Dense fermion systems in the center of compact stars

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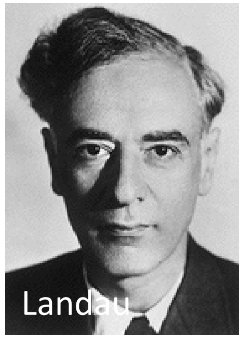
<u>Part I:</u>

- Rational Units
- Fermion gas

Cube of Physical Theories

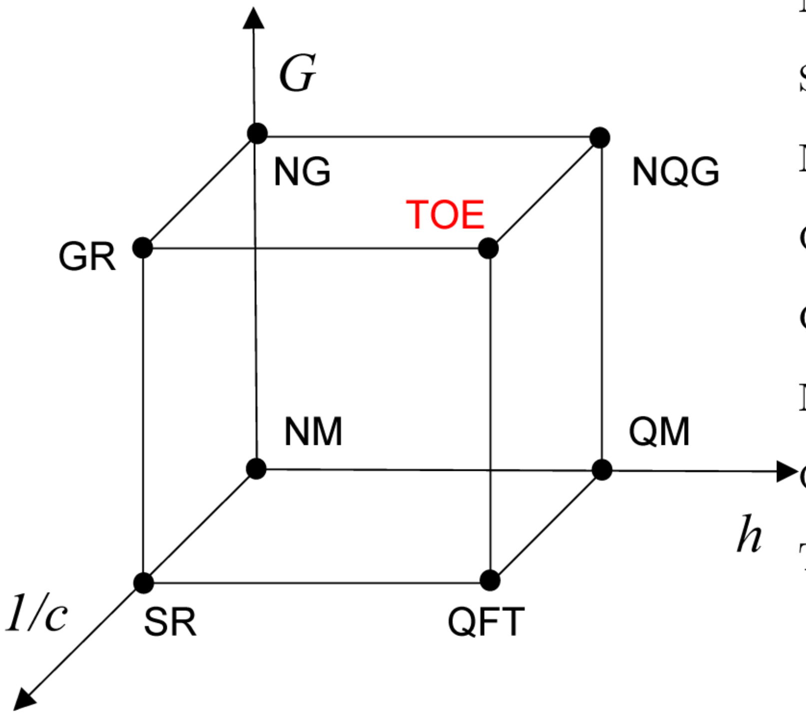


- Role of fundamental physical constants *c*, *G*, *h*

[Г. Гамов, Д. Иваненко и Л. Ландау, ЖРФХО. Ч. Физ. **60** (1928) 13]

[G. Gamow, D. Ivanenko, L. Landau, Journal of Russian Physicochemical Society, Ser. Physics **LX** (1928) 13]

- Bronshtein-Zel'manov's cube of physical theories



NM = Newton Mechanics

SR = Special Relativity

NG = Newton Gravity

QM = Quantum Mechanics

GR = General Relativity

NQG = Non-relativistic Quantum Gravity

QFT = Quantum Field Theory

TOE = Theory Of Everything

speed of light

$$c = 3 \times 10^{10}\ \frac{\mathrm{cm}}{\mathrm{s}}$$

gravitational constant

$$G = 6.8 \times 10^{-8}\ \frac{\mathrm{cm}^3}{\mathrm{g\,s}^2}$$

Planck constant

$$\hbar = 1.055 \times 10^{-27}\ \mathrm{erg\,s}$$

Planck units

$$m_{\mathrm{P}} = \left(\frac{\hbar\, c}{G}\right)^{1/2}$$

$$l_{\mathrm{P}} = \frac{\hbar}{m_{\mathrm{P}}\, c} \qquad t_{\mathrm{P}} = \frac{\hbar}{m_{\mathrm{P}}\, c^2}$$

practical units

$$\mathrm{m} \rightarrow \mathrm{cm}\,, \quad \mathrm{kg} \rightarrow \mathrm{g}\,, \quad \mathrm{Newton} = \frac{\mathrm{kg{\cdot}m}}{s^2} \rightarrow \mathrm{dyne} = \frac{\mathrm{g{\cdot}cm}}{s^2} = 10^{-5}\,\mathrm{Newton}, \quad \mathrm{Joule} = \frac{\mathrm{kg{\cdot}m^2}}{s^2} \rightarrow \mathrm{erg} = \frac{\mathrm{g{\cdot}cm^2}}{s^2} = 10^{-7}\,\mathrm{Joule}$$

Rational Units

electromagnetic interaction: Coulomb law $F = k\frac{q_1 q_2}{r^2}$ $k = 9\times 10^9 \frac{\mathrm{N\cdot m^2}}{\mathrm{C^2}}$ (SI) $k=1$ (CGSE)

electric charge: $e = 4.80\times 10^{-10}\left(\mathrm{CGSE} = \mathrm{dyn}^{1/2}\cdot\mathrm{cm} = \mathrm{g}^{1/2}\,\mathrm{cm}^{3/2}\,\mathrm{s}^{-1}\right) = 1.60\times 10^{-19}\,\mathrm{C}$

Energy units: $1\,\mathrm{Volt} = \frac{\mathrm{Joule}}{\mathrm{C}} = \frac{\mathrm{N\cdot m}}{\mathrm{C}}$ $1\,\mathrm{eV} = 1.60\times 10^{-19}\left(\mathrm{N\cdot m} = \mathrm{Joule}\right)$

“electron-Volts” $1\,\mathrm{MeV} = 1.60\times 10^{-13}\mathrm{Joule} = 1.60\times 10^{-6}\mathrm{erg}$ $1\ \mathrm{erg} = 6.3\times 10^5\ \mathrm{MeV}$

electromagnetic interaction: $k\,e^2 = 1.44\,\mathrm{MeV\cdot fm}$ Planck constant: $\hbar c = 197.327\,\mathrm{MeV\cdot fm} = 197.327\,\mathrm{eV\cdot nm}$

electrodynamic fine structure constant: $\alpha = \frac{k\,e^2}{\hbar c} \approx \frac{1}{137}$

Atomic units $e,\ \hbar,\ \text{and}\ m_e c^2 = 0.511\,\mathrm{MeV}$

Unit of length $r_\mathrm{B} = \frac{\hbar^2}{e^2 m_e} = \frac{\hbar^2 c^2}{e^2 m_e c^2} = \frac{\hbar c}{\alpha m_e c^2} = 0.529\times 10^{-10}\,\mathrm{m} = 0.529\,\text{Å}$ Bohr radius

Unit of energy $E_\mathrm{B} = \frac{e^2}{r_\mathrm{B}} = \frac{e^4 m_e}{\hbar^2} = \frac{e^4 m_e c^2}{\hbar^2 c^2} = \alpha^2 m_e c^2 = 27.2\,\mathrm{eV} = 2\,\mathrm{Ry}$ Rydberg (13.6 eV)

