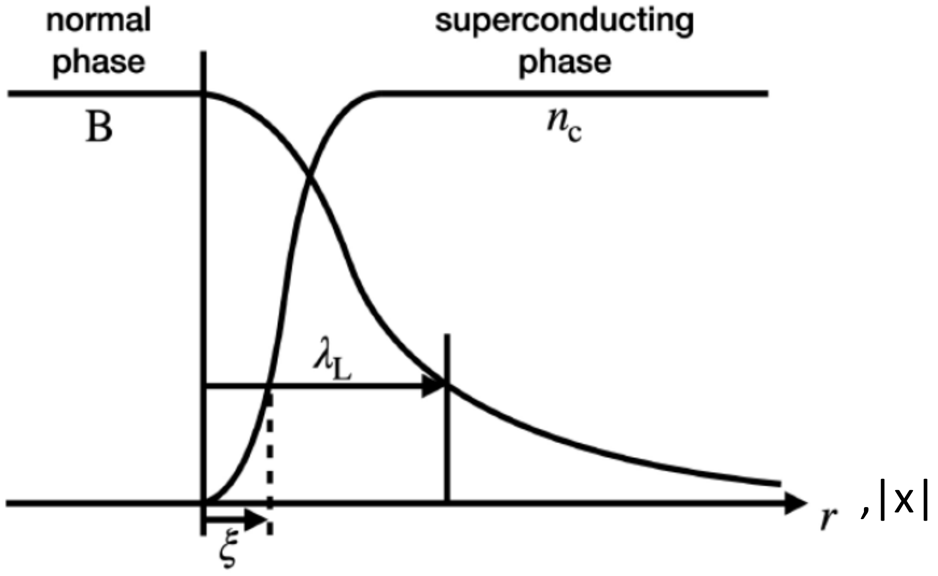
Superconductors of II-type for $\kappa > 1/\sqrt{2}$ In case of hadronic interactions λ~1 and

$$\kappa = \sqrt{\lambda/(8\pi e^2)} \gg 1.$$

➡ ***We deal with superconductor of II-type***

The inequality $1/l_\phi \gg eHl_h$ can be rewritten as $H \ll H_{\rm cr}$, where $H_{\rm cr} = \sqrt{4\pi}|m^2_{{\rm ef},0}|/\sqrt{\lambda}$, $H_{\rm cr}$ is named the thermodynamical critical field.

In condensed matter physics $l_\phi = \xi$ is named coherence length, $l_h = \lambda_{\rm L}$ London penetration depth.



Normal phase B and superconducting n_c phase in the type II superconductor. The external magnetic field H extends out of normal phase with the London penetration depth. The superconducting n_c phase extends out with the ξ coherence length.

Mixed Abrikosov state:

$$(\text{phase I: } \phi = f_0,\ \overline{h} = 0),\ \text{where } \vec{h} = \text{curl}\vec{A},$$

$$\overline{G}_{\text{I}} = -\frac{m_{\text{ef},0}^4}{2\lambda} + \frac{H^2}{8\pi}\,.$$

$$(\text{phase II: } \phi = 0,\ \overline{h} = H),$$

$$\overline{G}_{\text{II}} = 0\,.$$

for $H < H_{\text{cr}}$ the condensate phase is energetically favorable in the volume, since $\overline{G}_{\text{I}} < \overline{G}_{\text{II}}$

Mixed Abrikosov state: for $\kappa > 1/\sqrt{2}$ in interval of fields $H_{\text{cr}1} < H < H_{\text{cr}2}$

surface energy is decreased at formation of **normal vortices**, of transversal size l_ϕ directed parallel $\vec{H}$. Magnetic field decreases at a larger distance $r \sim l_h$

$r = \sqrt{x^2 + y^2}$ Gibbs free energy gain due to appearance of single vortex is $\sim -\pi l_h^2 d_z H^2/8\pi$ and the energy loss is $\sim \pi l_\phi^2 d_z m_{\text{ef},0}^4/2\lambda$.

Equate these quantities. ➡ Gibbs free energy is gained for $H > H_{cr1} \sim H_{\text{cr}}/\kappa$.

Also the value $H = H_{cr1}$ can be evaluated from equating $R_H(H_{cr1})$ and l_h.

For $H > H_{\rm cr1}$ vortices interact similarly to filamentary currents and form lattice.

Triangular lattice of vortices proves to be most energetically favorable .

With increasing *H* distance between vortices, *d*, decreases, the condensate field ϕ weakens and vanishes for $H=H_{cr2}$. For *H* slightly below H_{cr2} condensate field is weak and the equation of motion (*) can be linearized and presented as:

$$-\frac{(\nabla - ie\vec{A})^2\phi}{2m_{\rm aux}} \simeq -\frac{m_{\rm ef}^2\phi}{2m_{\rm aux}} \qquad (***) \qquad \vec{A} \simeq (0, Hx, 0).$$

We get Schroedinger equation for nonrelativistic spinless particle in uniform magnetic field, where m_{aux} is an auxiliary mass coefficient.

Spectrum of Eq. (***) is $E = |eH|(n+1/2)/(m_{\rm aux})$,

with $E_{\rm min} = \frac{|m^2_{\rm ef,0}|}{2m_{\rm aux}} = |e|H_{\rm cr2}/2m_{\rm aux}$ $\quad H_{\rm cr2} = |m^2_{\rm ef}/e| = H_{\rm cr}\sqrt{2}\kappa$.

Abrikosov state arises for $H_{\rm cr} < H_{\rm cr2}$ that happens for $\kappa > 1/\sqrt{2}$.

In the theory of electro-weak interactions there appears Abrikosov lattice of the fields of the *W* and *Z* bosons. Such a structure could occur in early Universe.

Also value $H_{\rm cr2}$ can be estimated by equating of the particle curvature radius $R_H(H_{\rm cr2})$ and the minimal length in the problem l_ϕ.

Nontrivial topology:

Let $\phi = \phi_0 e^{-i\Phi}$, where Φ is a phase of the field ϕ. Then the current density is

$$\vec{J} = -2e\phi_0^2(\nabla\Phi + (e/\hbar c)\vec{A})\,.$$

Presence of the term $\nabla\Phi$ allows to describe superfluid motion of the system in absence of the electromagnetic field. Deeply inside the interior of the superfluid we have $\vec{J} = 0$ and ${\rm curl}\vec{A} = 0$ but can be $A_i \neq 0$, cf. Aharonov-Bohm effect.

$$\oint \vec{A}d\vec{l} = \int \mathrm{curl}\vec{A}d\vec{S} = \int \vec{B}d\vec{S} = -\frac{\hbar c}{e}\oint \nabla\Phi d\vec{l} = \frac{\hbar c}{|e|}\delta\Phi \,.$$

From condition $\delta\Phi = 2\pi k$, where k is the integer number, we find the value of the minimal magnetic flux $(\int \vec{B}d\vec{S})_{min} = 2\pi\hbar c/|e| = 4\cdot 10^{-7}$ G·cm^2.

Discrete values of current associated with vortices.

Landau critical velocity:

Relation between energies in rest and lab. Frames $E' = E - \vec{p}\,\vec{V}$

p is momentum of Bose excitation in rest frame of fluid, *V* is speed of system.

Landau necessary condition of superfluidity is fulfilled for $pV < \omega(p)$

May hold for Goldstones and gapped modes, does not hold for $\omega(p)=p^2/2m$

Response of neutral superfluid on rotation— again vortices:

System described by the complex scalar field **with $m_0^2<0$** behaves as a superfluid.

Let vessel with superfluid rotates with constant angular velocity $\Omega \,||\, z$ and T=0.

For $\Omega < \Omega_{c1}$ superfluid does not rotate. For $\Omega > \Omega_{c1}$ there appear filament-vortices.

