Dense fermion systems in the center of compact stars

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Part II:

- Fermi liquids
- Green’s function technique

Fermi liquids

For real fermion systems, the particle interaction and the exclusion principle act simultaneously.

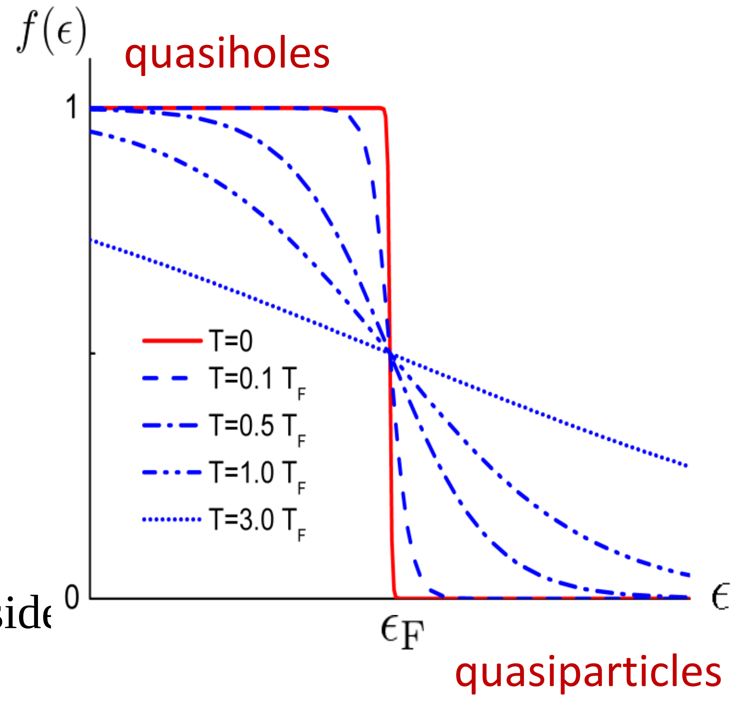
We consider degenerate Fermi liquids in which both effects are important.

In some systems, the nature of the degenerate gas is drastically modified by the particle interactions. Such is the case, for instance, in a superconducting electron gas.

Frequently, the interacting liquid retains many properties of the gas: it is then said to be **normal.**

A normal Fermi liquid at T = 0 has a sharply defined Fermi surface S_F

Its elementary excitations may be pictured as *quasiparticles* outside S_F and *quasiholes* inside S_F in close analogy with the single-particle excitations of a noninteracting Fermi gas.



Such a resemblance explains why so many properties of the liquid can be interpreted in terms of a "one-particle approximation."

Excitation in Fermi liquids

Let us now turn to the case of an interacting Fermi liquid. We are interested in the nature of its elementary excitations.

A "frontal“ attack on the problem involves the introduction of Green's functions, and the mathematical apparatus of many-body perturbation theory.

We start with an alternative approach, which consists in comparing the interacting "real" liquid with the noninteracting "ideal" gas; we establish a one-to-one correspondence between the eigenstates of the two systems.

Consider an eigenstate of the ideal system, characterized by a distribution function $n_{\boldsymbol{p}}$. In order to establish a connection with the real system, we imagine that the interaction between the particles is switched on infinitely slowly. Under such "adiabatic" conditions, the ideal eigenstates will progressively transform into certain eigenstates of the real interacting system.

However, there is no a priori reason why such a procedure should generate ***all*** real eigenstates. For instance, it may well happen that the real ground state may not be obtained in that way (superconductors!)
We ***assume*** that the real ground state may be adiabatically generated starting from some ideal eigenstate with a distribution $n^0_{\boldsymbol{p}}$.

This is the definition of a *normal* fermion system.

For reasons of symmetry, the distribution $n^0_{\boldsymbol{p}}$ of an isotropic system is ***spherical***. As a result, the spherical Fermi surface is not changed when the interaction between particles is switched on: the real ground state is generated adiabatically from the ideal ground state.

Let us now *add* a particle with momentum $\boldsymbol{p}$ to the ideal distribution $n^0_{\boldsymbol{p}}$ and, again, turn on the interaction between the particles adiabatically. We generate an *excited state* of the real liquid, which likewise has momentum $\boldsymbol{p}$, since momentum is conserved in particle collisions.

As the interaction is increased, we may picture the bare particle as slowly perturbing the particles in its vicinity; if the change in interaction proceeds sufficiently slowly, the entire system of N + 1 particles will remain in equilibrium.

Once the interaction is completely turned on, we find that our particle moves together with the surrounding particle distortion brought about by the interaction. In the language of field theory, we would say that the particle is *"dressed“ with a self-energy cloud*. We shall consider the "dressed" particle as an independent entity, which we call a quasipartide.

The above excited state corresponds to the real ground state plus a quasiparticle of momentum $\boldsymbol{p}$.

Let S_F be the Fermi surface characterizing the unperturbed distribution $n^0_{\boldsymbol{p}}$ from which the real ground state is built up. Because of the exclusion principle, quasipartide excitations can be generated only if their momentum $\boldsymbol{p}$ lies outside S_F. The quasipartide distribution in $\boldsymbol{p}$ space is sharply bounded by the Fermi surface S_F.

Using the same adiabatic switching procedure, we can define a quasihole, with a momentum **p** lying inside the Fermi surface SF; we may do likewise for higher configurations involving several excited quasipartieles and quasiholes. The quasiparticles and quasiholes thus appear as elementary excitationsof the real system which, when combined, give rise to a large class of eicited states. We have established our desired one-to-one correspondence between ideal and real eigenstates.

Landau Fermi liquid approach

interacting fermions

system of quasi-particles

quantized excitations in the system

quasi-particles =/= original "bare" fermions [constituents of the system]

Landau wrote the Boltzmann eq. for q.p distribution function: $n(\boldsymbol{x}, \boldsymbol{p}, t)$

$$\frac{dn}{dt} = \frac{\partial n}{\partial t} + \dot{\boldsymbol{x}} \frac{\partial n}{\partial \boldsymbol{x}} + \dot{\boldsymbol{p}} \frac{\partial n}{\partial \boldsymbol{p}} = I(n)$$

equations of motion for q.p.

$$\dot{\boldsymbol{x}} = \frac{\partial \epsilon(\boldsymbol{p}, \boldsymbol{x})}{\partial \boldsymbol{p}}, \qquad \dot{\boldsymbol{p}} = -\frac{\partial \epsilon(\boldsymbol{p}, \boldsymbol{x})}{\partial \boldsymbol{x}}$$

"generalized" velocity   Newton's law

$$\frac{\partial n}{\partial t} + \frac{\partial \epsilon(\boldsymbol{p}, \boldsymbol{x})}{\partial \boldsymbol{p}} \frac{\partial n}{\partial \boldsymbol{x}} - \frac{\partial \epsilon(\boldsymbol{p}, \boldsymbol{x})}{\partial \boldsymbol{x}} \frac{\partial n}{\partial \boldsymbol{p}} = I(n)$$

[G.E. Brown, RMP 43, 1] *"I am indebted to Professor I. M. Khalatniltov for explaining this (Landau's reasoning) to me."*

$$\boldsymbol{\mathcal{F}}(\boldsymbol{x},t) = \int \boldsymbol{p}\, n(t,\boldsymbol{x},\boldsymbol{p})\, \frac{d^3p}{(2\pi)^3}$$

momentum flux density

Aim is to obtain conservation of total quasiparticle momentum

$$\frac{\partial n}{\partial t} + \frac{\partial \epsilon(\boldsymbol{p},\boldsymbol{x})}{\partial \boldsymbol{p}}\frac{\partial n}{\partial \boldsymbol{x}} - \frac{\partial \epsilon(\boldsymbol{p},\boldsymbol{x})}{\partial \boldsymbol{x}}\frac{\partial n}{\partial \boldsymbol{p}} = I(n)$$

$$\int \boldsymbol{p}_i\, I(n)\, \frac{d^3p}{(2\pi)^3} = 0$$

momentum conservation in collisions

$$\begin{aligned}
\frac{\partial \boldsymbol{\mathcal{F}}_i}{\partial t} &= \int \boldsymbol{p}_i \frac{\partial n}{\partial t}\frac{d^3p}{(2\pi)^3} = \int \boldsymbol{p}_i\, I(n)\frac{d^3p}{(2\pi)^3} - \int \boldsymbol{p}_i \left[\frac{\partial \epsilon}{\partial \boldsymbol{p}_j}\frac{\partial n}{\partial \boldsymbol{x}_j} - \frac{\partial \epsilon}{\partial \boldsymbol{x}_j}\frac{\partial n}{\partial \boldsymbol{p}_j}\right]\frac{d^3p}{(2\pi)^3} \\
&= -\frac{\partial}{\partial \boldsymbol{x}_j}\int \boldsymbol{p}_i\, n \frac{\partial \epsilon}{\partial \boldsymbol{p}_j}\frac{d^3p}{(2\pi)^3} + \int \boldsymbol{p}_i\, n \frac{\partial^2 \epsilon}{\partial \boldsymbol{x}_j \partial \boldsymbol{p}_j}\frac{d^3p}{(2\pi)^3} - \int n \frac{\partial \boldsymbol{p}_i}{\partial \boldsymbol{p}_j}\frac{\partial \epsilon}{\partial \boldsymbol{x}_j}\frac{d^3p}{(2\pi)^3} - \int n\, \boldsymbol{p}_i \frac{\partial \epsilon}{\partial^2 \boldsymbol{p}_j \partial \boldsymbol{x}_j}\frac{d^3p}{(2\pi)^3} \\
&= -\frac{\partial}{\partial \boldsymbol{x}_j}\int \boldsymbol{p}_i\, n \frac{\partial \epsilon}{\partial \boldsymbol{p}_j}\frac{d^3p}{(2\pi)^3} - \int n \frac{\partial \epsilon}{\partial \boldsymbol{x}_i}\frac{d^3p}{(2\pi)^3} \\
&= -\frac{\partial}{\partial \boldsymbol{x}_j}\int \boldsymbol{p}_i\, n \frac{\partial \epsilon}{\partial \boldsymbol{p}_j}\frac{d^3p}{(2\pi)^3} - \frac{\partial}{\partial \boldsymbol{x}_i}\int n\, \epsilon \frac{d^3p}{(2\pi)^3} + \int \epsilon \frac{\partial n}{\partial \boldsymbol{x}_i}\frac{d^3p}{(2\pi)^3} \\
&= \frac{\partial}{\partial \boldsymbol{x}_j}\boldsymbol{\Pi}^{ij} + \int \epsilon \frac{\partial n}{\partial \boldsymbol{x}_i}\frac{d^3p}{(2\pi)^3}
\end{aligned}$$

momentum flux tensor $\boldsymbol{\Pi}^{ij} = \int n \left(\boldsymbol{p}_i \frac{\partial \epsilon}{\partial \boldsymbol{p}_j} + \delta_{ij}\epsilon\right)\frac{d^3p}{(2\pi)^3}$

[G.E. Brown, RMP 43, 1]

momentum conservation

$$0 = \frac{\partial}{\partial t}\int \boldsymbol{\mathcal{F}}_i d^3x = \cancel{\int \frac{\partial}{\partial x_j}\Pi^{ij}\, d^3x} + \int d^3x \int \epsilon \frac{\partial n}{\partial \boldsymbol{x}_i}\frac{d^3p}{(2\pi)^3}$$

Assuming Π to be zero on the surfaces of the spatial volume integrated over

$$\Rightarrow \int d^3x \underbrace{\int \epsilon \frac{\partial n}{\partial \boldsymbol{x}}\frac{d^3p}{(2\pi)^3}} = 0$$

if the $\partial/\partial \boldsymbol{x}_j$ can be taken outside the integral, in other words, if $e(p)\delta n(p)$ is the differential of some quantity E

$$\int \epsilon \frac{\partial n}{\partial \boldsymbol{x}}\frac{d^3p}{(2\pi)^3} = \frac{\partial}{\partial \boldsymbol{x}} E$$

$$\delta E = \int \epsilon\, \delta n \frac{d^3p}{(2\pi)^3}$$

E = energy density of the system

The quasiparticle energy $\bar{e}(p)$ is obtained by varying the energy with respect to quasiparticle number. We see that Landau was led to this assumption by his construction of conservation laws for the quasiparticle momentum.

$$\frac{\delta E}{\delta n(\boldsymbol{p})} = \epsilon(\boldsymbol{p})$$

[G.E. Brown, RMP 43, 1]

single particle mechanism of excitation!

Now the energy of the system is a **functional of occupation number** of all of the quasiparticles. $E = E\{n_{\boldsymbol{p}_1,s_1}, n_{\boldsymbol{p}_2,s_2}, \dots\}$

(states are quantified by their momentum $\boldsymbol{p}$ and spin s)

If one varies many of the *n(p)* away from their equilibrium value by, e.g., exciting a collective excitation, then the resulting energy is a functional $E' = E'\{\delta n_{\boldsymbol{p}_1),s_1}, \delta n_{\boldsymbol{p}_2,s_2}, \dots\}$

The quasiparticle energy $\epsilon_{\boldsymbol{p},s} = \epsilon_{\boldsymbol{p},s}\{n_{\boldsymbol{p},s}\}$, which is a functional of the quasiparticle distributions, is defined as the variaton of the system energy with respect to $n_{\boldsymbol{p},s}$.

$$\delta E = E' - E = \frac{1}{V}\sum_{\boldsymbol{p},s} \epsilon_{\boldsymbol{p},s}\delta n_{\boldsymbol{p},s}$$

A variation of the distribution function produces a variation of the quasiparticle energy given by

$$\delta\epsilon_{\boldsymbol{p},s} = \frac{1}{V}\sum_{\boldsymbol{p}',s'} f_{\boldsymbol{p}s,\boldsymbol{p}'s'}\delta n_{\boldsymbol{p}',s'}$$

$f_{\boldsymbol{p}s,\boldsymbol{p}'s'}$ quantifies the interaction between quasiparticles. It can be viewd as the second variation of the total energy $f_{\boldsymbol{p}s,\boldsymbol{p}'s'} = V^2\dfrac{\delta^2 E}{\delta n_{\boldsymbol{p},s}\delta n_{\boldsymbol{p}',s'}}$. Of course, the quasiparticle interaction energy f is itself a functional of the distribution function. Usually one assumes that f is evaluated for the equilibrium distributions $f\{n^{(0)}_{\boldsymbol{p}'',s''}\}$.