Unit of time $t_\mathrm{B} = \frac{\hbar}{E_\mathrm{B}} = \frac{\hbar c}{E_\mathrm{B}\,c} = \frac{7.25\mathrm{nm}}{c} = 2.41\times 10^{-17}\,\mathrm{s} = 24.1\,\mathrm{as}$ (attosecond)

43 attoseconds: the shortest pulses of laser light yet created

Nuclear units $(\hbar = 1 \text{ and } c = 1)$ $\hbar c = 197.327\,\text{MeV} \cdot \text{fm} = 1$ $\longrightarrow$ $1\,\text{fm} = \frac{1}{197.33\text{MeV}}$

The Yukawa potential of the strong interaction (long-range part) $V(r) = \frac{g^2}{4\pi}\frac{e^{-m_\pi c r/\hbar}}{r}$ $m_\pi c^2 = 140\,\text{MeV}$

Hence, the typical radius of nuclear forces is $r_{\text{nuc. force}} \sim \frac{\hbar}{m_\pi c} = 1.4\text{ fm}$ $1\,\text{fm} = 0.71\,m_\pi^{-1}$

Compare with the coefficient in the empirical formula for the nucleus size $R(A) = 1.2\,A^{1/3}$ fm

nucleon mass: $m_N = 940\,\text{MeV} = 6.7\,m_\pi$ nucleon number density: $n_{\text{nucl}} = (0.16\text{–}0.17)\,\text{fm}^{-3} \sim 0.5\,m_\pi^3$

nuclear matter density: $\varepsilon_0 \approx m_N\,n_{\text{nucl}} = 940\,\text{MeV}/c^2 \times 0.17\,\text{fm}^{-3} = 2.8 \times 10^{14}\,\frac{\text{g}}{\text{cm}^3}$

matter density: $\frac{m_\pi^4}{\hbar^3} = 48.58\frac{\text{MeV}}{c^2\text{fm}^3} = 8.65 \times 10^{13}\,\frac{\text{g}}{\text{cm}^3}$ pressure: $m_\pi^4\frac{c^2}{\hbar^3} = 48.58\frac{\text{MeV}}{\text{fm}^3} = 7.78 \times 10^{34}\,\frac{\text{dyne}}{\text{cm}^2}$

Geometrized units $(c = 1 \text{ and } G = 1)$ $G = 6.7 \times 10^{-8}\,\frac{\text{cm}^3}{\text{g}\,\text{s}^2}$ $c = 3 \times 10^{10}\,\frac{\text{cm}}{\text{s}}$

$1\,\text{s} = c = 3 \times 10^{10}\,\text{cm}$ $1\,\text{erg} = 8.23 \times 10^{-50}\,\text{cm}$ $1\,\text{dyne} = 8.23 \times 10^{-50}$

$1\text{g} = G/c^2 = 7.41 \times 10^{-29}\,\text{cm}$ $\frac{\text{g}}{\text{cm}^3} = 7.41 \times 10^{-29}\,\text{cm}^{-2}$ $\frac{\text{dyne}}{\text{cm}^2} = 8.23 \times 10^{-50}\,\text{cm}^{-2}$

$n_{\text{nucl}} = 1.95 \times 10^{-14}\,\text{cm}^{-2} = 1.95 \times 10^{-4}\,\text{km}^{-2}$

solar mass: $M_\odot = 2.0 \times 10^{33}\,\text{g} = 1.5\,\text{km}$

Temperature units

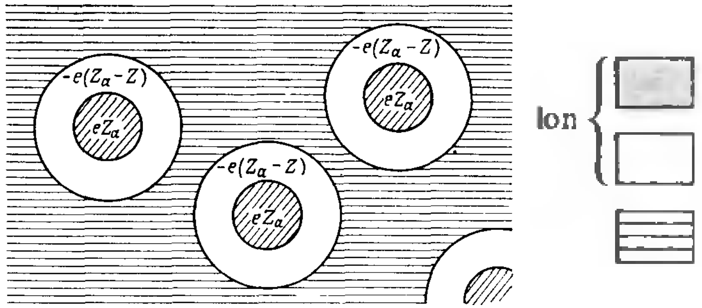
<u>Boltzmann constant</u> $$k_{\mathrm{B}} = 1.380649 \times 10^{-23} \frac{\mathrm{Joule}}{\mathrm{Kelvin}} = 8.617333262 \times 10^{-5} \frac{\mathrm{eV}}{\mathrm{Kelvin}}$$

$$k_{\mathrm{B}} = 1 \quad \Longrightarrow \quad 1\,\mathrm{eV} = 11604.5\,\mathrm{Kelvin} \approx 10^4\mathrm{Kelvin}$$

Fermionic systems

electron $m_e = 0.5\,\mathrm{MeV}, \quad s = \frac{1}{2}$

electron gas in metals



atomic nucleus eZ_a

atomic core $-e(Za-Z)$

conductivity electrons $-eZ$

$$n = Z\frac{\rho}{M_m}N_A \qquad \text{for Fe} \qquad \rho_{\mathrm{Fe}} = 7.874\,\frac{\mathrm{g}}{\mathrm{cm}^3}$$

$$N_A = 6.022 \times 10^{23}\,\frac{\text{atom}}{\text{mole}} \qquad M_m = 55.845\,\frac{\mathrm{g}}{\text{mole}} \qquad Z = 2\ (4s^2\ \text{electrons}) \qquad n = 17.0 \times 10^{22}\,\frac{\text{atoms}}{\mathrm{cm}^3}$$

typical number densities $n \sim (1\text{–}20) \times 10^{22}\,\mathrm{cm}^{-3}$

nucleons: protons and neutrons $m_N = 940\,\mathrm{MeV}, \quad s = \frac{1}{2}$

nuclear matter (Coulomb interaction is neglected)

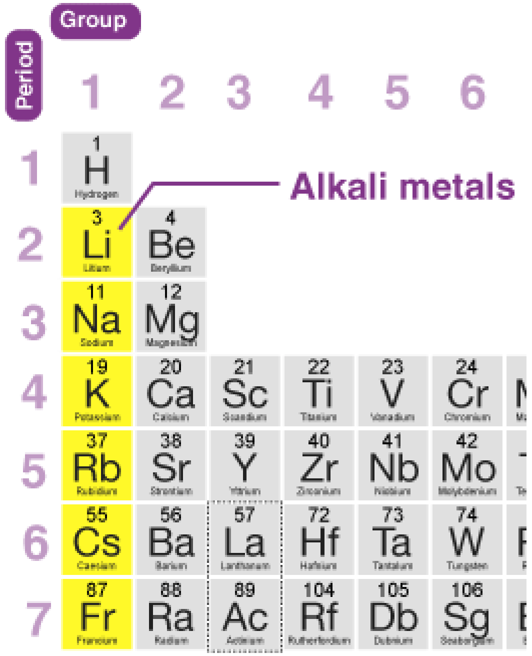
nucleus size $R(A) = r_0 A^{1/3}$, $r_0 = 1.1\text{–}1.2\,\mathrm{fm}$

$$n_{\mathrm{nucl}} \sim \left(\frac{4\pi}{3}r_0^3\right)^{-1} = 0.14\text{–}0.18\,\mathrm{fm}^{-3}$$

Helium-3 (n p p + e +e) 5 particles with spin ½ → total spin ½ **liquid helium 3**

Alkali atoms

The ground-state electronic structure of alkali atoms is simple: all electrons but one occupy closed shells, and the remaining one is in an s orbital in a higher shell.