To find vortex solution seek $\phi = f_0 \chi(r) e^{i\xi(\varphi)}$,

φ is the azimuthal angle, f_0 and ξ are real. Field ϕ should not change at replacement

$\varphi \to \varphi + 2\pi k$, $k = \pm 1, \pm 2, \ldots$ Integrating $\nabla \xi = m_{\rm qp} \vec{v}$ over the circle in x, y plane,

➡ $\oint d\vec{l} \nabla \xi = 2\pi k$,and thus we should take $\xi = k\varphi$, $\nabla_\varphi \xi = k/r$, $\nabla = (\partial_r, r^{-1}\partial_\varphi, \partial_z)$

➡ $v_\varphi = k/(m_{\rm qp} r)$ In our case $m_{qp} = \sqrt{2} m_0$, in case He-II, it is mass of He atom.

$$(\partial_x^2 + x^{-1}\partial_x \boxed{- k^2/x^2})\chi + \chi - \chi^3 = 0\,,\ \chi(x \to 0) \to 0 \text{ and } \chi(x \to \infty) \to 1.$$

$x = r/l_\phi$ and χ are dimensionless variables, cf. slide 29. Asymptotic solutions are:

$$\chi = 1 - k^2/(2x^2) \text{ and for } x \to 0 \text{ we get } \chi \propto x^{|k|}$$

At large distances the matter is in the equilibrium state $\phi = \phi_0$.

Energy of the vortex is $\mathcal{E}_{\rm kin} = \int d^3x |\nabla\phi|^2 = d_z 2\pi k^2 f_0^2 \ln(R/l_\phi)$ cf. slide 17,

d_z is the length in longitudinal z direction. We cut integration on transversal size R of rotating system in case of the single vortex line and on the distance R_L between vortices otherwise. Other terms in the energy density change on the size $r \sim l_\phi$ and can be dropped for $R \gg l_\phi$.

Angular momentum of vortex line **at a slow rotation**

$$\vec{L} = \int d^3x [\vec{p} \times \vec{r}]\,|\,, \quad p_\varphi = m_{\rm qp} k |\phi|^2 / r\,, \qquad L_z \simeq k\pi d_z R^2 f_0^2 m_{\rm qp}$$

In rotating frame $E' = E - \vec{L}\vec{\Omega},$ At $T \neq 0$: $G = F - \vec{L}\vec{\Omega}\,,$

$\mathcal{E}_{\rm kin} - \vec{L}\vec{\Omega} < 0$ yields $\Omega > \Omega_{c1} = 2|k| \ln(R/l_\phi)/(R^2 m_{\rm qp})\,,$

where we should put $|k|=1$, since it corresponds to minimum energy of vortex.

With increasing rotation velocity vortices form triangular lattice and system begins rotate as the rigid body with linear velocity $v=\Omega R$ provided number of vortices in lattice $N_v=R^2/R_L^2$ is given by $N_v\kappa = n_v\pi R^2\kappa = 2\pi R\cdot\Omega R$,

$\kappa = 2\pi k/m_{\rm qp}$, $2\pi R\cdot\Omega R$ is the total circulation

The distance between vortices, R_L, decreases with increasing Ω. Minimal distance

$R_L \sim l_\phi$ → Condensate field disappears for $\Omega > \Omega_{c2} \sim |k|/(l_\phi^2 m_{\rm qp})$

Nucleons in nuclei and neutron stars form Cooper pairs at $T < T_{cr} \lesssim (0.1-1)$ MeV.

For neutron stars, R=10 km, $\Omega_{c1} \sim 10^{-14}$ sec^{-1}. For all pulsars $\Omega \gg \Omega_{c1}$. Rigid body.

For Vela pulsar, period P=0.083 sec., distance between vortices is $3\cdot 10^{-3}$ cm.

For all millisecond pulsars $\Omega<10^4$ Hz.

$\Omega_{c2} \sim 10^{21}$ sec^{-1}. For such large Ω causality relation, $v=\Omega R<c$, is spoiled.

However for $\Omega>10^4$ Hz centrifugal force at $r=R$ enlarges gravitational one and pulsars with such Ω cannot exist.

Winding number *k* is conserved: vortices are produced together with antivortices.

Antivortices can be absorbed on walls of rotating vessel.

In absence of rotation vortex is not energetically favorable but if produced together with antivortex, it ``lives'' due to conservation of *topological charge k*.

Vortices and antivortices $\leftrightarrow$ baryons & antibaryons (skyrmions).

Cf. slide 20 Goldstone model. ➡

In a sense the Goldstone model allowing for presence of the very massive vortices, middle-heavy ρ', and massless χ particles can be considered as the simplest model unifying the "strong" and "electromagnetic" interactions. The vortices play a role of massive protons, neutral ρ' play a role of σ mesons and χ of massless (scalar) photons.

We studied essence of Higgs-Meissner effect, and vortices at rotation.
Now study meson-baryon interaction and meson mean fields.

Models of meson-baryon interaction

The simplest choice for the *NN* interaction could be the local contribution

$\mathcal{L}_{NN} = g_{NN}(\bar{N}N)^2 \qquad N^\dagger = (p, n),$ dimensionality of the $\bar{N}N$ is $1/l^3$

$m_N \simeq 938$ MeV. Perturbation theory parameter $g_{NN}E^2$, diverges for $E \to \infty$

Such a theory is not renormalizable, e.g., Fermi theory of weak interactions was replaced by renormalizable Weinberg-Salam theory of electro-weak interactions where fermions interact via exchange by the massive *W, Z* bosons.

Meson responsible for *NN* attraction at large distances is pseudoscalar iso-vector pion, $m_\pi \approx 140$ MeV, at intermediate distances, σ scalar meson, $m_\sigma \approx 600$ MeV,

$\mathcal{L}_{\sigma N} = -g_{\sigma N}\sigma\rho_s = -g_{\sigma N}\sigma\bar{N}N$, coupling is *g* dimensionless,

pseudoscalar *NπN* attraction $\mathcal{L}^{\rm ps}_{\pi NN} = -ig_{\pi N}\bar{N}\gamma^5\vec{\tau}\vec{\pi}N$. τ –isospin Pauli matrices,

$\gamma^5 = \gamma_5 = -i\gamma^0\gamma^1\gamma^2\gamma^3$ is the Dirac matrix. Coupling is real & dimensionless.

Simplest coupling between vector field A_μ and spinor field is constructed as scalar product of A_μ and 4-flux density, $A^\mu \bar{N} \gamma^\mu N$. For electromagnetic field interacting with protons, *p*, we get $\mathcal{L}_{Ap} = -eA_\mu j_p^\mu = -eA_\mu \bar{p}\gamma^\mu p\,,$ where $e^2 = 1/137$

Repulsion of the neutral vector meson Ω with the baryons (here nucleons):

$\mathcal{L}_{\Omega NN} = -g_{\Omega N} \bar{N} \gamma^\mu \Omega_\mu N\,,$ with $m_\Omega \approx 783$ MeV, real dimensionless coupling.

For pseudovector field B_μ interacting with nucleons $\mathcal{L}_{BNN} = -g_{BN} \bar{N} \gamma^5 \gamma^\mu N B_\mu\,.$

A pseudovector can be constructed from pseudoscalar pion field using derivative $\partial_\mu \pi$. $\mathcal{L}^{\mathrm{pv}}_{\pi NN} = f_{\pi N} \bar{N} \gamma^\mu \vec{\tau} \partial_\mu \vec{\pi} \gamma^5 N\,,$ $f_{\pi N}$ has dimensionality of l.

Theory is not renormalizable, but is used as a phenomenological theory.

Divergences are cut at (quark) distances ~ 0.2-0.3 fm.

$\rho^{\pm 0}$ meson field acts as vector in Lorentz sense and vector in the isospin space:

$\mathcal{L}_{\rho NN} = -g_{\rho N} \bar{N} \gamma^\mu \vec{\tau}\, \vec{\rho}_\mu N\,.$

Relativistic mean-field (Walecka) model

Construct a simple mean-field model of nuclear eq. of state matter at $\rho \sim \rho_0 \approx 0.16$ fm^{-3} employing σ, Ω interactions with nucleons. Pseudoscalar pions do not contribute in mean-field approximation.

$$\mathcal{L}_{N\sigma\Omega} = \frac{\partial_\mu \sigma \partial^\mu \sigma}{2} - \frac{m_\sigma^2 \sigma^2}{2} + \bar{N}(i\gamma^\mu \partial_\mu - m_N)N - g_{\sigma N} \bar{N} N \sigma$$

$$- \frac{\Omega_{\mu\nu}\Omega^{\mu\nu}}{4} + \frac{m_\Omega^2 \Omega_\mu \Omega^\mu}{2} - g_{\Omega N} \bar{N} \gamma^\mu \Omega_\mu N \,,$$

Heaviside units, $\Omega_{\mu\nu} = \partial_\mu \Omega_\nu - \partial_\nu \Omega_\mu$ Eqs. of motion:

$$\partial_\mu \partial^\mu \sigma + m_\sigma^2 \sigma + g_{\sigma N} \bar{N} N = 0 \,, \quad \partial_\mu \partial^\mu \Omega^\nu + m_\Omega^2 \Omega^\nu - g_{\Omega N} \bar{N} \gamma^\nu N = 0 \,,$$

$$(i\gamma^\mu \partial_\mu - m_N - g_{\sigma N}\sigma - g_{\Omega N}\gamma^\mu \Omega_\mu)N = 0 \,.$$

σ, Ω_0 condensates appear as responses on sources of scalar and baryon charges.