The variation of the energy due to a variation, $\delta n_{\boldsymbol{p},s}$, of the distribution function from its ground state form can be written as

$$E = E_0 + \frac{1}{V}\sum_{\boldsymbol{p}s} \epsilon^{(0)}_{\boldsymbol{p},s}\delta n_{\boldsymbol{p},s} + \frac{1}{2}\frac{1}{V^2}\sum_{\boldsymbol{p}s,\boldsymbol{p}'s'} f_{\boldsymbol{p}s,\boldsymbol{p}'s'}\delta n_{\boldsymbol{p},s}\delta n_{\boldsymbol{p}',s'}$$

$$\epsilon^{(0)}_{\boldsymbol{p}s} = \epsilon_{\boldsymbol{p}s}\{n^{(0)}_{\boldsymbol{p}'s'}\}$$

Distribution function for quasiparticles $n_{\boldsymbol{p},s}$

For any variation about thermodynamic equilibrium at finite temperature $\delta E = T\delta s + \mu \delta n$ V is constant

Particle density $n = \frac{1}{V}\sum_{\boldsymbol{p},s} n_{\boldsymbol{p},s}$ Entropy density $s = -\frac{1}{V}\sum_{\boldsymbol{p}s}\{n_{\boldsymbol{p}s}\ln n_{\boldsymbol{p}s} + (1-n_{\boldsymbol{p}s})\ln(1-n_{\boldsymbol{p}s})\}$

$$\delta n = \frac{1}{V}\sum_{\boldsymbol{p},s}\delta n_{\boldsymbol{p},s} \qquad \delta s = -\frac{1}{V}\sum_{\boldsymbol{p}s}\delta n_{\boldsymbol{p}s}\ln\frac{n_{\boldsymbol{p}s}}{1-n_{\boldsymbol{p}s}}$$

$$\delta E = T\delta s + \mu\delta n \Rightarrow \frac{1}{V}\sum_{\boldsymbol{p},s}\epsilon_{\boldsymbol{p},s}\delta n_{\boldsymbol{p},s} = -\frac{T}{V}\sum_{\boldsymbol{p}s}\delta n_{\boldsymbol{p}s}\ln\frac{n_{\boldsymbol{p}s}}{1-n_{\boldsymbol{p}s}} + \frac{\mu}{V}\sum_{\boldsymbol{p},s}\delta n_{\boldsymbol{p},s}$$

$$\Rightarrow \frac{1}{V}\sum_{\boldsymbol{p},s}\left[\epsilon_{\boldsymbol{p},s} + T\ln\frac{n_{\boldsymbol{p}s}}{1-n_{\boldsymbol{p}s}} - \mu\right]\delta n_{\boldsymbol{p},s} = 0 \Rightarrow \epsilon_{\boldsymbol{p},s} + T\ln\frac{n_{\boldsymbol{p}s}}{1-n_{\boldsymbol{p}s}} - \mu = 0$$

$$\Rightarrow n_{\boldsymbol{p}s} = \frac{1}{e^{\frac{\epsilon_{\boldsymbol{p}s}-\mu}{T}}+1}$$

This is a complicated implicit equation for $n_{\boldsymbol{p}s}$

For $T = 0$
$$n^{(0)}_{\boldsymbol{p}s} = \begin{cases} 1 & \epsilon^{(0)}_{\boldsymbol{p}s} < \mu \\ 0 & \epsilon^{(0)}_{\boldsymbol{p}s} > \mu \end{cases}$$

The Fermi momentum p_{F} at which $\epsilon^{(0)}_{\boldsymbol{p}_{\mathrm{F}}s} = \mu$

Quasiparticle energy $\epsilon^{(0)}_{\boldsymbol{p}s} = \epsilon_{\boldsymbol{p}s}\{n^{(0)}_{\boldsymbol{p}'s'}\} \neq \frac{\boldsymbol{p}^2}{2m}$ Some complicated function of p

Close to the Fermi surface $\epsilon^{(0)}_{\boldsymbol{p}s} \approx \mu + \left(\frac{\partial \epsilon_{\boldsymbol{p}s}}{\partial p}\right)_{p=p_{\mathrm{F}}} (p - p_{\mathrm{F}})$ Fermi momentum is related to the density $\frac{p_{\mathrm{F}}^3}{3\pi^2} = n$

Fermi velocity $\left(\frac{\partial \epsilon_{\boldsymbol{p}s}}{\partial p}\right)_{p=p_{\mathrm{F}}} = v_{\mathrm{F}} = \frac{p_{\mathrm{F}}}{m^*}$ Effective mass of the quasiparticle $\frac{1}{m^*} = \left(\frac{\partial^2 \epsilon_{\boldsymbol{p}s}}{\partial p^2}\right)_{p=p_{\mathrm{F}}}$

$\Rightarrow$ $p \sim p_{\mathrm{F}}$ $\epsilon^{(0)}_{\boldsymbol{p}s} \approx \mu + \frac{p_{\mathrm{F}}}{m^*}(p - p_{\mathrm{F}})$ with the same precision O(p-p_{F}) $\epsilon^{(0)}_{\boldsymbol{p}s} \approx \mu + \frac{p^2 - p_{\mathrm{F}}^2}{2m^*}$

Density of states $D(\epsilon) = \frac{1}{V}\sum_{\boldsymbol{p}s} \delta\big(\epsilon_{\boldsymbol{p},s} - \epsilon\big)$

at the Fermi surface $N_0 = D(\mu) = \frac{1}{V}\sum_{\boldsymbol{p}s} \delta\big(\epsilon_{\boldsymbol{p},s} - \mu\big) = \frac{m^* p_{\mathrm{F}}}{\pi^2}$

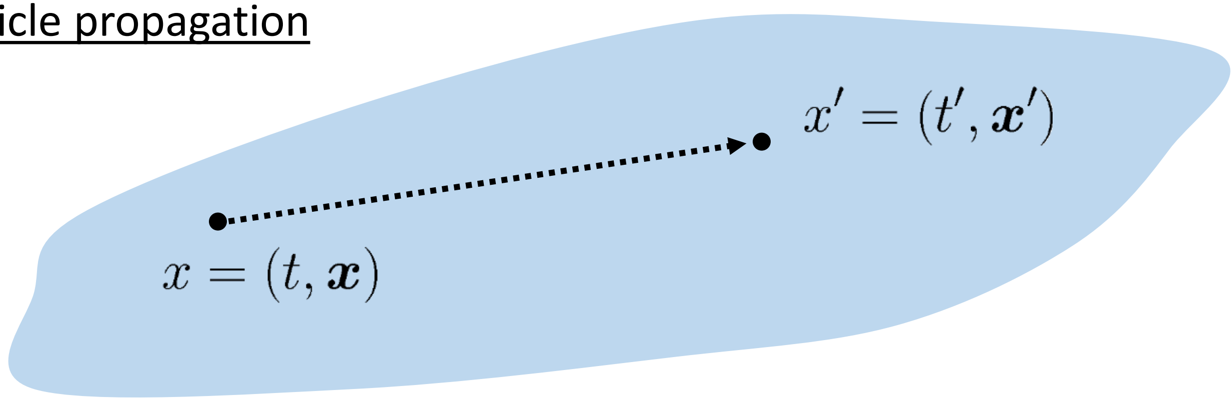
$\Rightarrow$ Heat capacity $C_V = \frac{m^* p_{\mathrm{F}}}{3} T$

Green's functions

N-body system: wave function of the whole system $\Psi(x_1,x_2,\ldots,x_N)$
encodes the dynamics of all particles and is very complicated

Introduce the object which describes the dynamics of the reduced number of particles of interest

<u>one-particle propagation</u>



Amplitude of particle transition from a point (x,t) to a point (x',t')

$$\Psi(\boldsymbol{x}', t') = \int d\boldsymbol{x}\, G^{(+)}(\boldsymbol{x}', t'; \boldsymbol{x}, t)\Psi(\boldsymbol{x}, t) \qquad t' > t$$

for $t' = t + 0$ $\qquad \Psi(\boldsymbol{x}', t+0) = \int d\boldsymbol{x}\, G^{(+)}(\boldsymbol{x}', t+0; \boldsymbol{x}, t)\Psi(\boldsymbol{x}, t)$

$$G^{(+)}(\boldsymbol{x}', t+0; \boldsymbol{x}, t) = \delta(\boldsymbol{x}' - \boldsymbol{x})$$

If *Ψ(**x**,t)* obeys the Schrödinger equation $\qquad [i\partial_t - H(\boldsymbol{x})]\,\Psi(\boldsymbol{x}, t) = 0$

$$[i\partial_t - H(\boldsymbol{x})]\,G^{(+)}(\boldsymbol{x}, t; \boldsymbol{x}', t') = i\,\delta(t - t')\,\delta(\boldsymbol{x} - \boldsymbol{x}')$$

for homogeneous system : $\qquad G^{(+)}(\boldsymbol{x}', t'; \boldsymbol{x}, t) = G^{(+)}((\boldsymbol{x}' - \boldsymbol{x})^2, t' - t > 0)$

eigenfunctions: $\qquad H\,\varphi_\lambda(\boldsymbol{x}) = \epsilon_\lambda(\boldsymbol{x})\,\varphi(\boldsymbol{x})$

$$G^{(+)}(\boldsymbol{x}', \boldsymbol{x}, \tau = t' - t) = -\sum_\lambda \int_{-\infty}^{\infty} \frac{d\epsilon}{2\,\pi\,i}\, e^{-i\epsilon\,\tau} \frac{\varphi_\lambda(\boldsymbol{x}')\,\varphi_\lambda^*(\boldsymbol{x})}{\epsilon - \epsilon_\lambda + i\,0}$$

$$G^{(+)}(\boldsymbol{x}', t'; \boldsymbol{x}, t) =< N|\hat{\Psi}(\boldsymbol{x}', t')\,\hat{\Psi}^\dagger(\boldsymbol{x}, t)|N >$$

$$\hat{\Psi}(\boldsymbol{x}, t) = \sum_\lambda \varphi_\lambda(x)\,\hat{a}_\lambda\, e^{-i\,\epsilon_\lambda\, t} \qquad |N >= a_1^\dagger\, a_2^\dagger\, a_3^\dagger \ldots\, a_N^\dagger\,|0 >$$

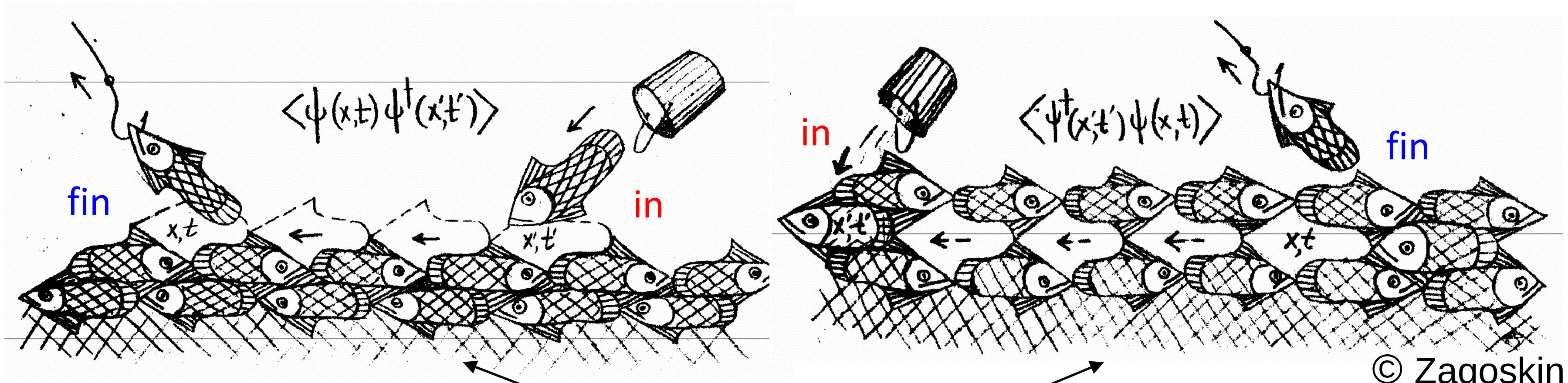
$a_i,\ a_i^\dagger$ annihilation and creation operator

Green's function of non-interacting fermions

$$i\,G(\boldsymbol{x},t;\boldsymbol{x}',t') =< N|T\{\hat{\Psi}(\boldsymbol{x},t)\,\hat{\Psi}^{\dagger}(\boldsymbol{x}',t')\}|N >$$

fin in

$$=< N|\hat{\Psi}(\boldsymbol{x},t)\,\hat{\Psi}^{\dagger}(\boldsymbol{x}',t')|N > \theta_{t-t'} - < N|\hat{\Psi}^{\dagger}(\boldsymbol{x}',t')\,\hat{\Psi}(\boldsymbol{x},t)|N > \theta_{t'-t}$$



no interaction

$$G(\boldsymbol{x},t;\boldsymbol{x}',t') = G(t-t',\boldsymbol{x}-\boldsymbol{x}')$$

$$G_0(\epsilon,p) = \frac{\mathbf{1-n_p}}{\epsilon-\epsilon_p+i\,0} + \frac{\mathbf{n_p}}{\epsilon+\epsilon_p^h-i\,0}$$

$$T=0$$

$$n_p = \theta(p_{\mathrm{F}}-p)$$

$$\epsilon_p^h = -\epsilon_p$$

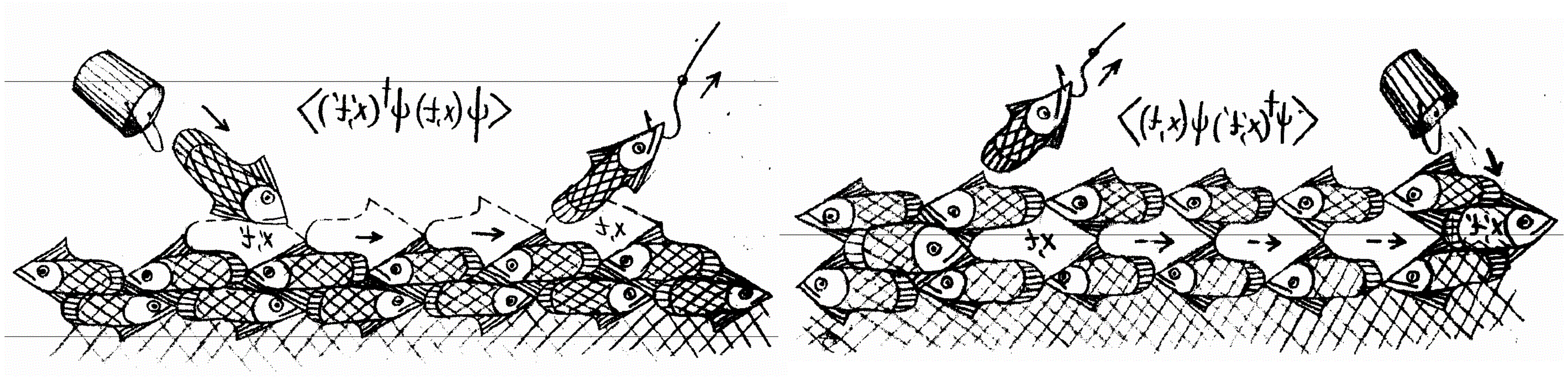
$$G(t,\boldsymbol{x}) = \int \frac{\mathrm{d}\epsilon\,\mathrm{d}^2p}{(2\pi)^4} G(\epsilon,\boldsymbol{p})e^{-i\epsilon t+i\boldsymbol{p}\boldsymbol{x}}$$

$$G_0(\epsilon,\boldsymbol{p}) = \frac{1}{\epsilon-\epsilon_p+i0\,\mathrm{sign}(\epsilon)}$$

$$\epsilon_p = \frac{p^2-p_{\mathrm{F}}^2}{2\,m}$$

G(ε,p)

particle



hole

$G^h(\varepsilon,p)=G(-\varepsilon,p)$

Momentum distribution of particles

$$G(\boldsymbol{x},t;\boldsymbol{x}',t') = -i < N|T\{\hat{\Psi}(\boldsymbol{x},t)\,\hat{\Psi}^\dagger(\boldsymbol{x}',t')\}|N >= G(t-t',\boldsymbol{x}-\boldsymbol{x}')$$