Element	Z	Electronic spin	Electron configuration
H	1	1/2	1s
Li	3	1/2	$1s^2 2s^1$
Na	11	1/2	$1s^2 2s^2 2p^6 3s^1$
K	19	1/2	$1s^2 2s^2 2p^6 3s^2 3p^6 4s^1$
Rb	37	1/2	$(Ar) 3d^{10} 4s^2 4p^6 5s^1$
Cs	55	1/2	$(Kr) 4d^{10} 5s^2 5p^6 6s^1$

The nuclear spin is coupled to the electronic spin by the hyperfine interaction.

$$H_{\rm hf} = A\boldsymbol{I}\cdot\boldsymbol{J}$$

$\boldsymbol{I}$ nuclear spin

$\boldsymbol{J}$ electron angular momentum

$\boldsymbol{J} = \boldsymbol{L} + \boldsymbol{S}$

$L = 0$ and $S = 1/2$

good qunatum number is the total spin $\boldsymbol{F} = \boldsymbol{I} + \boldsymbol{J}$

$F = I \pm 1/2$

$$\boldsymbol{I}\cdot\boldsymbol{J} = \frac{1}{2}[F(F+1) - J(J+1) - I(I+1)]$$

$$\Delta E_{\rm hf} = E(F = I + 1/2) - E(F = I - 1/2)$$

$$= h\nu_{\rm hf} = \left(I + \frac{1}{2}\right)A$$

Isotope	Z	N	I	μ/μ_N	$\nu_{\rm hf}$ (MHz)
^{1}H	1	0	1/2	2.793	1420
^{6}Li	3	3	1	0.822	228
^{7}Li	3	4	3/2	3.256	804
^{23}Na	11	12	3/2	2.218	1772
^{39}K	19	20	3/2	0.391	462
^{40}K	19	21	4	−1.298	−1286
^{41}K	19	22	3/2	0.215	254
^{85}Rb	37	48	5/2	1.353	3036
^{87}Rb	37	50	3/2	2.751	6835
^{133}Cs	55	78	7/2	2.579	9193

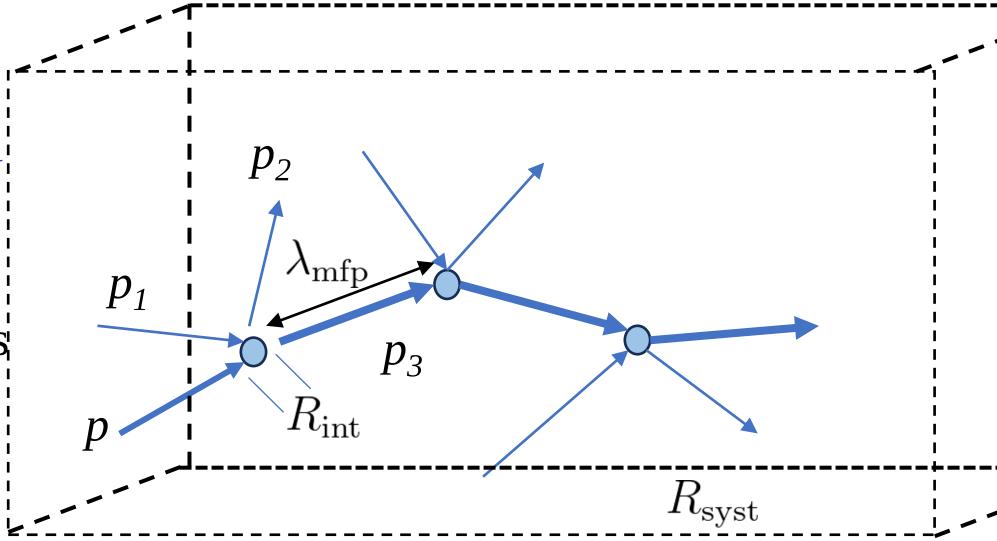
Neutral atoms contain equal numbers of electrons and protons, and therefore the statistics that an atom obeys is determined solely by the number of neutrons N: if N is even, the atom is a boson, and if it is odd, a fermion.

$$\hbar = 1.055\times10^{-27}\,\text{erg s} = 6.65\times10^{-16}\,\text{eV}\cdot\text{s} = 7.71\times10^{-12}\,\text{K}\cdot\text{s} = 7.71\times10^{-3}\,\text{K/GHz}$$

$$h = 2\pi\hbar = 4.8\times10^{-2}\,\text{K/GHz}$$

Dilute fermion gas

Let us consider a gas of neutral atoms interacting through a short-range binary potential. At high enough temperatures and low enough pressures, the gas is dilute.
Each atom moves as if it were essentially free, apart from infrequent collisions with other atoms or with the container walls.
The system is well described by the elementary kinetic theory of gases.