Solutions: perform averaging over the ground state $\sigma = -g_{\sigma N} \langle \bar{N} N \rangle / m_\sigma^2 \,,$

$\rho_{s,N} = \langle \bar{N} N \rangle$ is the scalar nucleon density.

$\Omega^0 = g_{\Omega N} \langle N^\dagger N \rangle / m_\Omega^2 \,.$ $\vec{\Omega}$ leads to increase of the energy, since $\langle \bar{N} \vec{\gamma} N \rangle = 0$.

$m_N^* = m_N + g_{\sigma N}\sigma = m_N - \rho_{s,N} g_{\sigma N}^2 / m_\sigma^2$ effective nucleon mass in the medium.

Consider media *N*=*Z* and *Z*=0 as applications to nuclei and neutron stars

Nucleon (baryon) density: $$\rho_N = \langle N^\dagger N \rangle = \gamma \int \frac{d^3p}{(2\pi)^3} n_N(p)\,,$$

$\gamma = 4$ for $N = Z$ and $\gamma = 2$ for $Z = 0$, at zero temperature $n_N = \theta(p_{\mathrm{F}N} - p)$, $p_{\mathrm{F}N}(n_N)$ is the nucleon Fermi momentum.

Static mean-field contribution to energy density is $E_{\mathrm{MF}} = -\langle \mathcal{L}_{\mathrm{MF}} \rangle$.

Adding kinetic fermion term:

$$E(\rho_{s,N}) = \frac{g_{\sigma N}^2 \rho_{s,N}^2}{2m_\sigma^2} + \frac{g_{\Omega N}^2 \rho_N^2}{2m_\Omega^2} + \frac{\gamma}{(2\pi)^3} \int d^3p \sqrt{m_N^{*2}(\rho_{s,N}) + p^2}\, n_N(p).$$

Employing $\partial E(\rho_{s,N})/\partial \rho_{s,N} = 0$ ⇒ $$\rho_{s,N} = \frac{\gamma}{(2\pi)^3} \int \frac{d^3p\, m_N^*(\rho_{s,N})}{\sqrt{m_N^{*2}(\rho_{s,N}) + p^2}} n_N(p).$$

Energy depends on two fitting parameters $$C_\sigma^2 = \frac{g_{\sigma N}^2 m_N^2}{\hbar c^3 m_\sigma^2}\,, \quad C_\Omega^2 = \frac{g_{\Omega N}^2 m_N^2}{\hbar c^3 m_\Omega^2}\,.$$

the nuclear saturation density $\rho_N = \rho_0 \simeq 0.193\ \mathrm{fm}^{-3}$.

$\mathcal{E} = E/\rho_N = m_N c^2 - 15.75\ \mathrm{MeV}$ ➡ $C_\sigma^2 = 266.9\,,\quad C_\Omega^2 = 195.7\,.$

$m_N^*(\rho_0) \simeq 0.56 m_N$, $E = K(\rho - \rho_0)^2/(18\rho_0^2)$ near $\rho \simeq \rho_0$ proves to be $K \simeq 540$ MeV.

Maximum neutron star mass $M \simeq 2.57 M_\odot$.

Attractive features of Walecka model: simple-- only 2 fitting parameters, Lorentz invariant, describes qualitatively *NN* potential and behavior of equation of state of nuclear matter. Deficiency: *K* is too large, *m** is too small.

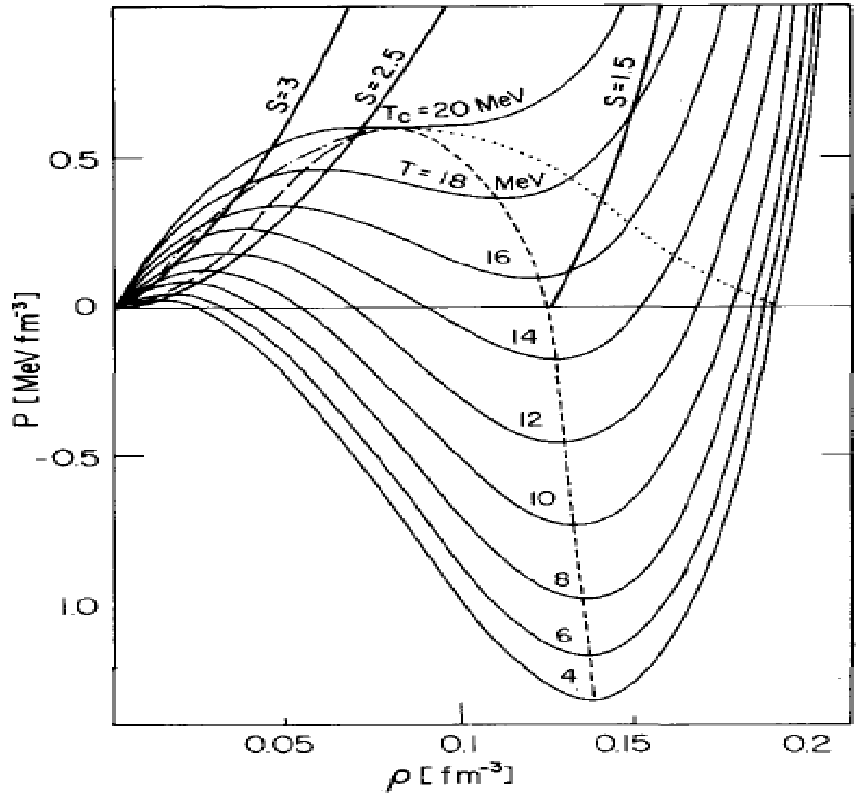
Generalization to finite *T*: Nucleon/antinucleon distributions $n_N(T,\mu_N^*) + \bar{n}_N(T,\mu_N^*)$,

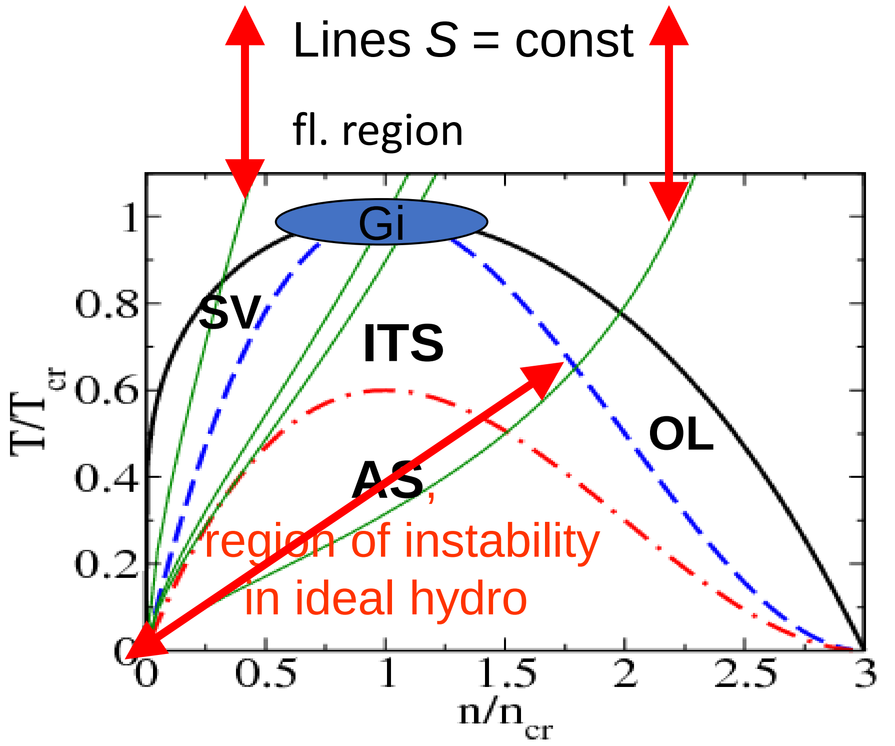
$$n_N(T,\mu_N) = \frac{1}{1+\exp[(\sqrt{m_N^{*2}(\rho_{s,N},T)+p^2} - \mu_N^*)/T]}\,,\ \bar{n}_N(T,\mu_N^*) = n_N(T,-\mu_N^*)$$

Massive bosons are less affected by *T* than fermions occu[ying Fermi sea ➡ meson fluctuation term is disregarded.