$t' = t+0$ $$G(-0,\boldsymbol{x}-\boldsymbol{x}') = i < N|\hat{\Psi}^\dagger(\boldsymbol{x}',t+0)\hat{\Psi}(\boldsymbol{x},t)\,|N >$$

$$G(0,-0) = i < N|\hat{\Psi}^\dagger(\boldsymbol{x},t+0)\hat{\Psi}(\boldsymbol{x},t)\,|N >= i\frac{N}{V}$$ particle density in the homogenous system

Particle momentum distribution $$n(\boldsymbol{p}) = \int(-i)G(t\to -0,\boldsymbol{x})e^{-i\boldsymbol{x}\boldsymbol{p}}\mathrm{d}^3\boldsymbol{x}$$

$$n(\boldsymbol{p}) = \int(-i)G(t\to -0,\boldsymbol{x})e^{-i\boldsymbol{x}\boldsymbol{p}}\mathrm{d}^3\boldsymbol{x} = \lim_{t\to -0}\int G(\epsilon,\boldsymbol{p})e^{-i\epsilon t}\frac{\mathrm{d}\epsilon}{2\pi i}$$

Free Green's function $$n(\boldsymbol{p}) = \lim_{t\to -0}\int\frac{1}{\epsilon-\frac{p^2-p_\mathrm{F}^2}{2m}+i\,0\mathrm{sign}(\epsilon)}e^{-i\epsilon t}\frac{\mathrm{d}\epsilon}{2\pi i}$$

for $t<0$ we have $-i\epsilon t = i\epsilon|t|$

we have to close the contour for $\mathrm{Im}\,\epsilon>0$
positive direction of the contour

$\mathrm{Im}\,\epsilon = 0(-)\mathrm{sign}(p^2-p_\mathrm{F}^2)>0$ if $p<p_\mathrm{F}$

$$n(\boldsymbol{p}) = \theta(p_\mathrm{F}-p)$$

Diagram technique

Ground state:

$$iG(x,y) = < N|\widehat{T}\{\widehat{\Psi}(x)\widehat{\Psi}^\dagger(y)\}|N > =< N|\widehat{S}^{-1}\widehat{T}\{\widehat{\Psi}_I(x)\widehat{\Psi}_I^\dagger(y)\}\widehat{S}|N >$$

in interaction picture: $$iG =< N|\widehat{T}\{\widehat{\Psi}_I(x)\widehat{\Psi}_I^\dagger(y)\}\widehat{S}|N >< \widehat{S}^{-1} >$$

transition from the ground state to the ground state under action of evolution operator

$$\widehat{S} = \underset{\substack{\uparrow \\ \text{time ordering}}}{\widehat{T}} \exp\left\{ -i\int_{-\infty}^{\infty} \widehat{V}_I(t)dt \right\}$$

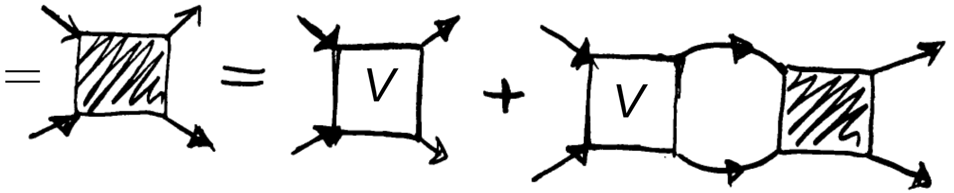
$$\widehat{V}_I(t) = e^{i\widehat{H}_0(\mu)\,t}\widehat{V}\,e^{-i\widehat{H}_0(\mu)t}$$

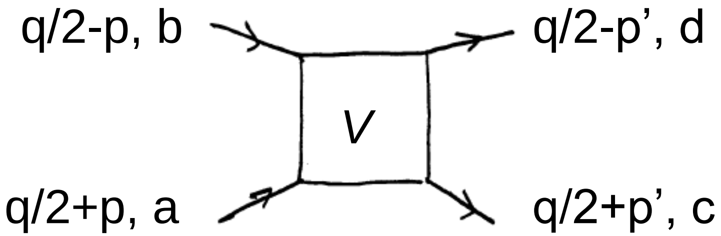
$$\widehat{H}_0(\mu) = H_0 - \sum_a \mu_a \widehat{N}_a$$

Only one type of Green's functions

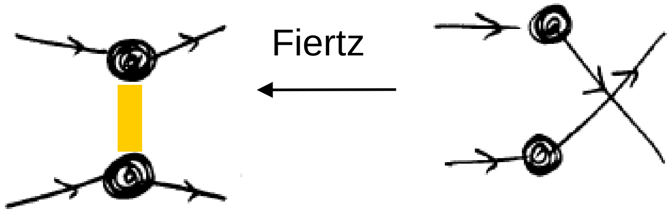
Particle-particle interaction

$$-i\,T_{pp}(p,p';q) =$$

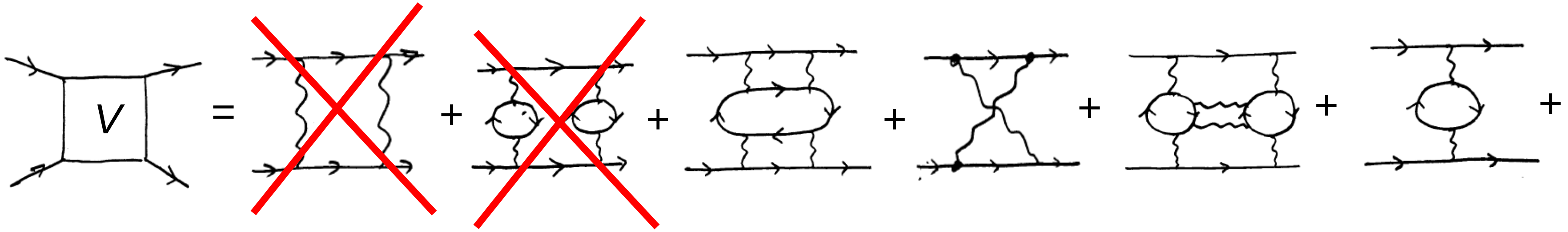




$$\left[\widehat{V}(p,p',q)\right]_{cd,ab} = V_0(p,p',q)(i\sigma_2)_{dc}(i\sigma_2)_{ab} + V_1(p,p',q)(\boldsymbol{\sigma} i\sigma_2\,)_{dc}(i\sigma_2\boldsymbol{\sigma})_{ab}$$



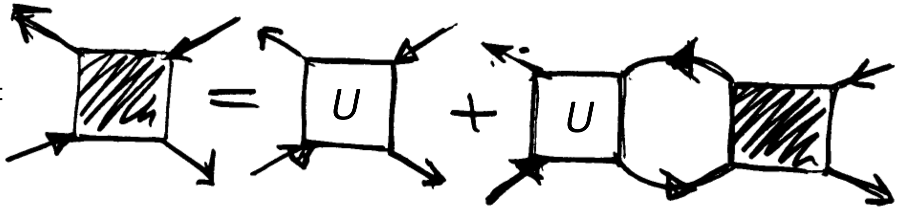
two-particle irreducible
interaction

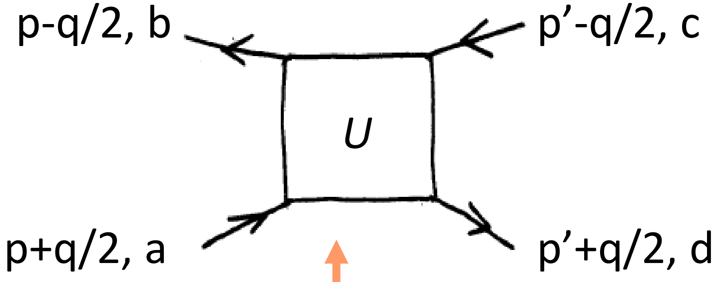


$$\widehat{T}_{\rm pp}(p,p',q) = \widehat{V}(p,p',q) + \int \frac{\mathrm{d}^4 p''}{(2\pi)^4\, i} \widehat{V}(p,p'',q)\,\widehat{G}(q/2+p'')\,\widehat{G}(q/2-p'')\,\widehat{T}_{\rm pp}(p'',p',q)$$

Particle-particle interaction

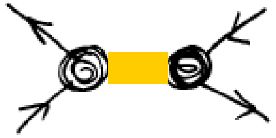
$$-i\,T_{\mathrm{ph}}(p,p';q) =$$

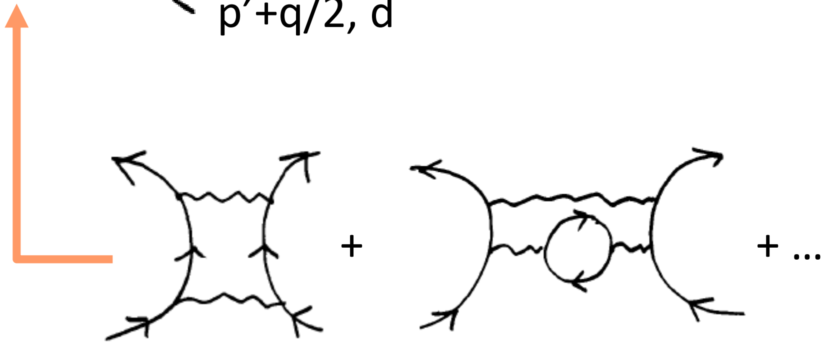




particle-hole irreducible interaction

$$\left[\widehat{U}(p,p',q)\right]^{ac}_{bd} = U_0(p,p',q)\,\delta^a_b\,\delta^c_d + U_1(p,p',q)\,(\boldsymbol{\sigma}\,)^a_b\,(\boldsymbol{\sigma}\,)^c_d$$

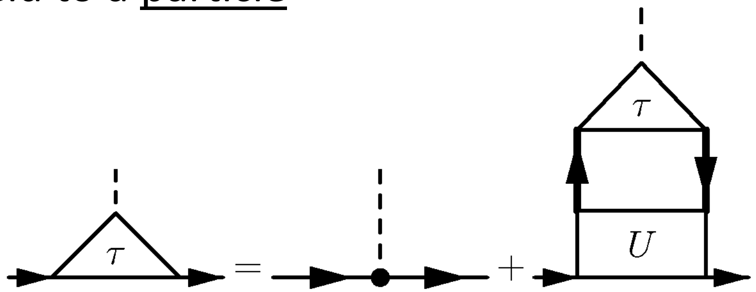




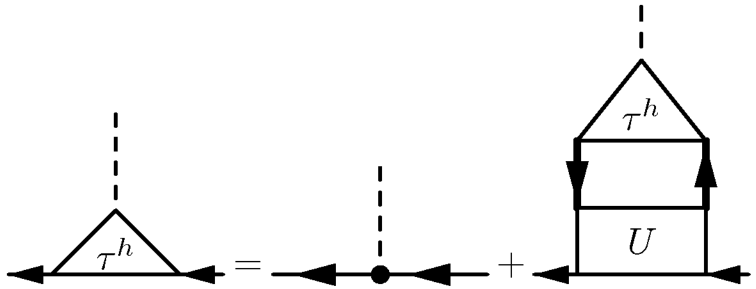
$$\widehat{T}_{\mathrm{ph}}(p,p',q) = \widehat{U}(p,p',q) + \int \frac{\mathrm{d}^4p''}{(2\pi)^4\,i}\widehat{U}(p,p'',q)\,\widehat{G}(q/2+p'')\,\widehat{G}^h(q/2-p'')\,\widehat{T}_{\mathrm{ph}}(p'',p',q)$$

"Charge" of particle and hole

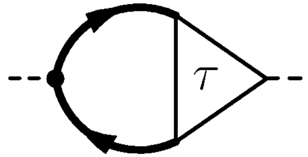
Coupling of the external field to a particle