Equation for the particle distribution $n_f(\boldsymbol{x}, \boldsymbol{p}, t)$

$$\frac{\mathrm{d}n_f}{\mathrm{d}t} = I[n_f]$$

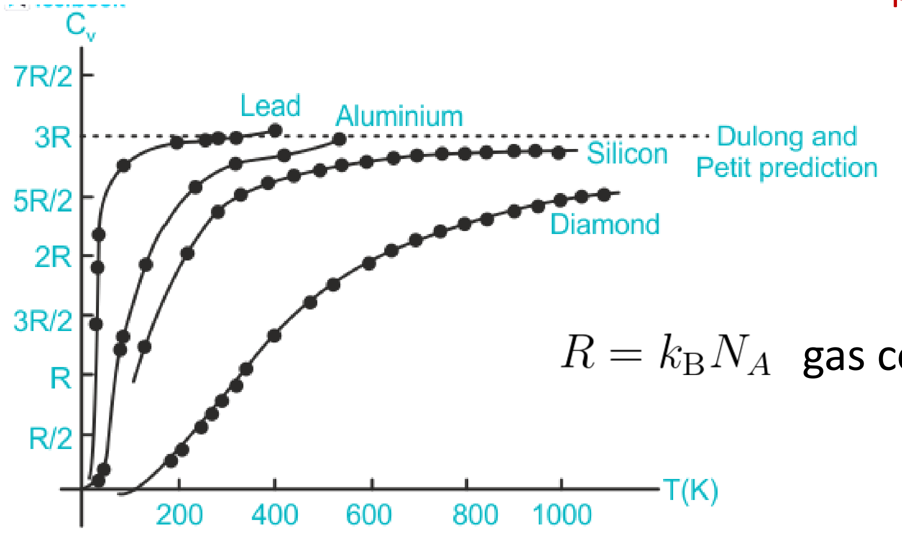
$$I[n_{\boldsymbol{p}}] = \frac{1}{V^2} \sum_{p_1} \sum_{p_2} W(p, p_1; p_2, p_3) \delta(\epsilon_{\boldsymbol{p}} + \epsilon_{\boldsymbol{p}_1} - \epsilon_{\boldsymbol{p}_2} - \epsilon_{\boldsymbol{p}_3}) \times \delta^{(3)}(\boldsymbol{p} + \boldsymbol{p}_1 - \boldsymbol{p}_2 - \boldsymbol{p}_3) \left[n_{f,\boldsymbol{p}_2} n_{f,\boldsymbol{p}_3} - n_{f,\boldsymbol{p}} n_{f,\boldsymbol{p}_1} \right]$$

transition probability

$$\frac{\mathrm{d}n_f}{\mathrm{d}t} = \frac{\partial n_f}{\partial t} + \dot{\boldsymbol{x}} \frac{\partial n_f}{\partial \boldsymbol{x}} + \dot{\boldsymbol{p}} \frac{\partial n_f}{\partial \boldsymbol{p}} = I[n_f]$$

$$R_{\text{syst}} \gg \lambda_{\text{mfp}} \sim \frac{1}{n\sigma} \gg R_{\text{int}} \qquad \sigma \sim 4\pi R_{\text{int}}^2 \qquad n R_{\text{int}}^3 \ll 1$$

It displays the usual properties of a classical gas;
the specific heat C_V is temperature independent



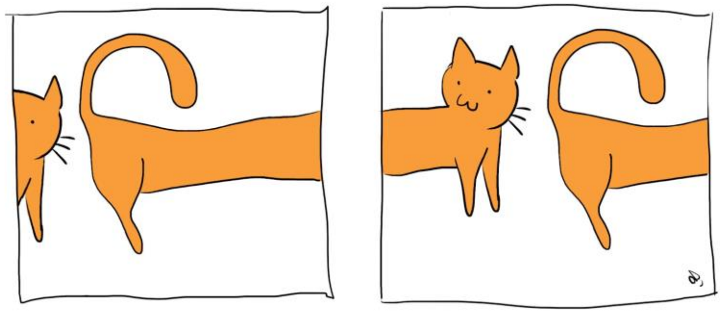
$R = k_{\mathrm{B}} N_A$ gas constant

$$C_V = \left(\frac{\delta Q}{\delta T}\right)_V = T\left(\frac{\partial S}{\partial T}\right)_V = \left(\frac{\partial E}{\partial T}\right)_V$$

$$n_f(p) = \frac{N}{V}\left(\frac{2\pi}{mT}\right)^{3/2} e^{-\frac{p^2}{2mT}} \qquad E = V\int n_f(p) \frac{p^2}{2m} \frac{2\mathrm{d}^3 p}{(2\pi)^3} = \frac{3}{2} T 2N$$

$$\mathrm{d}E = T\mathrm{d}S - P\mathrm{d}V$$

Thermodynamics of a Fermi gas



one type of fermions

single particle Hamiltonian $\hat{H}_1 = \frac{\hat{p}^2}{2m}$ box $L_x \times L_y \times L_z$

Box with periodic boundary conditions

$$\Psi_{\boldsymbol{p}}(\boldsymbol{r}) = \frac{1}{\sqrt{V}} e^{i\boldsymbol{pr}}$$

$$\boldsymbol{p_l} = \hbar \boldsymbol{k_l} = \hbar\left(2\pi \frac{l_x}{L_x}\boldsymbol{i} + 2\pi \frac{l_y}{L_y}\boldsymbol{j} + 2\pi \frac{l_z}{L_z}\boldsymbol{k}\right) \qquad (l_x, l_y, l_z) \text{ integers}$$

The total wave function is a Slater determinant made up of *N* such plane waves. All the eigenstates of the system can be characterized by the distribution function n_{ps} , which is equal to 1 if the state ***ps*** , is occupied, to zero otherwise.

Probability that the system has the particular distribution of particles on various levels

$$\{n_p\} = (n_{p1+}, n_{p1-}, n_{p2+}, n_{p2-}, n_{p3+}, n_{p3-}, \ldots, n_{p_n} \ \ldots)$$

$$P(\{n_p\}) = \frac{1}{Z_\mu^{(\mathrm{f})}} \exp\left\{-\frac{1}{T}\left(\sum_p (\epsilon_p - \mu) n_{p+} + (\epsilon_p - \mu) n_{p-}\right)\right\}$$

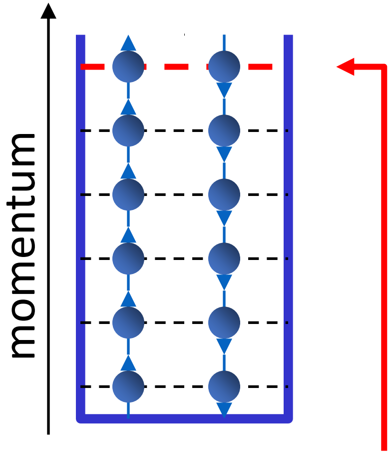
here μ is the Legandre coefficients, which fix the averaged number of particles. Quantity $Z_\mu^{(\mathrm{f})}$ is the statistical sum, which is the normalization factor for the probability.