I-order liquid-vapor phase transition. Van der Waals-like EoS, nonequilibrium effects: spinodal and fluctuations

Liquid-vapor tr. in Walecka model, Schulz et al PhLB 1983





- - isothermal spinodal (ITS) unstable region, isothermal compressibility K_T<0, - . - . - adiabatic spinodal (AS), solid line - Maxwell construction

Liquid-nuclear clusters, vapor-nucleon gas

1-order phase transitions in HIC: nuclear vapor-liquid & feasibly HQ

Nuclear liquid-vapor tr. at low HIC energies:

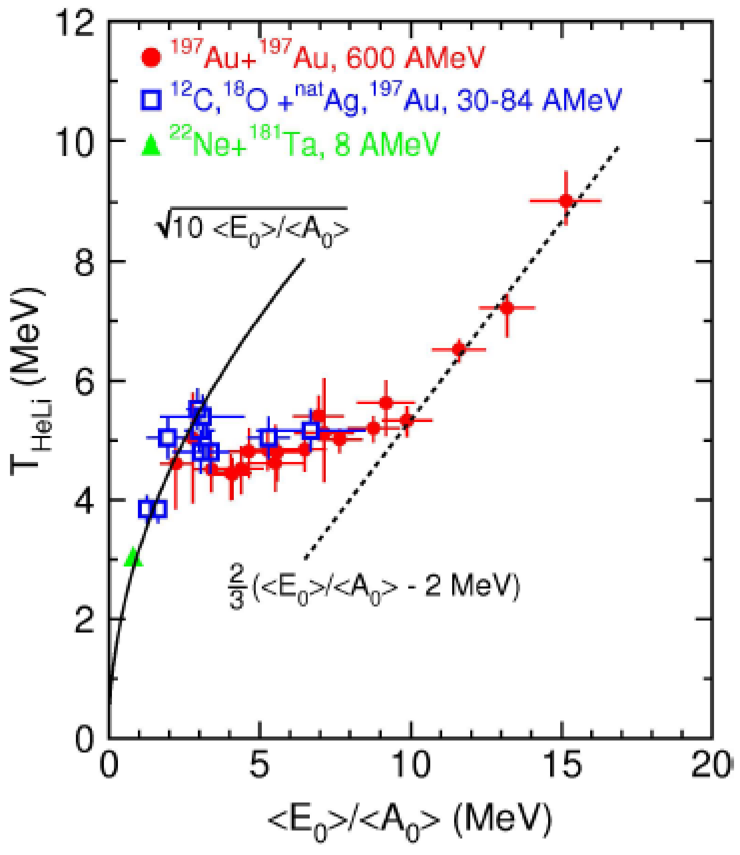
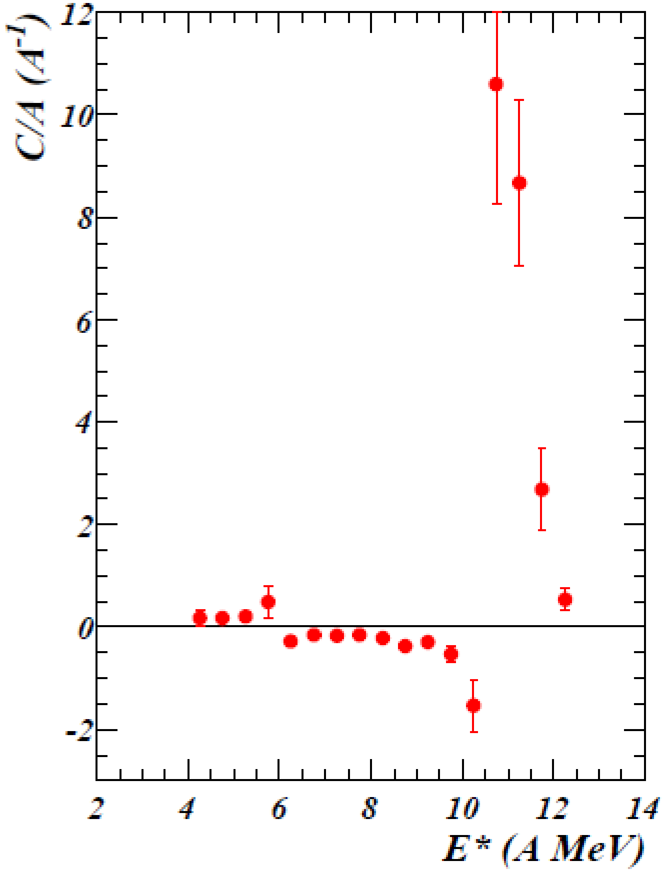


Fig. 27. Caloric curve of nuclei determined by the dependence of the isotope temperature on the excitation energy per nucleon (see text). From [165]. Pochodzalla et al (1995)

Region with $C_p < 0$



Borderie et al., EPJA, **56**, 101 (2020)

Negative heat capacity which signs a first order phase transition for finite systems is observed

We studied meson-baryon interaction and meson mean fields.
Now study nonrelativistic limit and phenomenology of superfluidity and superconductivity.

Transition to nonrelativistic limit. Condensation of nonrelativistic bosons in potential wells

Take $\phi = e^{-imt}\Phi$, where Φ is a smoothly varying function on the time scale $t \sim 1/m$. Besides that consider the case, when the scalar field $|U| \ll m^2$, and electromagnetic fields are $|V| \ll m$, $|\vec{A}| \ll m$. Retaining first derivatives $\dot{\Phi}$

$$\mathcal{L} = im\Phi^*\dot{\Phi} - im\Phi\dot{\Phi}^* - 2Vm\Phi\Phi^* - (\nabla - ie\vec{A})\Phi(\nabla + ie\vec{A})\Phi^*$$
$$-\lambda_4|\Phi|^4/2 - U|\Phi|^2 - F_{\mu\nu}F^{\mu\nu}/(16\pi) - J_\mu^{\text{ext}} A^\mu + \dots,$$

$\lambda_1 = \lambda_4/(4m^2)$, employing notations $\Psi = \Phi\sqrt{2m}$, $u = U/(2m)$ ⇒

$$\mathcal{L} = \frac{i}{2}\Psi^*\dot{\Psi} - \frac{i}{2}\Psi\dot{\Psi}^* - (V+u)\Psi\Psi^* - \frac{(\nabla - ie\vec{A})\Psi(\nabla + ie\vec{A})\Psi^*}{2m}$$
$$-\lambda_1|\Psi|^4/2 - \frac{F_{\mu\nu}F^{\mu\nu}}{16\pi} - J_\mu^{\text{ext}} A^\mu,$$

Variation of the action in Ψ^* yields $i\dot{\Psi} = -\frac{(\nabla - ie\vec{A})^2\Psi}{2m} + v\Psi + \lambda_1|\Psi|^2\Psi$ (*)

Scalar and electric fields enter this equation in a combination $v = V + u$

Difference appears only as a relativistic effect!

Flux conservation: $\partial_t|\Psi|^2+\mathrm{div}\vec{\mathcal{F}}=0\,,\quad \vec{\mathcal{F}}=\frac{\hbar}{2mi}(\Psi^*\nabla\Psi-\Psi\nabla\Psi^*)+\frac{\hbar e\vec{A}}{m}|\Psi|^2\,,$

$\rho=-\partial\mathcal{L}/\partial V=-i\Psi\partial\mathcal{L}/\partial\dot{\Psi}+i\Psi^*\partial\mathcal{L}/\partial\dot{\Psi}^*=|\Psi|^2$ in our classical treatment is density of particles at time moment *t*. Consider spherical broad potential well: at $v=const<0$ for $r<R$ and zero for $r>R$ we have $|\Psi|^2V_3\simeq N$, where $V_3=4\pi R^3/3$. *N* is number of bosons put in potential well on ground state level.