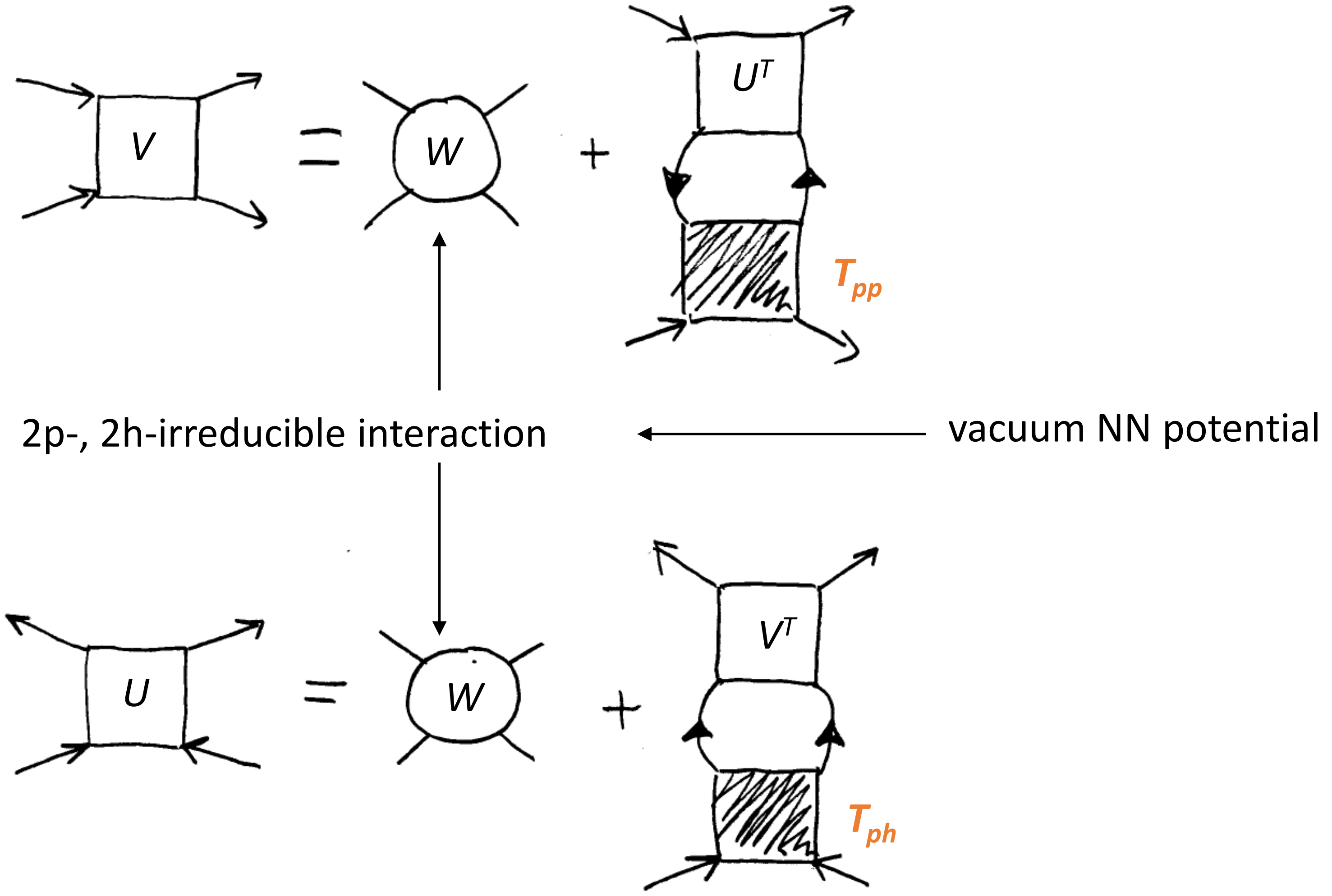


and a hole



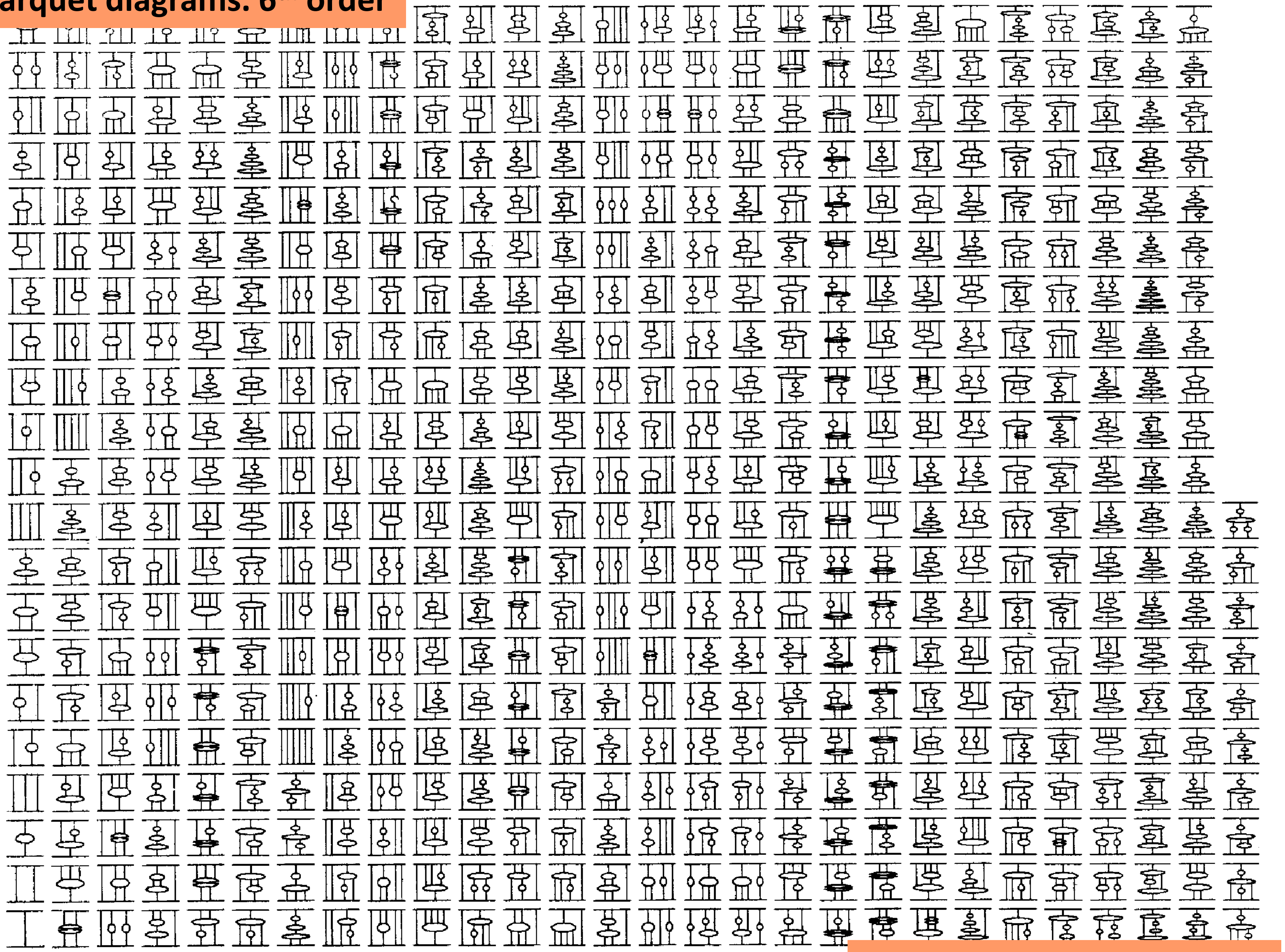
- **one-nucleon reaction**

$-i\,\Pi =$ 



[Wambach, Ainsworth,Pines NPA555]

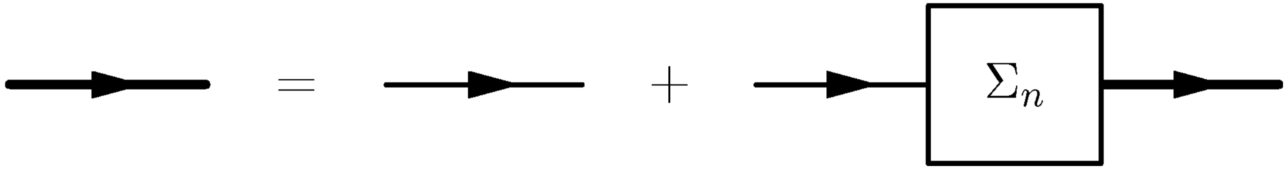
Parquet diagrams. 6th order



Jackson, Lande, Smith PR86

Full Green's function

<u>particle-line</u>

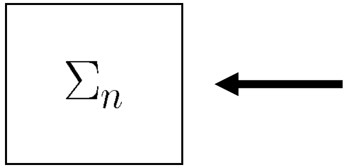


$$\widehat{G}_{\text{n.s.}} = \widehat{G}_0 + \widehat{G}_0\, \widehat{\Sigma}_{\text{n.s.}}\, \widehat{G}_{\text{n.s.}} = \left[[\widehat{G}_0]^{-1} - \widehat{\Sigma}_{\text{n.s.}} \right]^{-1}$$

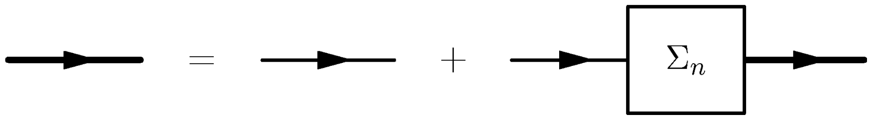
diagonal in spin-space

$$\hat{G}_0(\epsilon, \boldsymbol{p}) = \frac{\hat{1}}{\epsilon - \boldsymbol{p}^{\,2}/2\, m_N + i\, 0\, \text{sign}\, (\epsilon - \epsilon_{\text{F}})}$$

analogously for <u>the hole-line</u>



particle-particle, particle-hole hole-hole interactions



$$\widehat{G} = \widehat{G}_0 + \widehat{G}_0\,\widehat{\Sigma}\,\widehat{G} = \left[[\widehat{G}_0]^{-1} - \widehat{\Sigma}\right]^{-1}$$

$$T \ll \varepsilon_{\mathrm{F},n}, \varepsilon_{\mathrm{F,p}} \text{ and } \epsilon \sim \epsilon_{\mathrm{F}}\,,\ p \sim p_{\mathrm{F}}$$

pole residue

$$a^{-1} = 1 - \frac{\partial}{\partial \epsilon}\Sigma(\epsilon, 0, T)\Big|_{\epsilon \simeq \epsilon_{\mathrm{F}}}$$

$$G(\epsilon, \boldsymbol{p}) = \frac{a}{\epsilon - \epsilon_p + i\,\gamma\,\epsilon^2\,\mathrm{sign}\epsilon} + G_{\mathrm{reg}}(\epsilon, \boldsymbol{p})$$

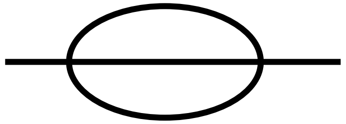
q.p. energy

q.p. width

$$\epsilon_p = \frac{p^2 - 2m_N\mu_N}{2\,m_N^*} \approx \frac{p^2 - p_{\mathrm{F}}^2}{2\,m_N^*} \approx v_{\mathrm{F}}\,(p - p_{\mathrm{F}})$$

$$\gamma = -\lim_{\epsilon \to 0} \mathrm{Im}\,\Sigma^R(\epsilon, p^2 = 2m_N\mu_N, T)/\epsilon^2$$

small for $T \ll T_F$



q.p. effective mass

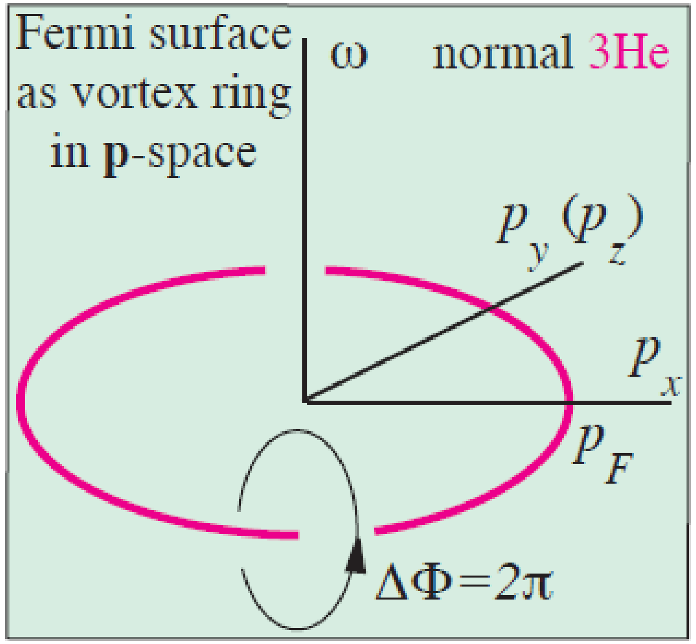
$$\frac{1}{m_N^*} = \frac{a}{m_N} + 2\,a\,\frac{\partial}{\partial p^2}\Sigma(\epsilon, \boldsymbol{p}, T)\Big|_{p=0, \epsilon \simeq \epsilon_{\mathrm{F}}}$$

$G_{\mathrm{reg}}(\epsilon, \boldsymbol{p})$ complicated background part

Fermi surface is a topological object.

ideal gas $$G_0(z=i\omega,p)=\frac{1}{i\omega-v_F(p-p_F)} \qquad v_F=p_F/m_N$$

In 4D space (ω,p) there is a singularity at (ω=0,p=p_F) [singular hyperline] where this function is not defined!

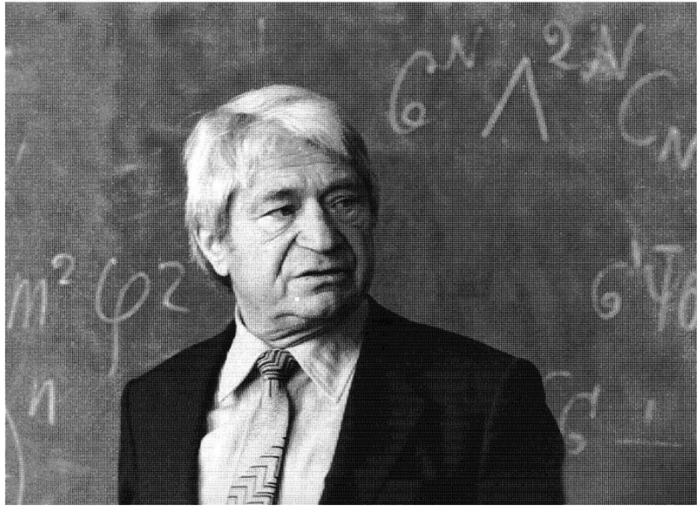


The phase of the Green's function changes by 2p when one goes along a contour encircling this singular line. One can define a topological invariant [see book y G.E. Volovik, The Universe in a helium droplet] The singular-line is topologically protected and thus robust against perturbations

(normal) Fermi liquid $$G(z=i\omega,p)=\frac{a}{i\omega-v_F(p-p_F)} \qquad v_F=p_F/m_N^*$$

Migdal jump

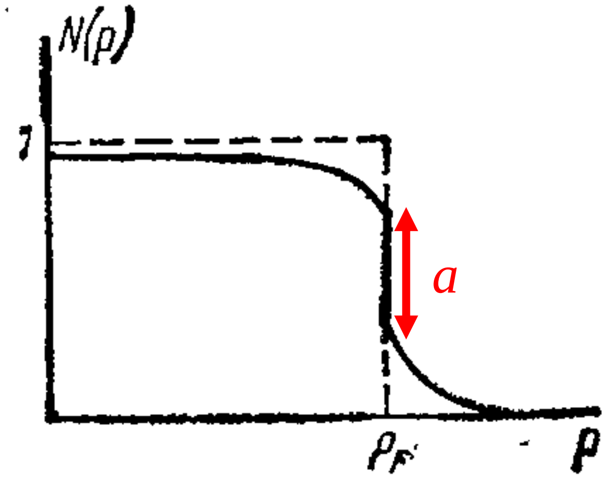
$$G(\epsilon,\boldsymbol{p}) = \frac{a}{\epsilon - v_{\rm F}\,(p-p_{\rm F}) + i\,\gamma\,\epsilon^2\,{\rm sign}\epsilon} + G_{\rm reg}(\epsilon,\boldsymbol{p})$$



$$n(\boldsymbol{p}) = \lim_{t\to -0}\int G(\epsilon,\boldsymbol{p})e^{-i\epsilon t}\frac{{\rm d}\epsilon}{2\pi i}$$

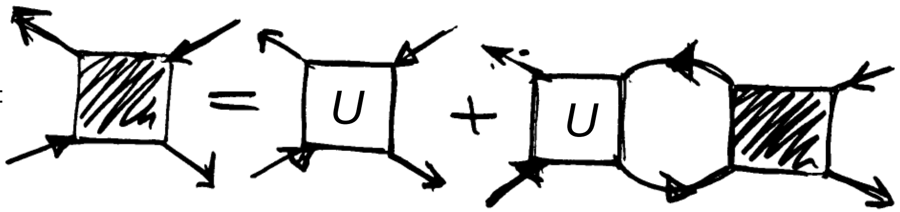
$$\lim_{q\to 0}\left[n(p_{\rm F}-q) - n(p_{\rm F}+q)\right] = \lim_{t\to -0}\int\Big\{\frac{a}{\epsilon + v_{\rm F}\,q - i\,\gamma\,\epsilon^2} - \underbrace{\frac{a}{\epsilon - v_{\rm F}\,q + i\,\gamma\,\epsilon^2}}_{=0} + \underbrace{G_{\rm reg}(\epsilon,p-q) - G_{\rm reg}(\epsilon,p+q)}_{\to 0}\Big\}e^{-i\epsilon t}\frac{{\rm d}\epsilon}{2\pi i}$$

$$= a$$



Fermi surface exists even for the strongly interacting systems!