Pauli exclusion principle

Two identical particles with uneven spin cannot occupy simultaneously the same quantum states, i.e. have the same quantum numbers

$T=0$

spin ½ particles in a box



$$Z_\mu^{(\mathrm{f})}=\sum_{n_{1,+}=0}^{1}\sum_{n_{1,-}=0}^{1}\sum_{n_{2,+}=0}^{1}\sum_{n_{2,-}=0}^{1}\cdots\sum_{n_{i,+}=0}^{1}\sum_{n_{i,-}=0}^{1}\ldots\exp\Big\{-\frac{1}{T}\Big(\sum_p(\epsilon_p-\mu)n_{p,+}(\epsilon_p-\mu)n_{p,-}\Big)\Big\})$$

For one sum of the particle numbers on the ith (enegy/momentum) level with spin $s=\pm$ we have

$$\sum_{n_{i,s}=0}^{1}e^{-\frac{1}{T}(\epsilon_{p_i}-\mu)n_{i,s}}=z_{p_i,s}^{(\mathrm{f})}\equiv 1+e^{-\frac{1}{T}(\epsilon_{p_i}-\mu)}$$

Then the statistical sum for the fermion system is equal to $Z_\mu^{(\mathrm{f})}=\prod_p z_{p,+}^{(\mathrm{f})}z_{p,-}^{(\mathrm{f})}$

and the thermodynamical potential for fermions is

$$\Omega^{(\mathrm{f})}=-T\sum_p\log Z_\mu^{(\mathrm{f})}=-TV\int\frac{2\mathrm{d}^3p}{(2\pi)^3}\log\Big(1+e^{-\frac{1}{T}(\epsilon_p-\mu)}\Big)$$

Fermi momentum p_F

Fermi energy $E_F=e(p_F)$

Thermodynamic limit $L\to\infty$, $V\to\infty$, $\langle N\rangle\to\infty$, but $\langle N\rangle/V=n=\mathrm{const.}$ $\quad\sum_p\to V\int\frac{\mathrm{d}^3k}{(2\pi)^3}=V\int\frac{\mathrm{d}^3p}{(2\pi\hbar)^3}\ (\hbar=1)$

Now we can define the probability that on the level p there will be exactly n particles as

$$w_p^{(\mathrm{f})}(n) = \langle \delta_{n_{p,+},0}\delta_{n_{p,-},0}\rangle_{\mathrm{f}}\delta_{n,0} + \Big(\langle \delta_{n_{p,+},1}\delta_{n_{p,-},0}\rangle_{\mathrm{f}} + \langle \delta_{n_{p,+},0}\delta_{n_{p,-},1}\rangle_{\mathrm{f}}\Big)\delta_{n,1} + \langle \delta_{n_{p,+},1}\delta_{n_{p,-},1}\rangle_{\mathrm{f}}\delta_{n,2}$$

we introduce the notation for the averaging with the probability $P(\{n\})$

$$\langle (\dots)\rangle_{\mathrm{f}} = \sum_{n_{1,+}=0}^{1}\sum_{n_{1,-}=0}^{1}\sum_{n_{2,+}=0}^{1}\sum_{n_{2,-}=0}^{1}\cdots\sum_{n_{i,+}=0}^{1}\sum_{n_{i,-}=0}^{1}\dots(\dots)P(\{n_p\})$$

$$\Rightarrow \quad w_p^{(\mathrm{f})}(n) = \frac{\delta_{n,0} + 2\delta_{n,1}e^{-\frac{1}{T}(\epsilon_p-\mu)} + \delta_{n,2}\left[e^{-\frac{1}{T}(\epsilon_p-\mu)}\right]^2}{\left(1+e^{-\frac{1}{T}(\epsilon_p-\mu)}\right)^2}$$

The averaged number of particles on the level p is

degeneracy factor

$$n_f(p) = \sum_{n=0}^{2} n w_p^{(\mathrm{f})}(n) = \frac{2e^{-\frac{1}{T}(\epsilon_p-\mu)} + 2\left[e^{-\frac{1}{T}(\epsilon_p-\mu)}\right]^2}{\left(1+e^{-\frac{1}{T}(\epsilon_p-\mu)}\right)^2} = 2\frac{e^{-\frac{1}{T}(\epsilon_p-\mu)}}{1+e^{-\frac{1}{T}(\epsilon_p-\mu)}} = \frac{2}{e^{\frac{1}{T}(\epsilon_p-\mu)}+1}$$

For particles with specific spin projection

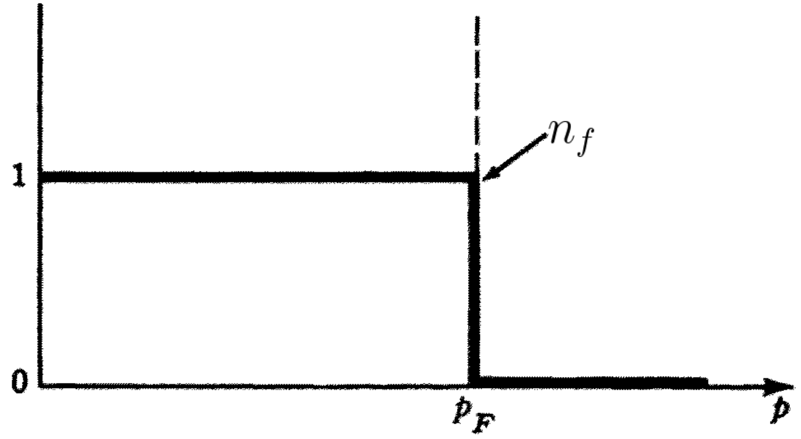
Fermi-Dirac distribution function

$$n_f(p,s) = \frac{1}{e^{\frac{1}{T}(\epsilon_p-\mu)}+1} \qquad s=\pm$$

Fermi gas at T=0

$$n_f(p,s) = \frac{1}{e^{\frac{1}{T}(\epsilon_p-\mu)}+1} \qquad s=\pm \qquad\qquad T\to 0 \qquad n_f(p,s) = \begin{cases} \frac{1}{0+1}=1 & \epsilon_p<\mu \\ \frac{1}{\infty+1}=0 & \epsilon_p>\mu \end{cases} = \theta(\mu-\epsilon_p) = \theta(p_{\rm F}-p)$$

Fully degenerate fermion system



p_F is the limiting momentum occupied by particles (Fermi momentum)

$\mu = \epsilon_{\rm F}$ limiting energy (Fermi energy) $\quad \epsilon_{\rm F} = \frac{p_{\rm F}^2}{2m}$

Fermi temperature $\quad T_{\rm F} = k_{\rm B}\epsilon_{\rm F}$

$$n = \frac{N}{V} = \sum_{s=\pm}\int\frac{d^3p}{(2\pi\hbar)^3}n_f(p,s) = 2\int_0^{p_{\rm F}}\frac{4\pi dp p^2}{(2\pi\hbar)^3} = \frac{p_{\rm F}^3}{3\pi^2\hbar^3}$$