For $R-r\gg l_<, |\Psi|^2=|\Psi_0|^2=-v\theta(-v)/\lambda_1$ ➡ $\mathcal{E}_0=-v^2V_3/(2\lambda_1)+O(R^2)\,.$

In relativistic problem potential had another dimensionality. In nonrelativistic problem sum of energies of particle and antiparticle is $2m+v$. Thus energy conservation does not allow for production of pairs in a shallow potential well. **However, if there are external bosons obeying conservation law of their number,** being put in the potential well, $v<0$, they may occupy ground state level forming Bose condensate. Value of condensate field is limited either by repulsive self-interaction $\lambda_1>0$, or, in case $\lambda_1 \ll e^2$, by electromagnetic interaction of particles.

Cold Bose gases: In approximation of weakly interacting particles wave function of system of bosons is approximately given by a product of single-particle functions

$$\Psi(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_N) = \psi(\mathbf{r}_1)\psi(\mathbf{r}_2)\ldots\psi(\mathbf{r}_N).$$

At a low temperature de Broglie wavelength is much larger than range of boson–boson interaction, and one deals with s-wave scattering and local interaction. Hamiltonian of system can be written as

$$H = \sum_{i=1}^{N}\left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial \mathbf{r}_i^2} + V(\mathbf{r}_i)\right) + \sum_{i<j}\frac{4\pi\hbar^2 a_s}{m}\delta(\mathbf{r}_i - \mathbf{r}_j),$$

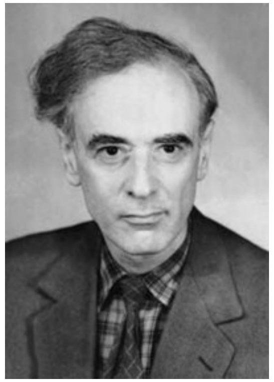
m is mass of boson, *V* is external potential, a_s is the boson–boson *s*-wave scattering length. Employing variational procedure we recover so called Gross-Pitaevskii equation describing cold Bose gases:

$$\left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial \mathbf{r}^2} + V(\mathbf{r}) + \frac{4\pi\hbar^2 a_s}{m}|\psi(\mathbf{r})|^2\right)\psi(\mathbf{r}) = \mu\psi(\mathbf{r})$$

Cf. with (*).

$\mu = gN/V_3$ is the chemical potential. $g = 4\pi\hbar^2 a_s/m$

Second-order phase transitions in condensed matter



Macroscopic wave function as order parameter:

In superfluid He and superconducting materials for $T<T_c$, or for pressure $P>P_c$, there occurs second order phase transition to condensate state. In ^{4}He neutral Bose atoms form superfluid Bose condensate. In metallic superconductors electrons become bound in Cooper pairs forming Bose condensate mean field, typical size of the Cooper pair ~10^{-4} cm >> a_B -Bohr radius.

For $0<T_c-T<<T_c$ (Landau condition), the typical length scale for the change of the classical field proves to be much larger than distance between atoms.

Free energy functional:

In mentioned cases free energy density, $F[\psi, T]$, can be expanded in Taylor series in a complex order parameter (the macroscopic wave function ψ). Parameters of expansion are further expanded in series in T_c-T. Also in quasi-uniform approximation the free energy density is expanded in $|\nabla\psi|^2$.

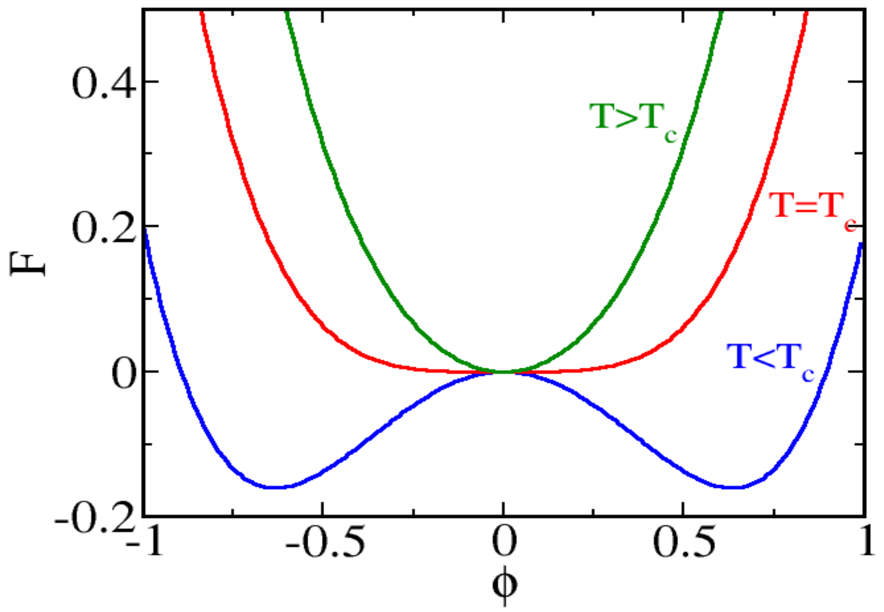
$E = \dot{\Psi}\partial\mathcal{L}/\partial\Psi + \dot{\Psi}^*\partial\mathcal{L}/\partial\Psi^* - \mathcal{L}$, does not depend on $\dot{\Psi}$ in linear approximation

$$F[\psi, T] = \frac{|\nabla\psi|^2}{2m^*(T_c)} + \boxed{\alpha(T_c)(T - T_c)}|\psi|^2 + \beta(T_c)|\psi|^4/2 + \ldots, \qquad \text{(L1)}$$

where we used expansion of the coefficients m^*, α and β in $T - T_c$ and put $T = T_c$ except coefficient proportional to $|\psi|^2$, since experiment shows that in case of ^4He and metallic superconductors we deal with the second-order phase transitions, and therefore equilibrium value of order parameter vanishes at $T=T_c$.

Thereby we expand $\alpha(T)$ and put $\alpha(T) = \alpha(T_c)(T - T_c)$-

Acts as T dependent squared mass term in $\lambda\phi^4$ theory, $\alpha(T_c)>0$. For the stability of ground state we also need $m^*(T_c)>0$, and self-interaction $\beta>0$.



Minimization of $F[\psi, T]$ yields order parameter

$$|\psi|^2 = \alpha(T_c)\theta(T_c - T)/\beta(T_c)\,. \quad F = -\,\alpha^2(T-T_c)^2/4\beta.$$

Ginzburg-Landau phenomenological theory of superconductivity

In superconductors field ψ interacts with electromagnetic field. Static magnetic field enters in combination

$$(\nabla - iq\vec{A})\psi, \text{ where } q = 2e$$

$|\psi|^2 \propto \Delta$, where Δ is the Cooper pairing gap, *q*=2*e* charge of pair.

$$F[\psi, \vec{A}] = \frac{|(\nabla - iq\vec{A})\psi|^2}{2m^*} + \boxed{\alpha(T_c)(T - T_c)}|\psi|^2 + \beta(T_c)|\psi|^4/2 + \frac{(\text{curl}\vec{A})^2}{8\pi}\,.$$

Cf. response of charged complex scalar field on external magnetic field which we have considered above in slides 47-58.

Fluctuation region near T_c . Ginzburg criterion

Probability of fluctuation $W \sim exp(-\delta F(T) V_{fl}/T)$,
where free energy loss $\delta\ F^{MF} \sim \alpha^2 (T-T_c)^2/\beta$, in minimal volume
$V_{fl} \sim l_0^3$, $l_0 \sim 1/(T_c-T)^{1/2}$ is coherence length. At $T \sim T_{fl}$ the fluctuation formed in a minimal volume $\sim l_0^3$ is probable ($W \sim 1$). Fluctuations are dominant for T near T_c , $|T_c - T_{fl}|/T_c < 1$.