$$-i\,T_{\rm ph}(p,p';q) =$$



$$\widehat{T}_{\rm ph}(p,p',q) = \widehat{U}(p,p',q) + \int \frac{\mathrm{d}^4p''}{(2\pi)^4\,i}\widehat{U}(p,p'',q)\,\widehat{G}(q/2+p'')\,\widehat{G}^h(q/2-p'')\,\widehat{T}_{\rm ph}(p'',p',q)$$

$$G(q/2+p)\,G^h(q/2-p) = G(q/2+p)\,G(p-q/2)$$

$$= \frac{a}{[\epsilon+\omega/2-\epsilon_{\boldsymbol{p}+\boldsymbol{q}/2}+i\,0\,\mathrm{sign}\,(\epsilon+\omega/2)]}\,\frac{a}{[\epsilon-\omega/2-\epsilon_{\boldsymbol{p}-\boldsymbol{q}/2}+i\,0\,\mathrm{sign}\,(\epsilon-\omega/2)]} + \tilde{B}(p,q)$$

$$\simeq a^2\,\delta(\epsilon)\int d\epsilon\,\frac{1}{[\epsilon+\omega/2-\epsilon_{\boldsymbol{p}+\boldsymbol{q}/2}+i\,0\,\mathrm{sign}\,(\epsilon+\omega/2)]}\,\frac{1}{[\epsilon-\omega/2-\epsilon_{\boldsymbol{p}-\boldsymbol{q}/2}+i\,0\,\mathrm{sign}\,(\epsilon-\omega/2)]} + B(p,q)$$

$$= -2\pi i\,a^2\,\delta(\epsilon)\frac{f(\boldsymbol{p}+\boldsymbol{q}/2)-f(\boldsymbol{p}-\boldsymbol{q}/2)}{\omega-\epsilon_{\boldsymbol{p}+\boldsymbol{q}/2}+\epsilon_{\boldsymbol{p}-\boldsymbol{q}/2}+i\,0} + B(p,q)$$

$$p \sim p_{\rm F}$$

Fermi liquid approximation

particle-hole propagator …. for $q \to 0$

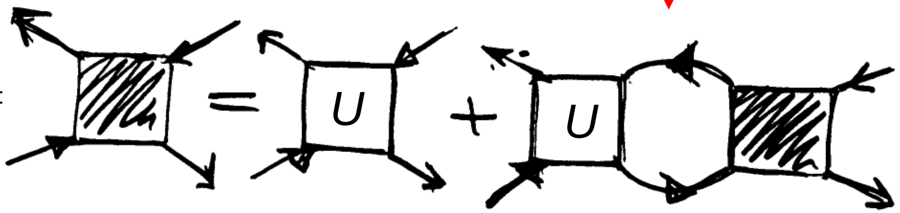
$$\boldsymbol{n} = \boldsymbol{p}/p$$

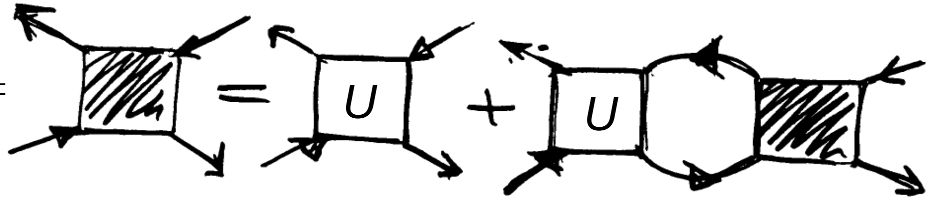
$$G(q/2+p)\, G^h(q/2-p) \simeq 2\pi i\, a^2\, \delta(\epsilon) \frac{v_{\mathrm{F}}\, \boldsymbol{q}\boldsymbol{n}}{\omega - v_{\mathrm{F}}\boldsymbol{q}\boldsymbol{n} + i\,0}\, \delta(p - p_{\mathrm{F}}) + B(p,q)$$

singular pole term

complicated background

$$-i\, T_{\mathrm{ph}}(p, p'; q) =$$



$$-i\,T_{\rm ph}(p,p';q) =$$ 

for $|\boldsymbol{p}| \simeq p_{\rm F} \simeq |\boldsymbol{p}'|$ and $|\boldsymbol{q}\boldsymbol{p}| << \omega << \epsilon_{\rm F}$

$$\widehat{T}_{\rm ph}(\boldsymbol{n},\boldsymbol{n}',q) = \widehat{\Gamma}^{\omega}(\boldsymbol{n},\boldsymbol{n}') - \int \frac{d\Omega_{p''}}{4\,\pi}\,\widehat{\Gamma}^{\omega}(\boldsymbol{n},\boldsymbol{n}')\,A(\boldsymbol{n},q)\,\widehat{T}_{\rm ph}(\boldsymbol{n},\boldsymbol{n}',q)$$

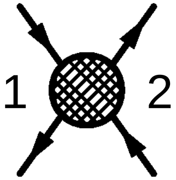
$$A(\boldsymbol{n},q) = a^2\,\frac{m^*\,p_{\rm F}}{\pi^2}\,\frac{v_{\rm F}\,\boldsymbol{q}\boldsymbol{n}}{\omega - v_{\rm F}\boldsymbol{q}\boldsymbol{n} + i\,0}$$

complicated dynamics is here:

$$\widehat{\Gamma}^{\omega}_{\rm ph}(\boldsymbol{n},\boldsymbol{n}') = \widehat{U}(\boldsymbol{n},\boldsymbol{n}') - \int \frac{d^4p''}{(2\pi)^4\,i}\widehat{U}(\boldsymbol{n},\boldsymbol{n}')\,B(p,\omega\to 0,\frac{\boldsymbol{q}}{\omega}\to 0)\,\widehat{\Gamma}^{\omega}_{\rm ph}(\boldsymbol{n},\boldsymbol{n}')$$

parameterize

Landau-Migdal parameters

1  2 $= f_{12}(\boldsymbol{n},\boldsymbol{n}') + g_{12}(\boldsymbol{n},\boldsymbol{n}')\,\boldsymbol{\sigma_1}\boldsymbol{\sigma_2}$

extracted from experiment

$$a^2 N\,\Gamma_0^\omega(\theta) = f(\theta) = \sum_l f_l P_l(\cos\theta)$$ $$N = \nu m^* p_{\rm F}/\pi^2$$

$$a^2 N\,\Gamma_1^\omega(\theta) = g(\theta) = \sum_l g_l P_l(\cos\theta)$$ *density of states at the Fermi surface*

$$\theta = \angle(\boldsymbol{n}, \boldsymbol{n}')$$

$\nu = 1, 2$ *number of fermion types* $n = \nu p_{\rm F}^3/3\pi^2$

neutron matter: $f = f_{nn}$ $g = g_{nn}$ (1 parameter in each channel)

nuclear matter: $f_{nn},\, f_{np},\, f_{pp}$ $g_{nn},\, g_{np},\, g_{pp}$ (3 parameters in each channel)

In matter of arbitrary isospin composition these parameters are independent.
Fermi-liquid renormalization is different for these parameters.

small isospin disballance $f_{nn} = f_{pp} = f + f'$ $f_{np} = f - f'$

$$a^2 N\Gamma^\omega = f + f'\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2$$ (2 parameters in each channel)

In nuclear physics one uses also the normalization on the nuclear Fermi surface

$$\tilde f(\boldsymbol{n}', \boldsymbol{n}) = a^2\, N_0 \Gamma_0^\omega(\boldsymbol{n}', \boldsymbol{n}) \qquad \tilde g(\boldsymbol{n}', \boldsymbol{n}) = a^2\, N_0 \Gamma_1^\omega(\boldsymbol{n}', \boldsymbol{n})$$

$N_0 = N(n = n_0)$ constant, independent of density $(N_0^{-1} = 300\,{\rm MeV\,fm}^3)$

Density dependence? Residual momentum dependence $\Gamma(\boldsymbol{n}', \boldsymbol{n}; q)$?

There are relations between some Landau parameters and bulk properties of the system

effective mass $m^* = m\left(1 + \frac{2}{3} f_1\right)$

compressibility $K = 6\,\frac{p_{\mathrm{F}}^2}{m^*}\,(1 + 2\,f_0)$

symmetry energy $E_{\mathrm{sym}} = \frac{1}{3}\,\frac{p_{\mathrm{F}}^2}{2\,m^*}\left(1 + 2\,f_0'\right)$

In general Landau parameter are to be fitted to empirical information (nucleus properties)

[Saperstein,Fayans, et al. 1995, 1998]

$$f \simeq 0,\ f' \simeq 0.5 - 0.6,\ g \simeq 0.05 \pm 0.1,\ g' \simeq 1.1 \pm 0.1$$

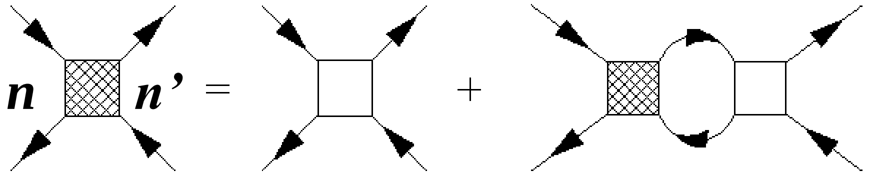
Sounds in Fermi liquid

system of strongly interacting fermions (no pairing)

single-particle excitation mechanism $\mathbf{G}(\epsilon,\boldsymbol{p}) = \dfrac{a}{\epsilon - v_{\rm F}(p-p_{\rm F})} + G_{\rm reg}$

quasiparticle

particle-hole interaction
on Fermi surface



short-range *long-range*

$$\widehat{T}_{\rm ph}(\boldsymbol{n}',\boldsymbol{n};q) = \widehat{\Gamma}^{\omega}(\boldsymbol{n}',\boldsymbol{n}) + \langle \widehat{\Gamma}^{\omega}(\boldsymbol{n}',\boldsymbol{n}'')\,\mathcal{L}_{\rm ph}(\boldsymbol{n}'';q)\,\widehat{T}_{\rm ph}(\boldsymbol{n}'',\boldsymbol{n};q)\rangle_{\boldsymbol{n}''}$$

particle-hole propagator

$$\mathcal{L}_{\rm ph}(\boldsymbol{n};q) = \int_{-\infty}^{+\infty}\frac{d\epsilon}{2\pi i}\int_{0}^{+\infty}\frac{dp\,p^2}{\pi^2}\,G(p_{\rm F+})\,G(p_{\rm F-})$$

pole parts

$$\langle\ldots\rangle_{\boldsymbol{n}} = \int\frac{d\Omega_{\boldsymbol{n}}}{4\pi}(\ldots)$$

$$p_{\rm F\pm} = (\epsilon \pm \omega/2, p_{\rm F}\,\boldsymbol{n} \pm \boldsymbol{k}/2)$$

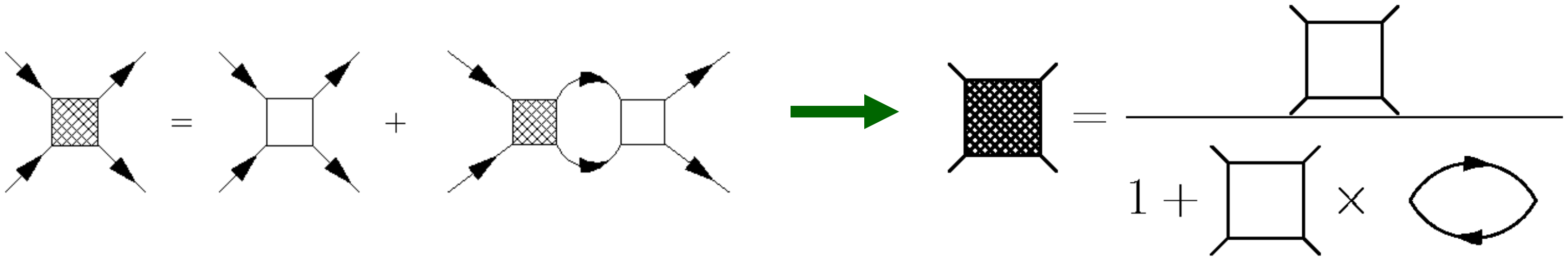
Landau parameters

$$\hat{\Gamma}^{\omega}(\boldsymbol{n}',\boldsymbol{n}) = \Gamma_0^{\omega}(\boldsymbol{n}'\boldsymbol{n})\,\sigma_0'\sigma_0 + \Gamma_1^{\omega}(\boldsymbol{n}'\boldsymbol{n})\,(\boldsymbol{\sigma}'\boldsymbol{\sigma})$$

Solutions for zeroth harmonics

$$\hat{\Gamma}^{\omega}(\boldsymbol{n}', \boldsymbol{n}) = \Gamma_0^{\omega}\, \sigma_0' \sigma_0 + \Gamma_1^{\omega}\, (\boldsymbol{\sigma}' \boldsymbol{\sigma})$$

keep only zeroth harmonics



$$\widehat{T}_{\rm ph}(\boldsymbol{n}', \boldsymbol{n}; q) = T_{{\rm ph},0}(q)\, \sigma_0'\, \sigma_0 + T_{{\rm ph},1}(q)\, (\boldsymbol{\sigma}' \boldsymbol{\sigma})$$

$$T_{{\rm ph},0(1)}(q) = \frac{1}{1/\Gamma^{\omega}_{0(1)} - \langle \mathcal{L}_{\rm ph}(\boldsymbol{n}; q)\rangle_{\boldsymbol{n}}}$$

$$\langle \mathcal{L}_{\rm ph}(\boldsymbol{n}; q)\rangle_{\boldsymbol{n}} = -a^2 N \Phi\left(\frac{\omega}{v_{\rm F}\, k}, \frac{k}{p_{\rm F}}\right)$$

Lindhard function

Lindhard function

$$\Phi(s,x) = \frac{z_-^2 - 1}{4(z_+ - z_-)} \ln \frac{z_- + 1}{z_- - 1} - \frac{z_+^2 - 1}{4(z_+ - z_-)} \ln \frac{z_+ + 1}{z_+ - 1} + \frac{1}{2} \qquad (z_\pm = s \pm x/2)$$

Imaginary part

$s = \omega / k v_{\mathrm{F}}$

$x = k / p_{\mathrm{F}}$

$$\Im\Phi(s,x) = \begin{cases} \frac{\pi}{2}\, s &, \quad 0 \le s \le 1 - \frac{x}{2} \\ \frac{\pi}{4x}(1 - z_-^2) &, \quad 1 - \frac{x}{2} \le s \le 1 + \frac{x}{2} \\ 0 &, \quad \text{otherwise} \end{cases}$$

Results of expansions depends on the expansion order:

$$\Phi(s,x) \approx -\frac{1}{3\, z_+\, z_-} = -\frac{1}{3} \frac{1}{s^2 - x^2/4} \qquad \text{for } s \gg 1$$

$$\Phi(s,x) \approx 1 + \frac{s}{2} \log \frac{s-1}{s+1} - \frac{x^2}{12\,(s^2-1)^2} \qquad \text{for } x \ll 1$$

Temperature corrections

$$\Phi_T(s,x,T) = \Phi(s,x) \left(1 - \frac{\pi^2}{12} \frac{T^2}{\epsilon_{\mathrm{F}}^2}\right)$$

Particle-hole interaction in the scalar channel

scalar channel zeroth harmonics

$$T_{\mathrm{ph},0} = \frac{1}{[\Gamma^{\omega}_{00}]^{-1} + N\,\Phi(\omega,\boldsymbol{q})} = \frac{N^{-1}}{f_0^{-1} + \Phi(\omega,\boldsymbol{q})}$$