Relation between the Fermi momentum and density $\quad p_{\rm F} = \hbar(3\pi^2 n)^{1/3} \qquad \epsilon_{\rm F} = \frac{\hbar^2(3\pi^2 n)^{2/3}}{2m}$

electron gas in metals (Fe) $\quad n_e = 17.0\times 10^{22}{\rm cm}^{-3} \qquad k_{\rm F} = \frac{p_{\rm F}}{\hbar} = (3\pi^2\cdot 17\times 10^{22})^{1/3}\,{\rm cm}^{-1} = 1.71\times 10^8\,{\rm cm}^{-1} = 17.1\,{\rm nm}^{-1}$

$$v_{\rm F} = \frac{\hbar k_{\rm F}}{m_e} = \frac{\hbar c k_{\rm F}}{m_e c^2}c = \frac{197.33\,{\rm eV}\cdot{\rm nm}\cdot 17.1\,{\rm nm}^{-1}}{0.511\times 10^6\,{\rm eV}}c \approx 0.6\times 10^{-2}c \qquad \epsilon_{\rm F} = \frac{\hbar^2 k_{\rm F}^2}{2m_e} = 11\,{\rm eV} \qquad T_{\rm F} = k_{\rm B}\epsilon_{\rm F} \simeq 10^5\,{\rm K}$$

$$\epsilon_{\rm F} \ll m_e$$

neutron matter $n_{\rm nucl} = 0.16\,{\rm fm}^{-3} \approx 0.45 m_\pi^3$

$$p_{\rm F} = (3\pi^2 n)^{1/3} = 2.4\, m_\pi \left(\frac{n}{n_{\rm nucl}}\right)^{1/3} = 331\,{\rm MeV} \left(\frac{n}{n_{\rm nucl}}\right)^{1/3} \qquad \frac{p_{\rm F}}{m_N} \simeq 0.35$$

relativistic effects could be important

$$\epsilon_{\rm F} = \frac{p_{\rm F}^2}{2m_N} = 0.43\, m_\pi \left(\frac{n}{n_0}\right)^{2/3} = 60\,{\rm MeV} \left(\frac{n}{n_0}\right)^{2/3}$$

Energy density

$$\frac{E}{V} = 2\int \frac{{\rm d}^3p}{(2\pi\hbar)^3}\theta(p_{\rm F}-p)\frac{p^2}{2m} = \frac{1}{\pi^2\hbar^3}\int_0^{p_{\rm F}} \frac{{\rm d}p p^4}{2m} = \frac{p_{\rm F}^5}{10m\pi^2\hbar^3} = \frac{3}{5}n\frac{p_{\rm F}^2}{2m} = \frac{3}{5}n\epsilon_{\rm F}$$

Pressure

$$P = -\frac{\Omega^{(\rm f)}}{V} = \lim_{T\to 0} TV\int \frac{2{\rm d}^3p}{(2\pi)^3}\log\left(1+e^{-\frac{1}{T}(\epsilon_p-\mu)}\right) = -\int \frac{2{\rm d}^3p}{(2\pi)^3}(\epsilon_p-\mu) = \mu n - \frac{E}{V} = n\,\epsilon_{\rm F} - \frac{3}{5}n\epsilon_{\rm F} = \frac{2}{5}n\epsilon_{\rm F}$$

$$PV = \frac{3}{2}E$$

for the electron gas in metals (Fe)

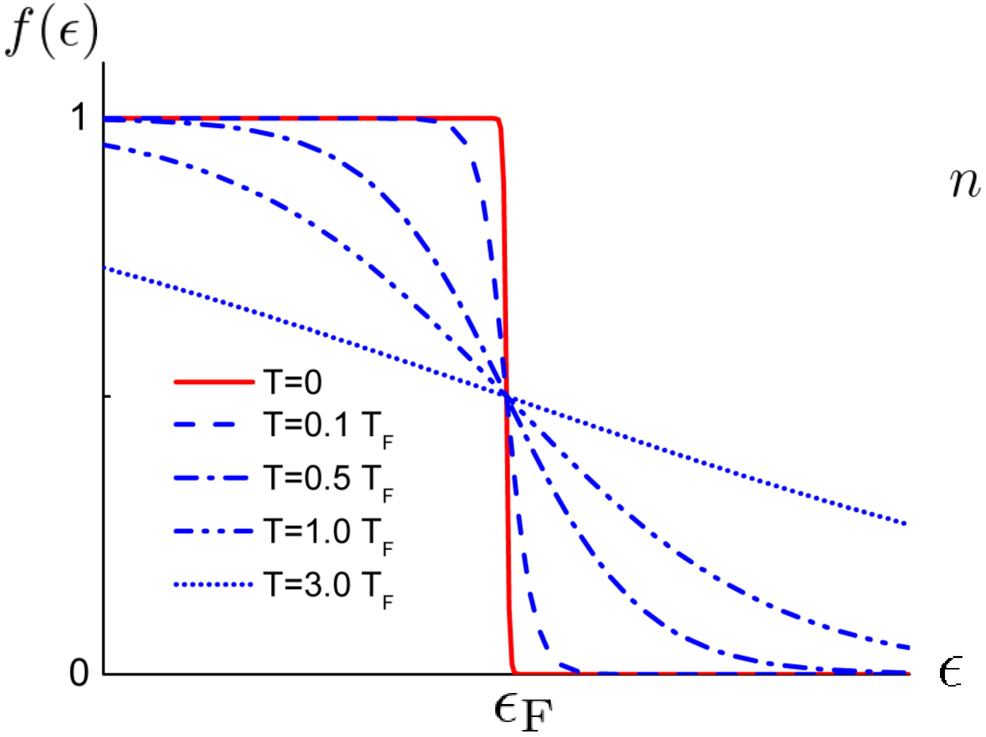
$$P = \frac{2}{5}(17\times 10^{22}\,{\rm cm}^{-3}\cdot 11\,{\rm eV}\cdot 1.6^{-19}\,{\rm J/eV}) \approx 1.2\times 10^5\frac{\rm J}{\rm cm^3} = 1.2\times 10^{11}\frac{\rm J}{\rm m^3} = 1.2\times 10^{11}\,{\rm Pa}$$

Fermi gas at T=/=0

$n_f(p) = f(\epsilon_p)$ where $f(\epsilon) = \frac{1}{e^{(\epsilon-\mu)/T}+1}$

at finite temperature $\mu \neq \epsilon_F$

μ fixes the total number of particles (particle density)



$$n = \frac{N}{V} = \sum_{s=\pm} \int \frac{d^3p}{(2\pi\hbar)^3} n_f(p,s) = 2 \int \frac{d^3p}{(2\pi\hbar)^3} f(\epsilon_p)$$

$$= \int_0^{+\infty} d\epsilon \int \frac{2d^3p}{(2\pi\hbar)^3} \delta(\epsilon - \epsilon_p) f(\epsilon) = \int_0^{+\infty} d\epsilon D(\epsilon) f(\epsilon)$$

$D(\epsilon)$ is the density of states $\quad D(\epsilon) = \frac{dn}{d\epsilon}$

$$D(\epsilon) = \int \frac{2d^3p}{(2\pi\hbar)^3}\delta(\epsilon-\epsilon_p) = \frac{1}{\pi^2\hbar^3}\int_0^\infty dp p^2 \delta\left(\epsilon - \frac{p^2}{2m}\right) = \frac{2m}{\pi^2\hbar^3}\int_0^\infty \frac{1}{2} du u^{1/2}\delta(2m\epsilon - u) = \frac{m}{\pi^2\hbar^3}\sqrt{2m\epsilon}$$