Fluctuation region is estimated by Ginzburg number $\quad Gi = |T_c - T_{fl}|/T_c$
In clean metalic superconductors Gi $\sim 10^{-8}$, fluctuation region is narrow, in He^4 , in color superconductors, at quark-hadron and hadron liquid-gas phase transitions, (strong interaction) fluctuation region is broad Gi >0.1-1.

Energy variance $\overline{(\Delta E)^2} = T^2 C_V$, where C is specific heat, Fluctuation contribution to C diverges in critical point of second-order phase transition.

In presence of singularities assumption that free energy is analytical function fails.
Therefore Taylor expansion, which we used to construct free energy density *F* becomes invalid in vicinity of critical point, in fluctuation region.

In case of ^4He fluctuation region proves to be broad and inclusion of fluctuations is important at all *T*. Free energy density can be presented now as

$$F[\psi, T] = \frac{|\nabla\psi|^2}{2m^*(T_c)} + \boxed{\alpha(T_c)|T - T_c|^\mu}|\psi|^2 + \beta|T - T_c|^\nu|\psi|^4/2 + \dots,$$

μ and ν are real (not integer) numbers.

Assuming that dependence on T-T_c should disappear from the Ginzburg criterion provided long-range fluctuations are properly incorporated and that
specific heat does not diverge at least as power-low (experimental fact) we find
$\mu = 4/3,\ \nu = 2/3$ being in good agreement with data.

Galilei transformation for nonrelativistic superfluid:

Galilei transformation $\vec{r} \to \vec{r}\,' + \vec{V} t'$ and $t = t'$, $\partial_{\vec{r}\,'} = \partial_{\vec{r}}$, $\partial_{t'} = \partial_t + \vec{V}\partial_{\vec{r}}$

$\Psi(t,\vec{r}) = e^{i\theta(t,\vec{r})}\Psi'(t',\vec{r}\,')$, In order Shroedinger eq. would not change in lab. system

$$\theta(t,\vec{r}) = \frac{m\vec{V}\vec{r}}{\hbar} - \frac{mV^2 t}{2\hbar} \quad \Rightarrow \quad \boxed{\left[\frac{(\hat{\vec{p}} + m\vec{V})^2}{2m} - \frac{mV^2}{2}\right]}\Psi'(t,\vec{r}-\vec{V}t) = i\hbar\partial_t\Psi'(t,\vec{r}-\vec{V}t)\,,$$

m is here effective quasiparticle mass.

Relation between energies in rest and lab. frames E´=E - $\vec{p}\,\vec{V}$ ⇒

Landau necessary condition of superfluidity fails for ω (*p*)- *pV*<0,

holds for Goldstones and gapped modes, does not hold for ω (*p*)=*p*^{2}/2*m*

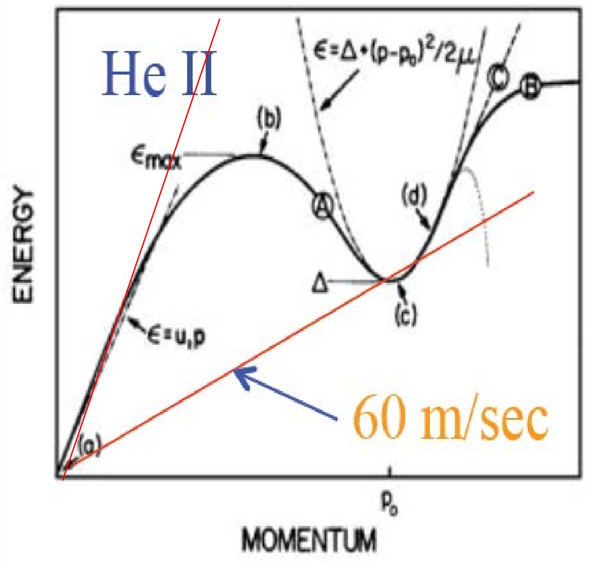
We studied nonrelativistic limit and phenomenology of superfluidity and superconductivity.
Now study condensation of Bose excitations in non-uniform state in uniformly moving media.

Condensation of Bose excitations in non-uniform state in uniformly moving media

The spectrum of excitations in the ^{4}He is shown in Fig. 1.

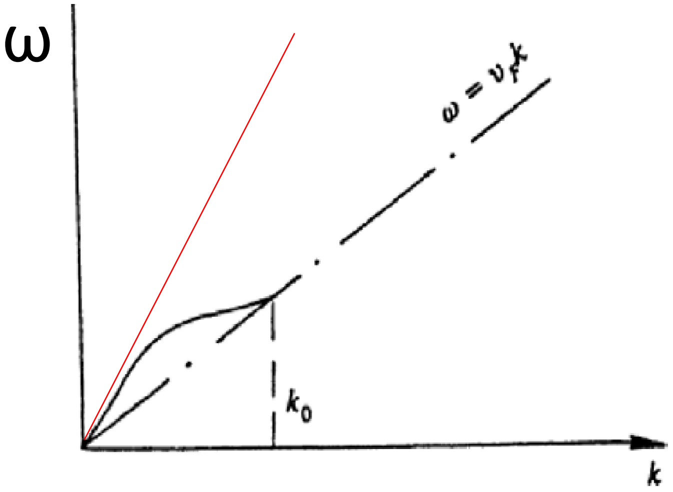
Landau critical velocity $V_c^{\mathrm{L}} = \min(\omega(k)/k)$ at $k \approx k_0 \neq 0$.

For $V > V^{\mathrm{L}}_c$ excitations may form condensate at $k \approx k_0$.



In Fermi liquids at $V > v_F$ where v_F is Fermi velocity, zero sound excitations may form condensate, shown in Fig.2.

Cf. levons in cold atomic gases.



Related phenomena in open systems:

shock wave after airplain at $V > v_{sound}$, Cherenkov radiation for $V > c_{med}$

The key idea: When a medium moves as a whole with velocity higher than V^L_c respectively laboratory frame, it may become energetically favorable to transfer a part of the momentum from ``normal'' particles of the moving medium or from moving walls to Bose condensate of excitations with $k_0 \neq 0$, if the spectrum of excitations is soft for $k \sim k_0 \neq 0$. In absence of excitations energy density of rectilinearly moving superfluid: $E^{\rm kin}_{\rm in} = \rho_M \vec{V}_{\rm in}^{\,2}/2,\ \rho_M = M/V_3$ mass density.

Seek field of condensate of excitations in the form $\phi' = \phi'_0 e^{-i\omega(\vec{k}_0)t + i\vec{k}_0\vec{r}}$

where ϕ'_0 is a real quantity. A part of initial momentum can be transferred to condensate of excitations: $\rho_M \vec{V}_{\rm in} = \rho_M \vec{V}_{\rm fin} + \vec{k}_0 \phi_0'^2$.

Energy density after condensation has occurred, is $E^{\rm kin}_{\rm fin} = \rho_M \vec{V}_{\rm fin}^{\,2}/2 + \omega(k_0)\phi_0'^2 + \lambda\phi_0'^4/2$,

the term $\lambda\phi_0'^4/2$ describes the self-interaction of the condensate

$$\delta E^{\rm kin} = E^{\rm kin}_{\rm fin} - E^{\rm kin}_{\rm in} = -(\vec{k}_0 \vec{V}_{\rm in} - \omega(k_0))\phi_0'^2 + \tilde{\lambda}\phi_0'^4/2\,, \quad \tilde{\lambda} = \lambda + k_0^2/\rho_M .$$

Energetically favorable is to take $\vec{k_0} \parallel \vec{V}$. Minimization yields

$$\phi_0'^2 = \frac{k_0(V_{\rm in} - V_c^{\rm L})}{2\tilde{\lambda}}\theta(V_{\rm in} - V_c^{\rm L}) \,, \;\; \delta E^{\rm kin} = -\frac{k_0^2(V_{\rm in} - V_c^{\rm L})^2}{2\tilde{\lambda}}\theta(V_{\rm in} - V_c^{\rm L}) \,.$$

for $V_{\rm in} > V_c^{\rm L} = \omega(k_0)/k_0$ formation of inhomogeneous condensate of excitations with $k = k_0$ becomes energetically profitable.

Second-order phase transition.

At $\lambda \to 0$, for arbitrary initial velocity $V > V^{\rm L}_c$ resulting speed of medium tends to $V^{\rm L}_c$

Vacuum instabilities and some phase transition phenomena have been studied on example of the scalar field theory.