$\Phi(\omega,k)$ Lindhard function

<u>solutions of equation:</u> $f_0^{-1} + \Phi(\omega,\boldsymbol{q}) = 0$ $\longrightarrow$ spectrum of excitations in the scalar channel $\omega(k)$

(zero-sound modes)

for $\omega \sim \omega(k)$

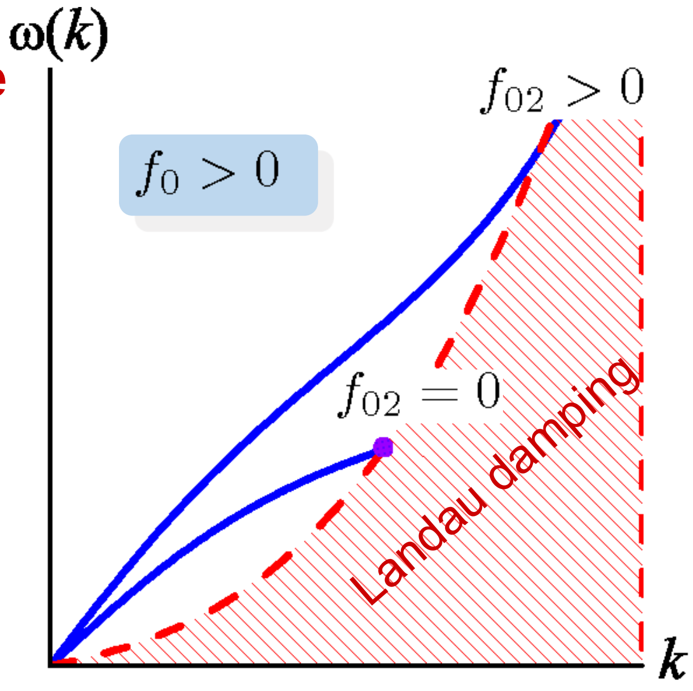
$$T_{\mathrm{ph},0} \approx \frac{V^2(k)}{\omega - \omega(k)} \quad \text{with} \quad V^{-2} = N\frac{\partial \mathrm{Re}\,\Phi}{\partial\omega}\Big|_{\omega(k)}$$

$$f_0^{-1} + \Phi(\omega, \boldsymbol{q}) = 0$$

Scalar modes in Fermi liquid

I. $\underline{f_0(k) = f_{00} + f_{02}\, k^2/p_{\rm F}^2 > 0}$ **zero-sound mode**

residual momentum dependence



$$\omega_s(k) = k\, v_{\rm F}\, s(x) \qquad s(x) \approx s_0 + s_2\, x^2 + s_4\, x^4$$

$$\frac{1 + f_{00}}{f_{00}} = \frac{s_0}{2} \ln \frac{s_0 + 1}{s_0 - 1}$$

$$s_2 = \frac{s_0(\alpha + f_{02})(s_0^2 - 1)}{f_{00}\,(1 + f_{00} - s_0^2)} \qquad \alpha = f_{00}^2/[12(s_0^2 - 1)^2]$$

$$s_4 > 0$$

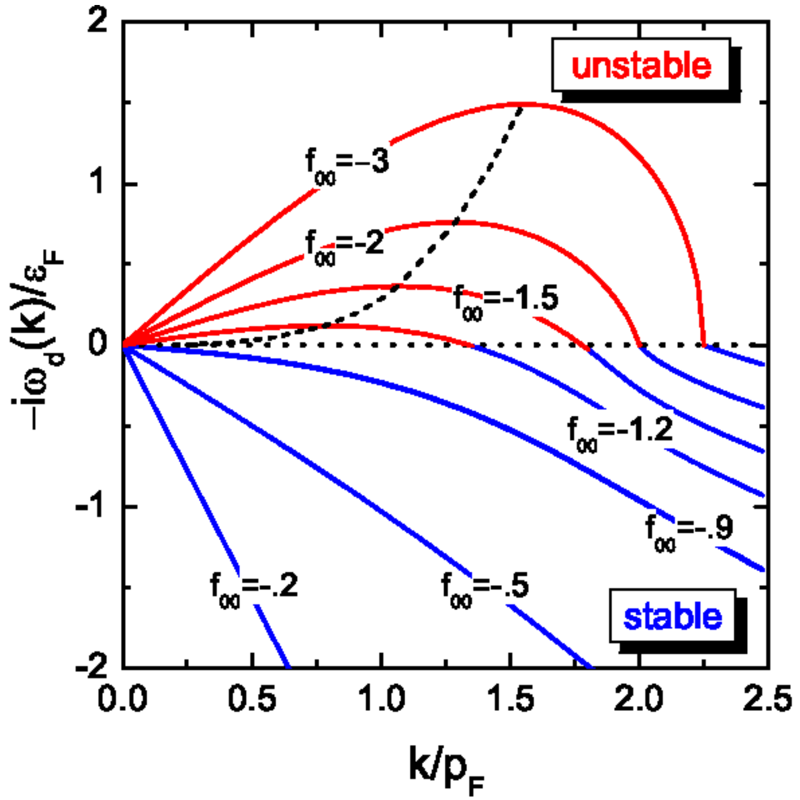
For $f_{02} < -\alpha$ the ratio $\omega_s(k)/k$ has a minimum at k_0 in which the group velocity of the excitation $v_{\rm gr} = {\rm d}\omega_s/{\rm d}k$ coincides with the phase one $v_{\rm ph} = \omega_s/k$.

II. $-1 < f_0 < 0$ **"diffuson"**

$$\omega_{\rm d} \approx -i\frac{2}{\pi}kv_{\rm F}\left[\frac{(1-|f_0|)}{|f_0|}+\frac{k^2}{12p_{\rm F}^2}\right]$$

$\varphi \sim e^{-i\omega t} \to 0$ $\quad t \to \infty$ stable mode

$k/p_F << 1$ expansion



III. $f_0 < -1$

$$\omega_{\rm d} \approx +i\frac{2}{\pi}kv_{\rm F}\left[1-\frac{1}{|f_0|}-\frac{k^2}{12p_{\rm F}^2}\right]$$

$\varphi \sim e^{-i\omega t} \to \infty$

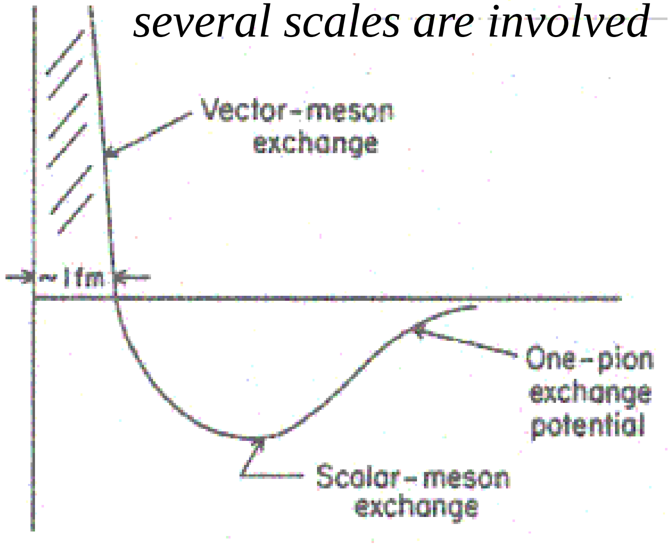
$t \to \infty$

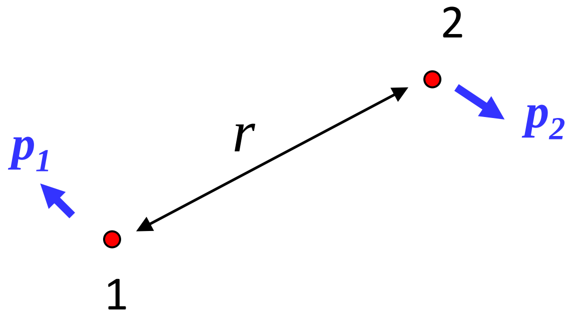
the mode growth rate is determined by f_0

$$-i\omega_m = \frac{8}{3\pi}v_{\rm F}p_{\rm F}\left(1-\frac{1}{|f_{00}|}\right)^{3/2}$$

Nucleon-nucleon interaction

several scales are involved





non-relativistic description

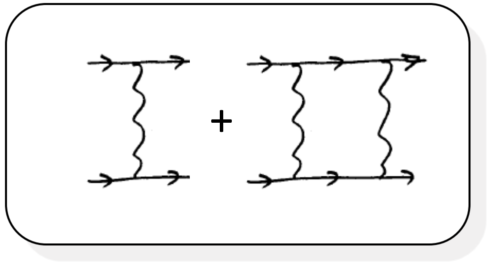
$$\begin{aligned}\hat{V}(r) &= V_C(r)\,\hat{1}\\ &+ V_{\rm spin}(r)\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2\\ &+ V_{LS}(r)\,(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\cdot\boldsymbol{L}\\ &+ V_{\rm tensor}(r)\,\hat{S}_{12}(r)\end{aligned}$$

vector mesons: $m_{\omega,\rho}$~800 MeV , ***r*~0.24 fm**

correlated 2π exchange: m~200-600 MeV ***r*~0.3—1fm**

1-pion exchange: $m_{\approx}$=140 MeV ***r*~1.4 fm**

relativistic description



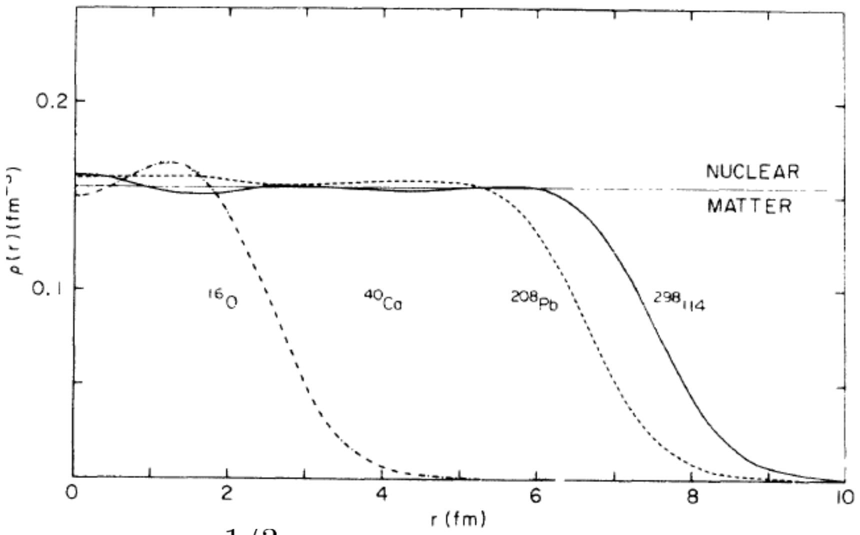
Equilibrium density of an atomic nucleus n_0=0.16 fm^{-3}

inter-nuclear distance $(n_0)^{-1/3}$=**1.8 fm**

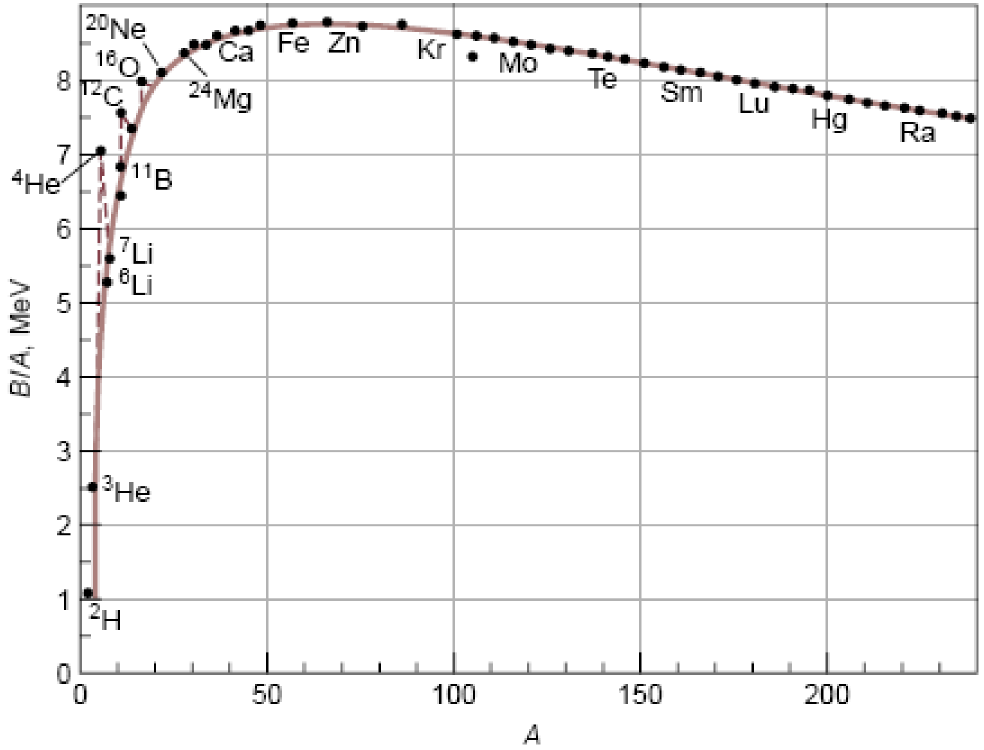
Nuclear equation of state

$$M(A,Z)\,c^2 \;=\; (A-Z)\,m_n\,c^2 + Z\,m_p\,c^2 - B(A,Z)$$

1932 Liquid-drop model of a nucleus:



$$R = r_0\,A^{1/3}$$



Weizsäcker’s semiempirical mass formula

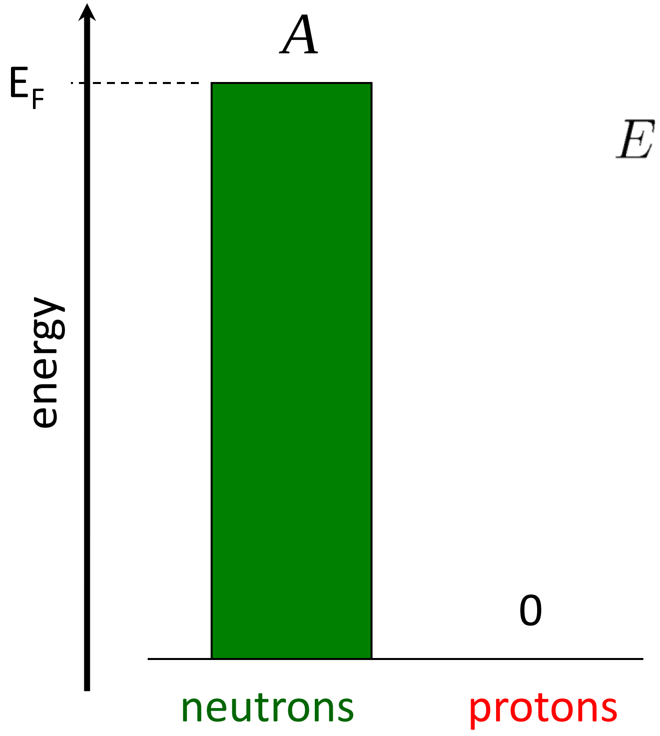
$$B(A,Z) = \quad a_{\mathrm{Vol}}\,A - a_{\mathrm{Surf}}\,A^{2/3} - a_{\mathrm{Sym}}\,\frac{(A-2\,Z)^2}{A} - a_{\mathrm{Coul}}\,\frac{Z\,(Z-1)}{A^{1/3}} + \delta(A,Z)$$

volume e. | surface e. | symmetry e. | Coulomb e. | pairing e.