Density of states on the Fermi surface $\quad D(\epsilon_F) = \left.\frac{dn}{d\epsilon}\right|_F = \frac{mp_F}{\pi^2\hbar^3}$

particle density

$$n = \int_0^{+\infty} \mathrm{d}\epsilon D(\epsilon) f(\epsilon) \quad \Rightarrow \quad \mu = \mu(n, T)$$

energy density

$$u = \frac{E}{V} = \int_0^{+\infty} \mathrm{d}\epsilon D(\epsilon) \epsilon f(\epsilon) \quad \Rightarrow \quad u = u(\mu, T) = u(\mu(n, T), T) = u(n, T)$$

entropy density

$$s = \frac{S}{V} = \int_0^{+\infty} \mathrm{d}\epsilon D(\epsilon) \sigma_f(\epsilon) \qquad \sigma_f(\epsilon) = -(1 - f(\epsilon)) \ln(1 - f(\epsilon)) - f(\epsilon) \ln f(\epsilon) \, ,$$

pressure

$$P = u - Ts - \mu n = \int_0^{\infty} \mathrm{d}\epsilon D(\epsilon) \left[\epsilon - T\sigma_f(\epsilon) - \mu \right]$$

How to calculate integrals $\int_0^\infty \frac{H(\epsilon)}{e^{\frac{\epsilon-\mu}{T}}+1}\mathrm{d}\epsilon$ for small temperatures?

$$\int_0^\infty \frac{H(\epsilon)}{e^{\frac{\epsilon-\mu}{T}}+1}\mathrm{d}\epsilon = T\int_{-\mu/T}^\infty \frac{H(\mu+zT)}{e^z+1}\mathrm{d}z = T\int_{-\mu/T}^0 \frac{H(\mu+zT)}{e^z+1}\mathrm{d}z + T\int_0^\infty \frac{H(\mu+zT)}{e^z+1}\mathrm{d}z = T\int_0^{\mu/T} \frac{H(\mu-zT)}{e^{-z}+1}\mathrm{d}z + T\int_0^\infty \frac{H(\mu+zT)}{e^z+1}\mathrm{d}z$$

$z = \frac{\epsilon-\mu}{T}$

$$\frac{1}{e^{-z}+1} = \frac{e^z}{e^z+1} = 1 - \frac{1}{e^z+1}$$

$$= T\int_0^{\mu/T} H(\mu-zT)\mathrm{d}z - T\int_0^{\mu/T} \frac{H(\mu-zT)}{e^z+1}\mathrm{d}z + T\int_0^\infty \frac{H(\mu+zT)}{e^z+1}\mathrm{d}z = \int_0^\mu H(\epsilon)\mathrm{d}\epsilon - T\int_0^{\mu/T} \frac{H(\mu-zT)}{e^z+1}\mathrm{d}z + T\int_0^\infty \frac{H(\mu+zT)}{e^z+1}\mathrm{d}z$$

low temperature expansion $T \ll \mu,\ z\sim 1$

$$H(\mu+zT) - H(\mu-zT) \approx 2H'(\mu)zT + \frac{1}{3}H'''(\mu)(zT)^3$$

$$\approx \int_0^\mu H(\epsilon)\mathrm{d}\epsilon + T\int_0^\infty \frac{H(\mu+zT)-H(\mu-zT)}{e^z+1}\mathrm{d}z$$

$$\int_0^\infty \frac{H(\epsilon)}{e^{\frac{\epsilon-\mu}{T}}+1}\mathrm{d}\epsilon = \int_0^\mu H(\epsilon)\mathrm{d}\epsilon + 2T^2H'(\mu)\int_0^\infty \frac{z\mathrm{d}z}{e^z+1} + \frac{1}{3}T^4H'''(\mu)\int_0^\infty \frac{z^3\mathrm{d}z}{e^z+1}$$

$$\int_0^\infty \frac{z^{x-1}\mathrm{d}z}{e^z+1} = \left(1-\frac{1}{2^{x-1}}\right)\Gamma(x)\zeta(x)$$

Gamma function: $\Gamma(x) = \int_0^\infty y^{x-1}e^{-y} = x\Gamma(x-1)$

$$\Gamma(n) = (n-1)! \qquad \Gamma(1) = 1\,,\Gamma(1/2) = \sqrt{\pi}$$

Riemann zeta-function: $\zeta(x) = \sum_{n=1}^\infty \frac{1}{n^x} \qquad \zeta(2) = \frac{\pi^2}{6}\,,\ \zeta(4) = \frac{\pi^4}{90}$

$$\int_0^\infty \frac{z\mathrm{d}z}{e^z+1} = \left(1-\frac{1}{2}\right)\Gamma(2)\zeta(2) = \frac{\pi^2}{12}$$

$$\int_0^\infty \frac{z^3\mathrm{d}z}{e^z+1} = \left(1-\frac{1}{2^3}\right)\Gamma(4)\zeta(4) = \frac{7\pi^4}{120}$$

$$\Rightarrow \int_0^\infty \frac{H(\epsilon)}{e^{\frac{\epsilon-\mu}{T}}+1}\mathrm{d}\epsilon = \int_0^\mu H(\epsilon)\mathrm{d}\epsilon + \frac{\pi^2}{6}T^2H'(\mu) + \frac{7(\pi T)^4}{360}H'''(\mu)$$

The chemical potential

$$n = \int_0^{+\infty} \mathrm{d}\epsilon D(\epsilon) f(\epsilon) \qquad D(\epsilon) = \frac{m}{\pi^2 \hbar^3}\sqrt{2m\epsilon} \qquad D'(\epsilon) = \frac{m}{\pi^2 \hbar^3}\frac{\sqrt{2m}}{2\sqrt{\epsilon}}$$

$$n = \frac{m}{\pi^2}(2m)^{1/2}\frac{2}{3}\mu^{3/2} + \frac{\pi^2}{6}T^2\frac{m}{\pi^2}\frac{(2m)^{1/2}}{2\sqrt{\mu}}$$

$$n = \frac{p_{\mathrm{F}}^3}{3\pi^2} = \frac{(2m\epsilon_{\mathrm{F}})^{3/2}}{3\pi^2}$$

$$\Rightarrow \quad \epsilon_{\mathrm{F}}^{3/2} = \mu^{3/2} + \frac{\pi^2}{8}\frac{T^2}{\sqrt{\mu}} + O(T^4/\mu^{3/2}) \qquad \mu(T=0) = \epsilon_{\mathrm{F}}$$