Thanks for attention!

Dynamical symmetry breaking. Quantum fluctuations

In theories dealing with massless particles symmetry breaking may appear, as consequence of radiative corrections, S. Coleman, E. Weinberg (1973).

Consider massless scalar field:

$$\mathcal{L} = \frac{\partial_\mu \phi \partial^\mu \phi}{2} - \frac{\lambda \phi^4}{4} \,.$$

$\phi = \phi_c + \phi'$, where ϕ_c is the long-range scaled classical field, ϕ' describes quantum fluctuations. In new variables

$$\mathcal{L} = \frac{\partial_\mu \phi_c \partial^\mu \phi_c}{2} - \frac{\lambda \phi_c^4}{4} + \frac{\partial_\mu \phi' \partial^\mu \phi'}{2} - \frac{6\lambda \phi_c^2 \phi'^2}{4} - \frac{4\lambda \phi_c^3 \phi'}{4} - \frac{4\lambda \phi_c \phi'^3}{4} - \frac{\lambda \phi'^4}{4} \,.$$

We are interested in quantities averaged over vacuum, such as $\langle \mathcal{L} \rangle$,

Then linear and cubic terms can be dropped due to averaging of oscillations and

$$\mathcal{L} = \frac{\partial_\mu \phi_c \partial^\mu \phi_c}{2} - \frac{\lambda \phi_c^4}{4} + \frac{\partial_\mu \phi' \partial^\mu \phi'}{2} - \frac{3\lambda \phi_c^2 \phi'^2}{2} + O(\phi'^4) \,.$$

Let us introduce the Euclidean action $S_E = -iS = -i\int d^4x\mathcal{L}$, $S_E = S_E^{\rm cl} + S'_E$,

$$S_E^{\rm cl} = -i\int d^4x\left(\frac{\partial_\mu\phi_c\partial^\mu\phi_c}{2} - \frac{\lambda\phi_c^4}{4}\right).$$

For constant fields $S_E^{\rm cl} = \Omega_4\frac{\lambda\phi_c^4}{4}$, $\Omega_4 = i\int d^4x = \int d^4\tilde{x}$, where $\tilde{x} = (\vec{r}, ix_0)$.

$$S'_E = -i\int d^4x\left(\frac{\partial_\mu\phi'\partial^\mu\phi'}{2} - \frac{3\lambda\phi_c^2\phi'^2}{2}\right) = \int d^4\tilde{x}\left(\frac{\tilde{\partial}_\mu\phi'\tilde{\partial}^\mu\phi'}{2} + \frac{3\lambda\phi_c^2\phi'^2}{2}\right)$$

expanded up to quadratic terms.

Employing the Fourier transformation $\phi'(x) = \int\frac{d^4k}{(2\pi)^4}\phi'_k e^{-ikx}$ we obtain

$$S'_E = \int\frac{d^4\tilde{k}}{(2\pi)^4}\frac{\tilde{k}^2 + 3\lambda\phi_c^2}{2}|\phi'(k)|^2,$$

$\tilde{k} = (\vec{k}, ik_0)$. Performing the functional integration with the help of the relation

$$e^{-S'_E} = N\int D\phi' e^{-S'_E[\phi']} = \prod_k dc'_k dc''_k e^{-(\tilde{k}^2+3\lambda\phi_c^2)(c'^2_k + c''^2_k)/2} = \prod_k\frac{2\pi N}{(\tilde{k}^2 + 3\lambda\phi_c^2)}$$

N is the normalization factor, $\phi'_k = c'_k + ic''_k$, with real c'_k, c''_k.

$$S'_E = -\frac{i}{2}\int \frac{d^4k}{(2\pi)^4}\Omega_4 \ln(k^2 - 3\lambda\phi_c^2) + const\,.$$

Introducing the effective potential, V, associated with the action, $S'_E = \Omega_4 V$,

$$V = V_{\rm cl} + V' = \frac{\lambda\phi_c^4}{4} - \frac{i}{2}\int \frac{d^4k}{(2\pi)^4} \ln(1 - 3\lambda\phi_c^2/k^2) + const\,. \qquad \text{(CW1)}$$

½ since integration is done over the half-space (real ϕ was expanded in Fourier complex integral). Taylor series of *ln*- allows for diagrammatic presentation

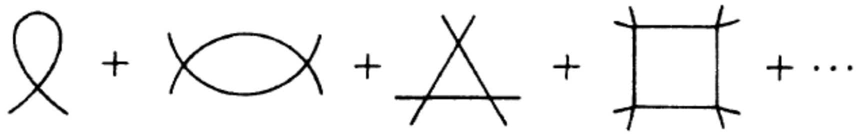
$$-iV' = \frac{1}{2}\int \frac{d^4k}{(2\pi)^4}\sum_{n=1}^{\infty}\frac{1}{n}\left[(-6i\lambda)\frac{i}{k^2}\frac{\phi_c^2}{2}\right]^n\,.$$

Integration is associated with the loop, vertex yields $(-6i\lambda)$

factor $\frac{i}{k^2}$ -line-Green function of massless particle,

$\frac{\phi_c^2}{2}$ corresponds to identical two external lines outgoing from each vertex, factor $1/n$ in each closed diagram counts number of vertices.

So, taking into account quadratic fluctuations is equivalent to summation of all one-loop contributions in effective potential:



Loop-expansion is equivalent to expansion over a formal parameter *a*,

as the parameter a we may use $\hbar$ $\mathcal{L}(\phi, \partial_\mu\phi, a) = a^{-1}\mathcal{L}(\phi, \partial_\mu\phi)$.(CW2)

Let P be the power of a in the graph. Then $P = I - v$,

I is the number of internal lines in the diagram, and v is the number of vertices.

Each propagator brings the factor *a*, since in Lagrangian it is inversely proportional to the operator $\phi'(\partial_\mu\partial^\mu + 3\lambda\phi_c^2)\phi'$, vertex enters together with the factor 1/*a*.

Number of loops: $L = I - v + 1$.

Each internal line brings one momentum integration, each vertex brings δ-function of conservation of the entering momenta, besides one δ –function, common for all diagrams, and *L* coincides with the number of remaining integrations. Thus

$$P = L - 1\,.$$

The loop expansion is convenient in a sense that result is not changed under shift of fields, since expansion goes over the pre-factor in front of the Lagrangian. So, expansion in *L* can be constructed independently, whether the vacuum is realized on the symmetry broken mode or not.

Since the term $\partial_\mu\phi\partial^\mu\phi$ enters as $\partial_\mu\phi\partial^\mu\phi/\hbar$, as the parameter a we may use $\hbar$ and the loop expansion can be considered as the semiclassical expansion. Employing Wick rotation

$$V' = \frac{1}{2}\int \frac{d^4\tilde{k}}{(2\pi)^4}\ln(1+3\lambda\phi_c^2/\tilde{k}^2)\,.\ \int_{-\infty}^{\infty} dk_0/i = \int_{-\infty}^{\infty} dk_4.\quad ik_0 = k_4 \qquad \text{(CW3)}$$

Using that $\int_0^\Lambda \tilde{k}^3 d\tilde{k}\ln(1+3\lambda\phi_c^2/\tilde{k}^2) = \frac{3\lambda\phi_c^2}{2}\Lambda^2 - \frac{9\lambda^2\phi_c^4}{8} - \frac{9\lambda^2\phi_c^4}{4}\ln\frac{\Lambda^2+3\lambda\phi_c^2}{3\lambda\phi_c^2}$

$$V' = \frac{3\lambda\phi_c^2}{32\pi^2}\Lambda^2 - \frac{(3\lambda\phi_c^2)^2}{64\pi^2}\left[\ln\frac{\Lambda^2}{3\lambda\phi_c^2} - \frac{1}{2}\right]. \qquad \text{(CW4)}$$

Renormalization can be performed adding to Lagrangian infinite contr-terms.

$$\mathcal{L} = \frac{\partial_\mu \phi_c \partial^\mu \phi_c}{2} - \frac{\lambda \phi_c^4}{4} - \frac{A \partial_\mu \phi_c \partial^\mu \phi_c}{2} - \frac{B \phi_c^2}{2} - -\frac{C \phi_c^4}{4} - V' .$$