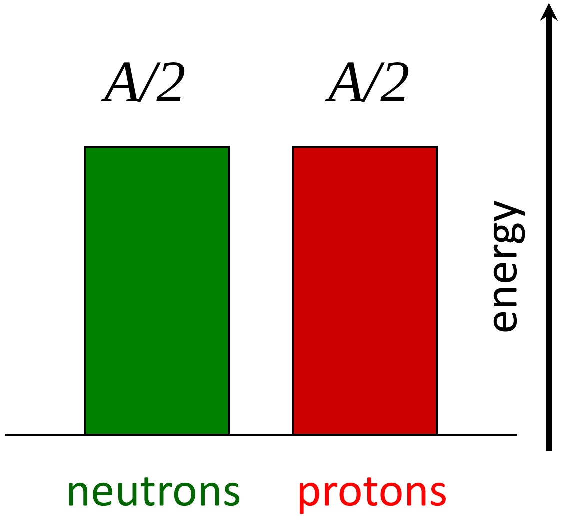
Pauli exclusion principle: nuclear symmetry energy

Neglect electric charge of protons: isospin symmetry.

We want to distribute A nucleons



$$E(A, Z=0) > E(A, Z=A/2)$$



Two Fermi seas are better than one Fermi sea!

Infinite nuclear matter

1) make A, V, big by keep $n_p = \frac{Z}{V}$, $n_n = \frac{A-Z}{V} = n - n_p$ fixed

2) switch off electromagnetic interaction

3) $m_n = m_p = m_N$

Energy density of the infinite nuclear matter as function of the proton and neutron densities:

$$\lim_{A\to\infty} \frac{M(A,Z)c^2}{A} = E(n_p, n_n) = m_N - \lim_{A\to\infty} \frac{B(A,Z)c^2}{A}$$

Binding energy per nucleon: $\varepsilon(n,x) = E(n,x)/n - m_N$

where, $n = n_p + n_n$ total density, $x = n_p/n$ proton fraction

chemical potentials:

$$\mu_n = \frac{\partial E(n_p,n_n)}{\partial n_n} = \frac{\partial E(n,x)}{\partial n} - \frac{x}{n}\frac{\partial E(n,x)}{\partial x}$$

$$\mu_p = \frac{\partial E(n_p,n_n)}{\partial n_p} = \frac{\partial E(n,x)}{\partial n} + \frac{1-x}{n}\frac{\partial E(n,x)}{\partial x}$$

Pressure: $P = \mu_n n_n + \mu_p n_p - E = n\frac{\partial E}{\partial n} - E$ T=0

Infinite nuclear matter. Symmetry energy

$$\varepsilon(n,x) = \varepsilon_0(n) + \varepsilon_S(n)\,(1-2\,x)^2 + \dots$$

ISM energy: $\varepsilon_0(n)$ Symmetry energy: $\varepsilon_S(n)$

Two definitions of the symmetry energy:

(1) $\varepsilon_S(n) = \frac{1}{8}\frac{\partial^2 \varepsilon(n,x)}{\partial x^2}\Big|_{x=1/2}$ and (2) $\varepsilon_S(n) = \varepsilon(n, x=0) - \varepsilon(n, x=1/2)$

local *NS applications*

If the derivative $\dfrac{\partial^4 \varepsilon(n,x)}{\partial x^4}$ is very small, then both definitions are equivalent

Equation of state of nuclear matter

The energy per nucleon of the nuclear matter

$$E(n_p, n_n) = \varepsilon_0(n) + \varepsilon_S(n)\frac{(n_p - n_n)^2}{n^2}$$

n_p – proton number density
n_n – neutron number density
$n = n_p + n_n$

- nuclear matter parameters

$$\varepsilon_0(n) = E_0 + 0 + \frac{K}{18}\frac{(n-n_0)^2}{n_0^2} + \frac{Q}{162}\frac{(n-n_0)^3}{n_0^3} + O\Big(\frac{(n-n_0)^4}{n_0^4}\Big)$$

symmetry energy

$$\varepsilon_S(n) = J + \frac{L}{3}\frac{n-n_0}{n_0} + \frac{K_{\mathrm{sym}}}{18}\frac{(n-n_0)^2}{n_0^2} + \frac{Q_{\mathrm{sym}}}{162}\frac{(n-n_0)^3}{n_0^3} + O\Big(\frac{(n-n_0)^4}{n_0^4}\Big)$$

S_0

There is a correlation among parameters: J, L, K_{sym}

- *low-density parameters*

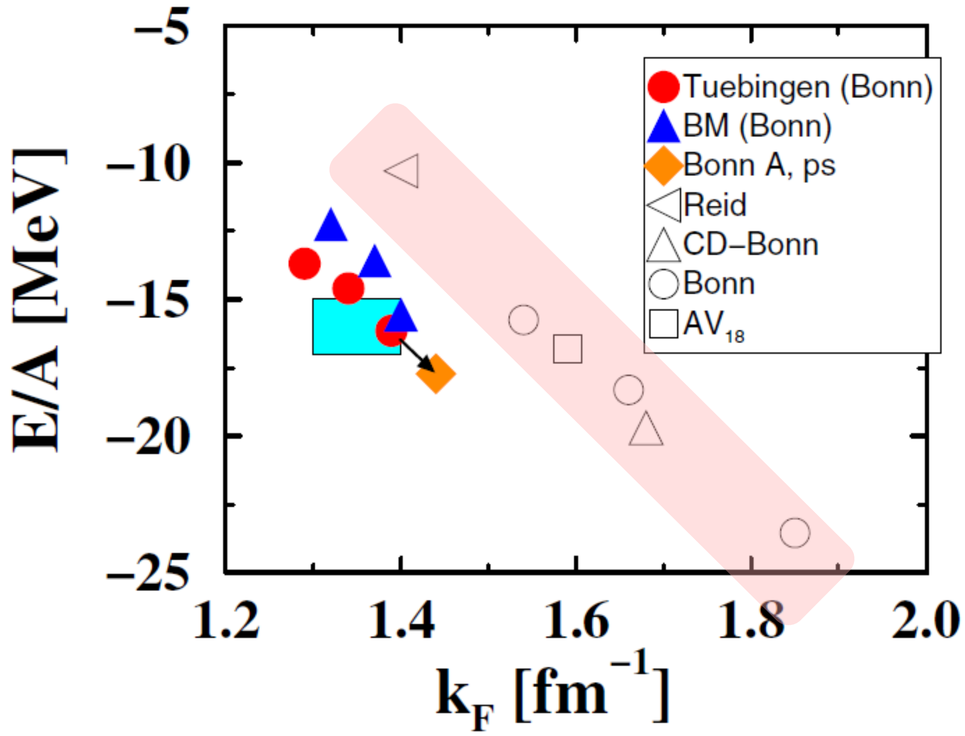
$$\varepsilon_0[n] = E_0 + \frac{K}{18}\frac{(n-n_0)^2}{n_0^2} - \frac{K'}{162}\frac{(n-n_0)^3}{n_0^3} + \dots$$

$$\varepsilon_S[n] = J + \frac{L}{3}\frac{n-n_0}{n_0} + \frac{K_{\rm sym}}{18}\frac{(n-n_0)^2}{n_0^2} + \dots$$

saturation density n_0 and binding energy E_0

$$n_0 \simeq 0.16 \pm 0.015 \ \mathrm{fm}^{-3}$$

$$E_0 \simeq -15.6 \pm 0.6\,\mathrm{MeV}$$



- *Correlations among parameters*

  n_0 vs E_0 –Coester line problem: role of TNF, relativistic effects, chiral forces

- *Stiffness of EoS*

  frequently characterized by compressibility modulus K

  Giant Monopole Resonance (GMR) $K = 240 \pm 20\,\mathrm{MeV}$

Correlations among parameters L-J

$$\varepsilon_S[n] = J + \frac{L}{3}\frac{n-n_0}{n_0} + \frac{K_{\rm sym}}{18}\frac{(n-n_0)^2}{n_0^2} + \dots$$

Masses: UNEDF0 Skyrme DF+BHF
[Kortelainen *et al.*, PRC **82**, 024313 (2010)]

Isobaric analog states+isovector skin:
[Danielewicz et al. NPA 958, 147 (2017)]

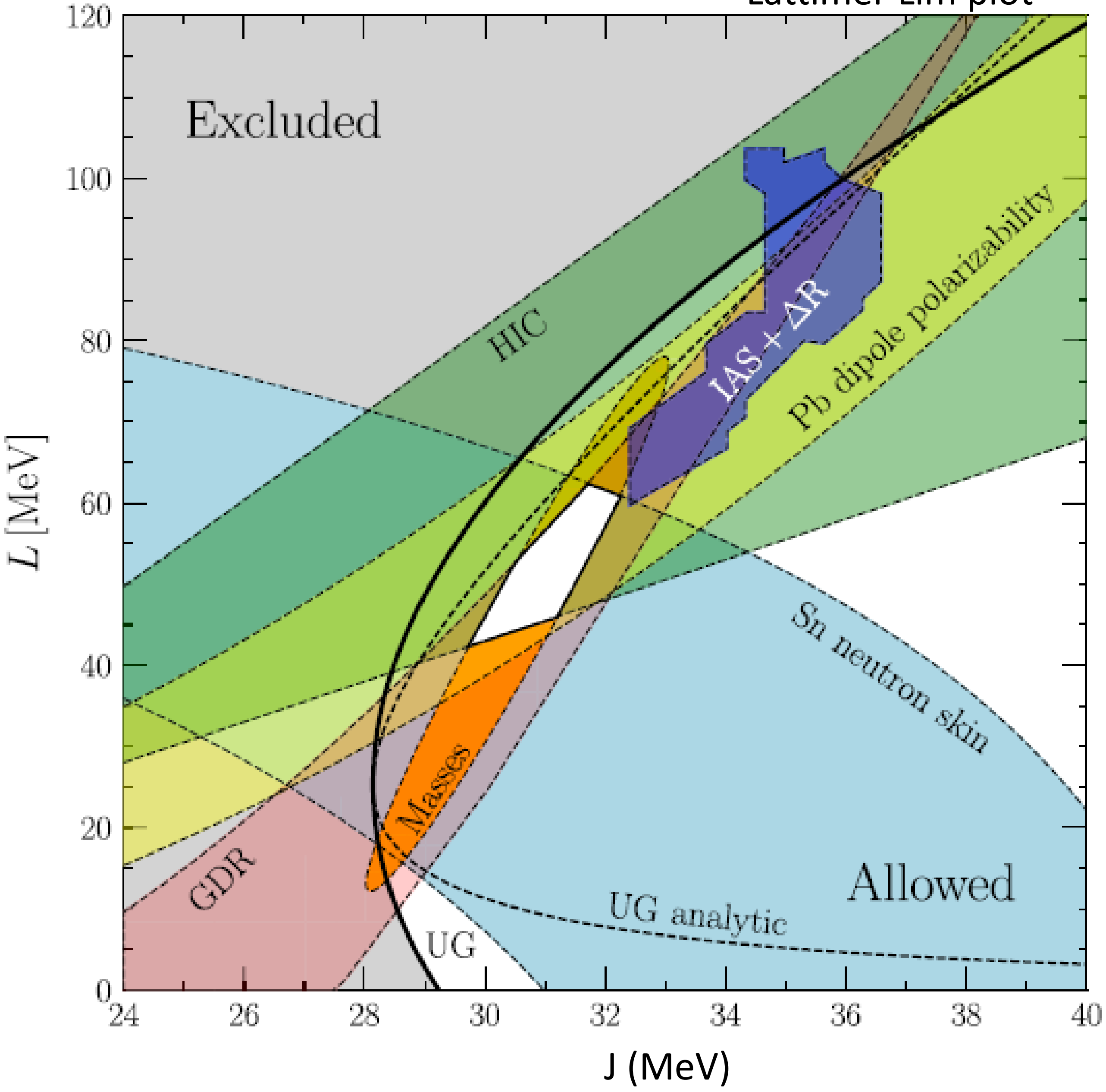
Pb dipole polarizability:
[Roca-Maza *et al.*, PRC **88**, 024316 (2013)]

Sn neutron skin:
[Chen et al., PRC **82**, 024321 (2010)]

GDR:
[Trippa et al., PRC **77**, 061304 (2008)]

Isospin defusion in HIC
[Tsang et al., PRL 102, 122701 (2009)]

Lattimer Lim plot



Behind all calculation are particular models for NN interactions and many-body techniques

- E_S **parameterization**

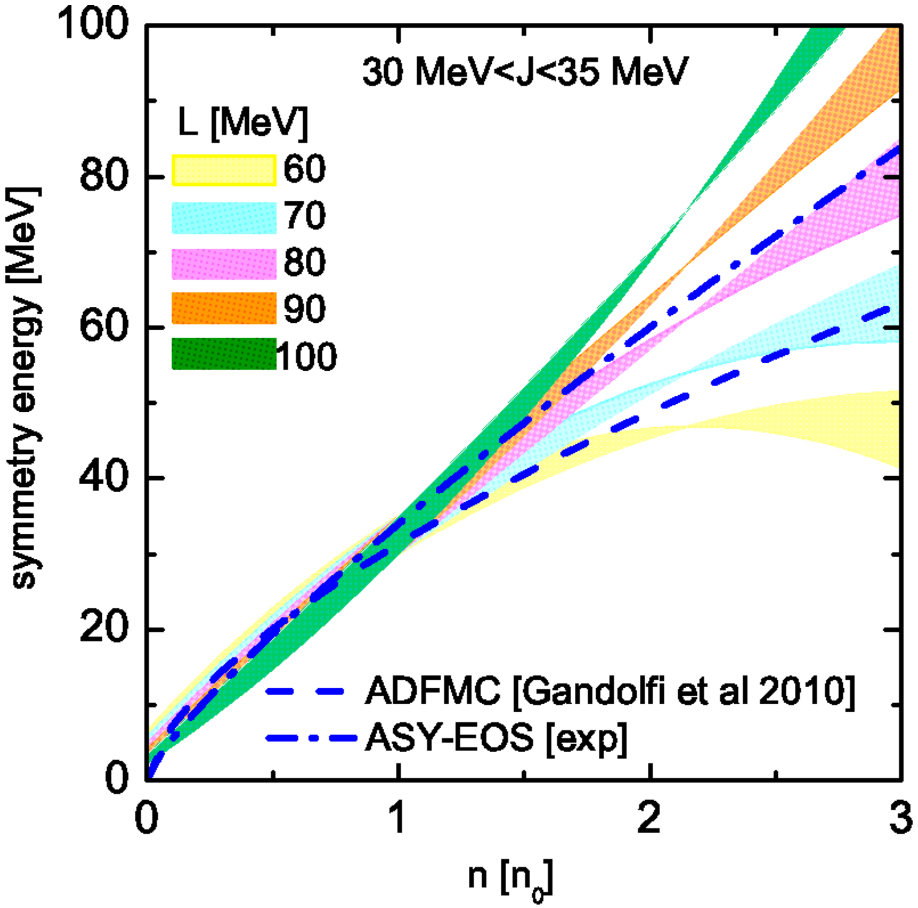
If we assume some model for the density dependence of the symmetry energy

$$\varepsilon_S(n) = C_k\,(n/n_0)^{2/3} + C_1\,n/n_0 + C_2\,(n/n_0)^{\gamma}$$

$$J = C_1 + C_2 + C_k \qquad 3L = C_1 + 2C_k + 3\gamma C_2 \qquad K = -2C_k + 9C_2(\gamma - 1)\gamma$$