We search for a solution in the form $\mu \approx \epsilon_{\mathrm{F}}\left(1 + a\frac{T^2}{\epsilon_{\mathrm{F}}^2} + O(T^4/\epsilon_{\mathrm{F}}^4)\right)$

$$\epsilon_{\mathrm{F}}^{3/2} \approx \epsilon_{\mathrm{F}}^{3/2}\left(1 + a\frac{T^2}{\epsilon_{\mathrm{F}}^2}\right)^{3/2} + \frac{\pi^2}{8}\frac{T^2}{\epsilon_{\mathrm{F}}^{1/2}}\left(1 + a\frac{T^2}{\epsilon_{\mathrm{F}}^2}\right)^{-1/2} \quad \Rightarrow \quad 0 \approx \frac{3}{2}a\frac{T^2}{\epsilon_{\mathrm{F}}^{1/2}} + \frac{\pi^2}{8}\frac{T^2}{\epsilon_{\mathrm{F}}^{1/2}} \quad \Rightarrow \quad a = -\frac{\pi^2}{12}$$

$$\mu \approx \epsilon_{\mathrm{F}}\left(1 - \frac{\pi^2}{12}\frac{T^2}{\epsilon_{\mathrm{F}}^2}\right) + O(T^4/\epsilon_{\mathrm{F}}^4)$$

Fermi temperature is the criterion if the Fermi gas is degenerate or not-degenerate

The energy density

$$u = \frac{E}{V} = \int_0^{+\infty} \mathrm{d}\epsilon D(\epsilon)\epsilon f(\epsilon) \approx \int_0^{\mu} D(\epsilon)\epsilon \mathrm{d}\epsilon + \frac{\pi^2}{6}T^2 \frac{\mathrm{d}}{\mathrm{d}\epsilon}\left[\epsilon D(\epsilon)\right]\Big|_{\mu}$$

$$D(\epsilon) = C\epsilon^{1/2}$$

$$\Rightarrow \quad \int_0^{\mu} D(\epsilon)\epsilon \mathrm{d}\epsilon = \int_0^{\mu} C\epsilon^{3/2}\mathrm{d}\epsilon = \frac{2}{5}C\mu^{5/2} = \frac{2}{5}\mu^2 D(\mu)$$

$$\frac{\mathrm{d}}{\mathrm{d}\epsilon}\left[\epsilon D(\epsilon)\right]\Big|_{\mu} = D(\mu) + \mu D'(\mu) = D(\mu) + \mu\frac{D(\mu)}{2\mu} = \frac{3}{2}D(\mu)$$

$$u \approx \frac{2}{5}\frac{\sqrt{2m}m}{\pi^2}\mu^{5/2} + \frac{\pi^2}{6}T^2\frac{3}{2}\frac{\sqrt{2m}m}{\pi^2}\mu^{1/2} \approx \frac{2}{5}\frac{\sqrt{2m}m}{\pi^2}\epsilon_{\mathrm{F}}^{5/2}\left(1 - \frac{\pi^2}{12}\frac{T^2}{\epsilon_{\mathrm{F}}^2}\right)^{5/2} + \frac{\pi^2}{6}T^2\frac{3}{2}\frac{\sqrt{2m}m}{\pi^2}\epsilon_{\mathrm{F}}^{1/2}$$

$$\approx \frac{2}{5}\frac{\sqrt{2m}m}{\pi^2}\epsilon_{\mathrm{F}}^{5/2} - \frac{\sqrt{2m}m}{\pi^2}\epsilon_{\mathrm{F}}^{1/2}\frac{\pi^2 T^2}{12} + \frac{\pi^2}{6}T^2\frac{3}{2}\frac{\sqrt{2m}m}{\pi^2}\epsilon_{\mathrm{F}}^{1/2}$$

$$\approx \frac{1}{5}\frac{(2m\epsilon_{\mathrm{F}})^{3/2}}{\pi^2}\epsilon_{\mathrm{F}} + \frac{(2m\epsilon_{\mathrm{F}})^{3/2}}{\pi^2}\epsilon_{\mathrm{F}}\left[-\frac{\pi^2 T^2}{24\epsilon_{\mathrm{F}}^2} + \frac{\pi^2}{12}\frac{T^2}{\epsilon_{\mathrm{F}}^2}\frac{3}{2}\right] = \frac{3}{5}n\epsilon_{\mathrm{F}} + n\epsilon_{\mathrm{F}}\frac{\pi^2}{4}\frac{T^2}{\epsilon_{\mathrm{F}}^2}$$

$$u \approx \frac{3}{5}n\epsilon_{\mathrm{F}}\left[1 + 5\frac{\pi^2}{12}\frac{T^2}{\epsilon_{\mathrm{F}}^2}\right]$$

Heat capacity of the fermion gas $C_V = \frac{\partial u}{\partial T}\Big|_V = \frac{\pi^2}{2}\frac{T}{\epsilon_F} n = \frac{m p_F}{3} T$

Heat capacity in metals has also a phonon contribution (oscillations of the ion lattice) $C_V = \gamma T + A T^3$

$$\Rightarrow \quad \frac{C_V}{T} = \gamma + A T^2$$

In order to separate out these two contributions it has become the practice to plot C_V/T against T^2

One finds γ by extrapolating the C_V/T curve linearly down to $T^2 = 0$, and noting where it intercepts the C_V/T-*axis*

SOME ROUGH EXPERIMENTAL VALUES FOR THE COEFFICIENT OF THE LINEAR TERM IN T OF THE MOLAR SPECIFIC HEATS OF METALS, AND THE VALUES GIVEN BY SIMPLE FREE ELECTRON THEORY

ELEMENT	FREE ELECTRON γ (in 10^{-4} cal-mole^{-1}-K^{-2})	MEASURED γ	RATIO* (m^*/m)
Li	1.8	4.2	2.3
Na	2.6	3.5	1.3
K	4.0	4.7	1.2
Rb	4.6	5.8	1.3
Cs	5.3	7.7	1.5
Cu	1.2	1.6	1.3
Ag	1.5	1.6	1.1
Au	1.5	1.6	1.1
Be	1.2	0.5	0.42
Mg	2.4	3.2	1.3
Ca	3.6	6.5	1.8
Sr	4.3	8.7	2.0
Ba	4.7	6.5	1.4
Nb	1.6	20	12
Fe	1.5	12	8.0
Mn	1.5	40	27
Zn	1.8	1.4	0.78
Cd	2.3	1.7	0.74
Hg	2.4	5.0	2.1
Al	2.2	3.0	1.4
Ga	2.4	1.5	0.62
In	2.9	4.3	1.5
Tl	3.1	3.5	1.1
Sn	3.3	4.4	1.3

In-medium "effective mass" of the electron can strongly deviates from the vacuum one!