To get the renormalized Green function we put *A*=0. In order to have zero renormalized mass we require

$$\left(\frac{d^2 V}{d\phi_c^2} \right)_{\phi_c = 0} = 0 , \quad \Rightarrow \quad B = -\frac{3\lambda\Lambda^2}{16\pi^2} .$$

Condition of the renormalization of the coupling constant can be rewritten as

$$\left(\frac{d^4 V}{d\phi_c^4} \right)_{\phi_c = M} = 6\lambda .$$

M cannot be taken zero, since it enters as *ln M*.

$$\Rightarrow \quad 6C = -\frac{27\lambda^2}{8\pi^2} \left(\ln \frac{3\lambda M^2}{\Lambda^2} + \frac{11}{3} \right) .$$

Finally $$V = \frac{\lambda \phi_c^4}{4} + \frac{9\lambda^2 \phi_c^4}{64\pi^2}\left(\ln \frac{\phi_c^2}{M^2} - \frac{25}{6}\right) + \text{higher loop terms}\,. \quad \text{(CW5)}$$

Minimization in ϕ_c yields $$\lambda \ln \frac{\phi_c^2}{M^2} \simeq -\frac{16\pi^2}{9} + O(\lambda^2)\,.$$

The resulting value is $|\lambda \ln \frac{\phi_c^2}{M^2}| \gg 1$

thereby two-loop correction terms are not small and cannot be dropped.

For massive particles (CW3) holds after performing replacement $\tilde{k}^2 \to \tilde{k}^2 + \mu^2$

Results (CW1), (CW4) hold after replacement $3\lambda\phi_c^2 \to \mu^2 + 3\lambda\phi_c^2$

In the limit $\mu \ll M$

$$V = \frac{\mu^2\phi_c^2}{2} + \frac{\lambda\phi_c^4}{4} + \frac{1}{64\pi^2}\left[(\mu^2 + 3\lambda\phi_c^2)^2 \ln(1 + \frac{3\lambda\phi_c^2}{\mu^2}) - 3\lambda\mu^2\phi_c^2 - \frac{75\phi_c^4}{2} + 9\lambda^2\phi_c^4 \ln \frac{\mu^2}{3\lambda M^2}\right] \quad \text{(CW6)}$$

Employing Eq. (CW6) and sum $\sum_{n=-\infty}^{\infty} (y^2+n^2)^{-1} = \pi y^{-1} \mathrm{cth}(\pi y)\,,$

and taking derivative $\frac{\partial V}{\partial \phi_c^2}$ we get $\phi_c^2 \frac{\partial V}{\partial \phi_c^2} = \phi_c^2 \frac{3\lambda T^2}{24}$ and recover

for the theory with potential $V(|\phi|^2) = \mu^2|\phi|^2 + \frac{\lambda_4|\phi|^4}{2}$

at $\mu^2<0$ the free energy density $F'[\phi_c] = -\frac{|m^2| - \lambda T^2/4}{2}\phi_c^2 + \frac{\lambda\phi_c^4}{4}\,.$

Cf. slide 35.

Return to model $V(|\phi|^2) = \mu^2|\phi|^2 + \frac{\lambda_4|\phi|^4}{2}$ including fluctuations perturbatively.

Let now $\mu^2=\alpha(T-T_c)<0$. Then Fourier component of free energy density is

$$F_k[\phi] = \frac{(k^2 + \alpha(T - T_c))|\phi_k|^2}{2} + \frac{\lambda|\phi_k|^4}{4}\,.$$

Separating fluctuation term (CW6) we have $V = \frac{\alpha(T - T_c)|\phi_c|^2}{2} + \frac{\lambda|\phi_c|^4}{4} + V'\,,$

$$V' \simeq \frac{T_c}{2}\sum_{n=-\infty}^{\infty}\int\frac{d^3k}{(2\pi)^3}\ln(\vec{k}^{\,2} + 3\lambda\phi_c^2(T) + \alpha(T - T_c) + 4\pi^2n^2T^2)\,.$$

Since fluctuations are important near critical point we have put $T=T_c$ in coefficient at Matsubara summation.

As a simplification, let us find the contribution of fluctuations to the specific heat assuming that it is still small compared to the contribution of the mean field.

put $\phi_c^2 = \alpha(T_c - T)/\lambda$

Since fluctuations are important at T near T_c, we have put $T = T_c$ in the coefficient at Matsubara sum but we should retain T dependence in the terms $\propto \alpha(T - T_c)$ and $\phi_c^2(T)$. Indeed, as we shall show, the converging part of the integral is concentrated near the pole at typical values $\vec{k}^{\,2} \sim \alpha(T_c - T)$. To separate this convergent part let us find T derivative of V'. So we find

$$\frac{\partial V'}{\partial T} = -T_c \alpha \sum_{n=-\infty}^{\infty} \int \frac{d^3k}{(2\pi)^3} \frac{1}{(\vec{k}^{\,2} + 2\alpha(T_c - T) + 4\pi^2 n^2 T_c^2)} \, .$$

For $4\pi^2 T_c^2 \gg \alpha(T_c - T)$ we may retain only one term $n = 0$ in the sum.

Calculate contribution to specific heat employing that $\int \frac{d^3k}{(2\pi)^3} \frac{1}{(\vec{k}^{\,2}+a^2)^2} = \frac{1}{8\pi a}$

$$C'_V = -T\frac{\partial^2 V'}{\partial T^2} \simeq 2\alpha^2 T_c^2 \int \frac{d^3k}{(2\pi)^3} \frac{1}{(\vec{k}^{\,2} + 2\alpha(T_c - T))^2} = \frac{T_c^2 \alpha^{3/2} 2^{1/2}}{8\pi\sqrt{|T_c - T|}}.$$

Now we are able to evaluate the width of the fluctuation region, where the fluctuation contribution to the specific heat, C'_V, exceeds the mean-field contribution

$$C_V^{\mathrm{MF}} = -T\frac{\partial^2 V^{\mathrm{MF}}}{\partial T^2} \simeq \frac{\alpha^2 T_c}{2\lambda}.$$

Equating $C'_V = C_V^{MF}(T_c)$ (criterion of Ginzburg–Levanyuk)

we find width of fluctuation region $\alpha|T_c - T_{\mathrm{fl}}| \simeq T_c^2\lambda^2/(8\pi^2)$. $Gi = |T_{\mathrm{fl}} - T_c|/T_c$

Fluctuation contribution for ^{4}He

$$F[\psi, T] = \frac{|\nabla \psi|^2}{2m^*(T_c)} + \alpha(T_c)|T - T_c|^{\mu}|\psi|^2 + \beta(T_c)|T - T_c|^{\nu}|\psi|^4/2 + \ldots,$$

$$\delta F \simeq \frac{T_c}{2}\int \frac{d^3k}{(2\pi)^3} \ln(c\vec{k}^{\,2} + 3\beta(T_c - T)^{\nu}|\phi_c^2(T) - \alpha(T_c - T)^{\mu}), \;\; c = \frac{1}{2m^*(T_c)}.$$

for $\phi_c^2 = \alpha(T_c - T)^{\mu-\nu}/\beta$ $\qquad \delta F \simeq \frac{T_c}{2}\int \frac{d^3k}{(2\pi)^3} \ln(c\vec{k}^{\,2} + 2\alpha(T_c - T)^{\mu}),$

$$C'_V = -T\frac{\partial^2 V'}{\partial T^2} \simeq (T_c \ll \alpha^{1/2}c^{3/2}/\beta\,; \frac{2\alpha^2\mu^2T_c^2}{(c\vec{k}^{\,2}+2\alpha(T_c-T)^{\mu})^2}$$

$$= (T_c - T)^{\frac{3}{2}\mu-2}\frac{T_c^2\alpha^{3/2}\mu^2 2^{1/2}}{8\pi c^{3/2}}.$$

$$C_V^{\rm MF} = -T\frac{\partial^2 V^{\rm MF}}{\partial T^2} \simeq (T_c - T)^{2\mu-\nu-2}\frac{\alpha^2(2\mu-\nu)(2\mu-\nu-1)T_c}{4\beta}.$$

For $\mu = 2\nu$, independently of the closeness to the critical point, for $T_c \ll \alpha^{1/2}c^{3/2}/\beta$, remaining fluctuation term $C'_V \ll C_V^{\rm MF}$.

END