Eliminate C_1 and C_2

$$L = \left(\frac{2}{3\gamma} - 1\right) C_k + 3\,J + \frac{K_{\rm sym}}{3\,\gamma}$$



$$\boxed{L = -19.5\ \mathrm{MeV} + 3\,J + \frac{K_{\rm sym}}{5.50}}$$

Analysis of 36 RMF models gives

[Dong, et al PRC85, 034308 (2012)]

Relativistic mean-field models

nucleon-nucleon interaction

vacuum: one boson-exchange for NN-potential
+ Lippmann-Schwinger equations

a model

$$\mathcal{L} = \sum_N \overline{\Psi_N} \left[i\,(\hat{\partial} + i\, g_{\omega\, N} \hat{\omega} + i\, g_{\rho\, N} \boldsymbol{\tau}\, \hat{\boldsymbol{\rho}}) \right] - (m - g_{\sigma\, N}\, \sigma) \Big] \Psi_N$$

$$+ \underbrace{\frac{1}{2}(\partial_\mu \sigma\, \partial^\mu \sigma - m_\sigma^2\, \sigma^2) - U(\sigma)}_{\text{scalar}} \underbrace{- \frac{1}{4}\omega_{\mu\nu}\,\omega^{\mu\nu} + \frac{1}{2} m_\omega\, \omega_\mu\, \omega^\nu}_{\text{vector}} \underbrace{- \frac{1}{4}\boldsymbol{\rho}_{\mu\nu}\, \boldsymbol{\rho}^{\mu\nu} + \frac{1}{2}\boldsymbol{\rho}_\mu\, \boldsymbol{\rho}^\mu}_{\text{iso-vector}}$$

$$\omega_{\mu\nu} = \partial_\mu \omega_\nu - \partial_\nu \omega_\mu \qquad \boldsymbol{\rho}_{\mu\nu} = \partial_\mu \boldsymbol{\rho}_\nu - \partial_\nu \boldsymbol{\rho}_\mu$$

Euler-Lagrange equations for $q \equiv q(\vec{x}, t) = \{\Psi, \sigma, \omega, \rho\}$ $\quad \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu q)} \right] - \frac{\partial \mathcal{L}}{\partial q} = 0$

$$[i\gamma_\mu(\partial^\mu + i g_{\omega N}\omega^\mu + i g_{\rho N}\boldsymbol{\tau}\boldsymbol{\rho}^\mu) - (m_N - g_{\sigma N}\sigma)]\,\Psi_N = 0$$

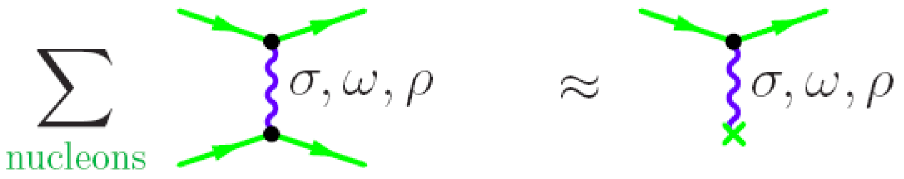
$$(\partial^2 + m_\sigma^2)\sigma + \frac{dU}{d\sigma} = g_{\sigma N} \sum_{N=p,n} \bar{\Psi}_N \Psi_N$$

$$(\partial^2 + m_\omega^2)\omega_\mu = g_{\omega N} \sum_{N=p,n} \bar{\Psi}_N \gamma_\mu \Psi_N$$

$$(\partial^2 + m_\rho^2)\boldsymbol{\rho}_\mu = g_{\rho N} \sum_{N=p,n} \bar{\Psi}_N \boldsymbol{\tau} \gamma_\mu \Psi_N$$

nucleon sources for meson fields

medium: mean-field approximation



$$\sigma(r,t) = \sigma$$
$$\omega_\mu(r,t) = \delta_{\mu,0}\,\omega_0$$
$$\rho^a_\mu(r,t) = \delta^{a,3}\,\delta_{\mu,0}\,\rho_0^{(3)}$$
constant fields

$$\omega_0 = \frac{g_{\omega N}}{m_\omega^2} < \Psi^\dagger \Psi > \equiv \frac{g_{\omega N}}{m_\omega^2}\, n_B = \frac{g_{\omega N}}{m_\omega^2}\,(n_p + n_n)$$

(vector) density

$$\rho_0^{(3)} = \frac{g_{\rho N}}{m_\rho^2} < \Psi^\dagger \tau^{(a)} \Psi > \equiv \frac{g_{\rho N}}{m_\rho^2}\, n_{\mathrm{iso}} = \frac{g_{\rho N}}{m_\rho^2}\,(n_p - n_n)$$

$$m_\sigma^2 \sigma_0 + \frac{dU}{d\sigma}\Big|_{\sigma_0} = g_{\sigma N} < \bar\Psi \Psi > \equiv g_{\sigma N} n_s = g_{\sigma N}(n_{s,p} + n_{s,n})$$

scalar density

$$\left[i\gamma_\mu \partial^\mu - g_{\omega N}\ \gamma^0 \omega_0 - g_{\rho N}\ \gamma^0 \rho_0^{(3)} - (m_N - g_{\sigma N}\sigma_0)\right]\Psi = 0$$

<u>nucleon spectrum in MF approximation</u>

$$E_N(p) = \sqrt{m_N^{*2} + p^2} + g_{\omega N}\,\omega_0 + g_{\rho N}\, I_N\, \rho_{03}$$

$$m_N^* = m_N - g_{\sigma N}\,\sigma$$

[Serot, Walecka]

pion dynamics falls out completely in this approx.

Energy-density functional

$$E[n_p, n_n; \sigma] = \frac{m_\sigma^2\,\sigma^2}{2} + U(\sigma) + C_\omega^2 \frac{(n_n + n_p)^2}{2\,m_N^2} + C_\rho^2 \frac{(n_n - n_p)^2}{8\,m_N^2}$$
$$+ \sum_N \int_0^{p_{\mathrm{F},N}} \frac{dp\,p^2}{\pi^2} \sqrt{(m_N - g_{\sigma N}\,\sigma)^2 + p^2}$$

evaluated for σ–field followed from the equation $\frac{\delta E[n_p, n_n, \sigma]}{\delta\sigma} = 0$

Paremeters $C_i^2 = \frac{g_{iN}^2\,m_N^2}{m_i^2}$ are adjusted to properties of nuclear matter at saturation

$$\begin{aligned} n_0 &\simeq 0.16 \pm 0.015\ \mathrm{fm}^{-3} \\ E_{\mathrm{bind}} &\simeq -15.6 \pm 0.6\ \mathrm{MeV} \\ m_N^*(\rho_0) &\simeq (0.75 \pm 0.1)\,m_N \\ K &\simeq 240 \pm 40\ \mathrm{MeV} \\ a_{\mathrm{sym}} &\simeq 32 \pm 4\ \mathrm{MeV} \end{aligned}$$

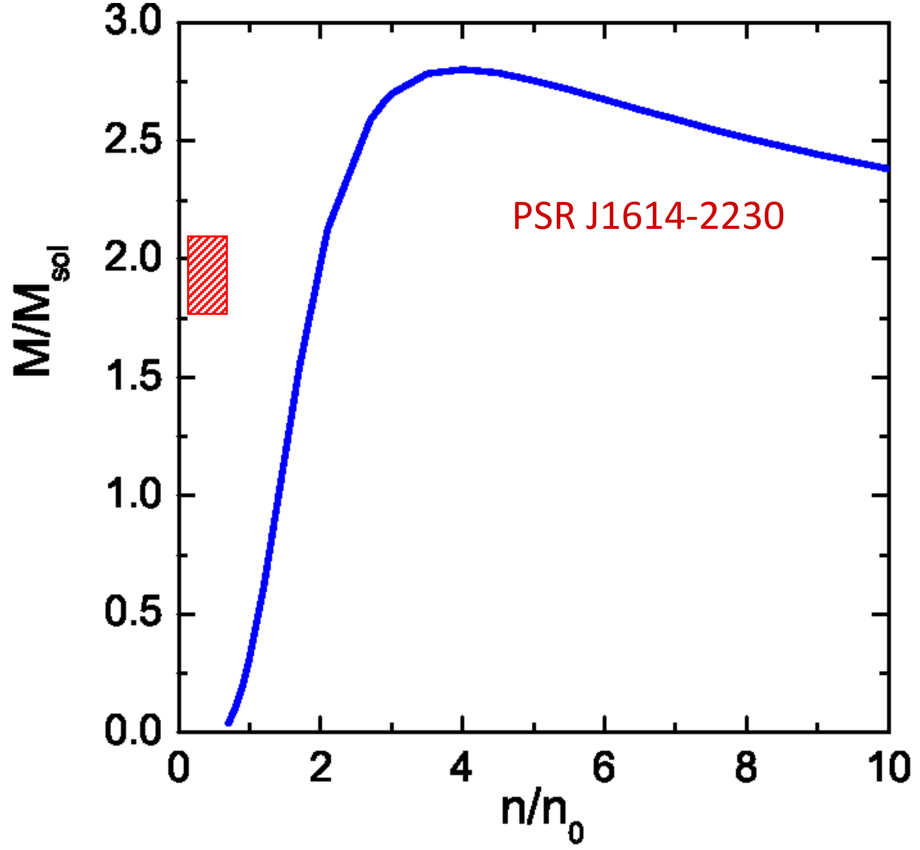
If we add gradient terms this energy density functional can be used for
a description of properties of atomic nuclei.

<u>(pure) Walecka model</u> $U(\sigma)=0$

$$n_0 = 0.16\mathrm{fm}^{-3}\,,\;\; E_{\mathrm{bind}} = -16\ \mathrm{MeV}$$

$$\downarrow$$

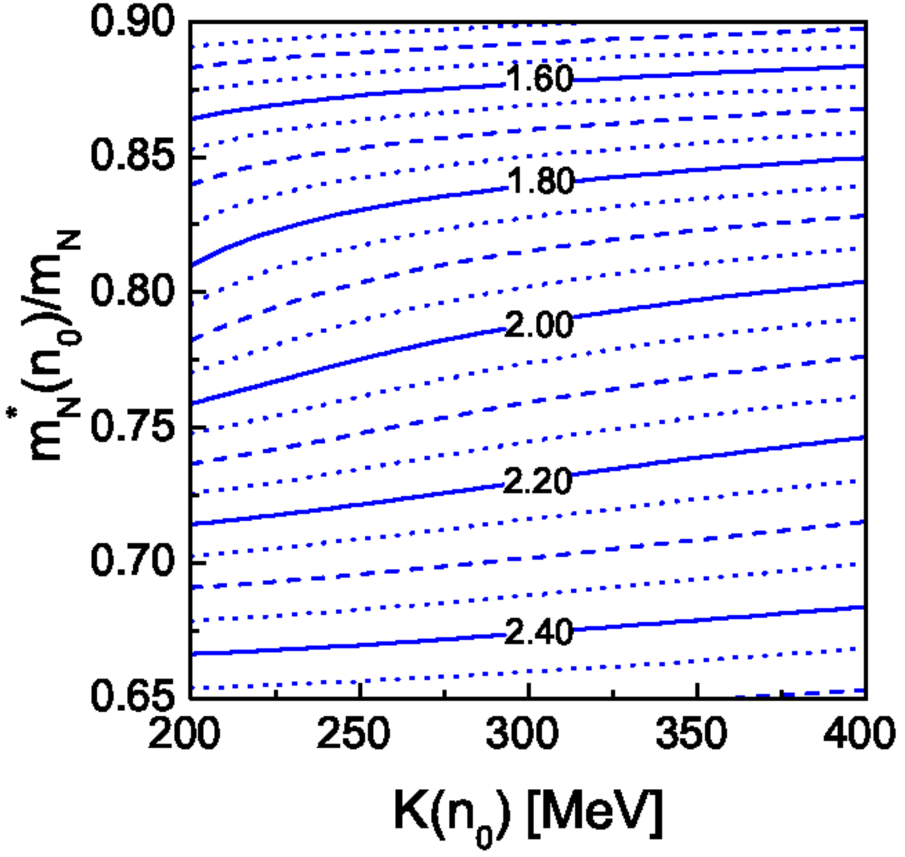
$$K = 553\ \mathrm{MeV}\,,\;\; m_N^*(n_0) = 0.54 m_N$$



Hardest EoS among RMF models

<u>modified Walecka</u> $U(\sigma)=a\sigma^3+b\sigma^4$

maximal mass of NS



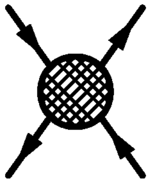
weak dependence on K !

strong dependence on m^*_N

Nuclear Fermi liquid. Approximations

- Quasiparticle approximation for nucleons, $T \ll \epsilon_{\rm FN}$.
  Only then diagrams with open nucleon lines make sence.
  Otherwise closed diagram technique

- Reduction of the more local interaction to the point-like interaction

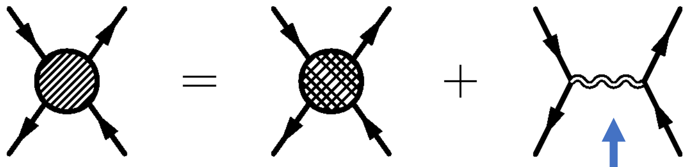


$$= C_0\left(f_{12} + g_{12}\boldsymbol{\sigma_1}\boldsymbol{\sigma_2}\right),$$

f_{12}, g_{12} are Landau-Migdal parameters.

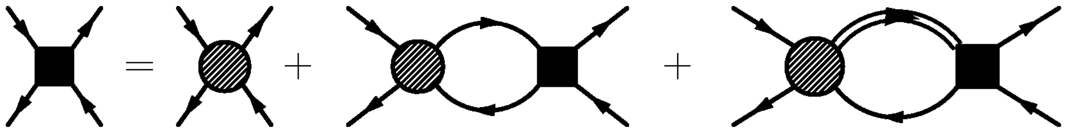
Constants!= rough approximation! But then eqs become algebraic!

- explicit pionic degrees of freedom

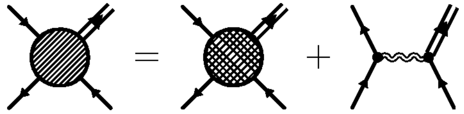


pion with residual (irreducible in NN^{-1} and ΔN^{-1}) s-wave π N interaction and $\pi\pi$ scattering``

- explicit Δ degrees of freedom

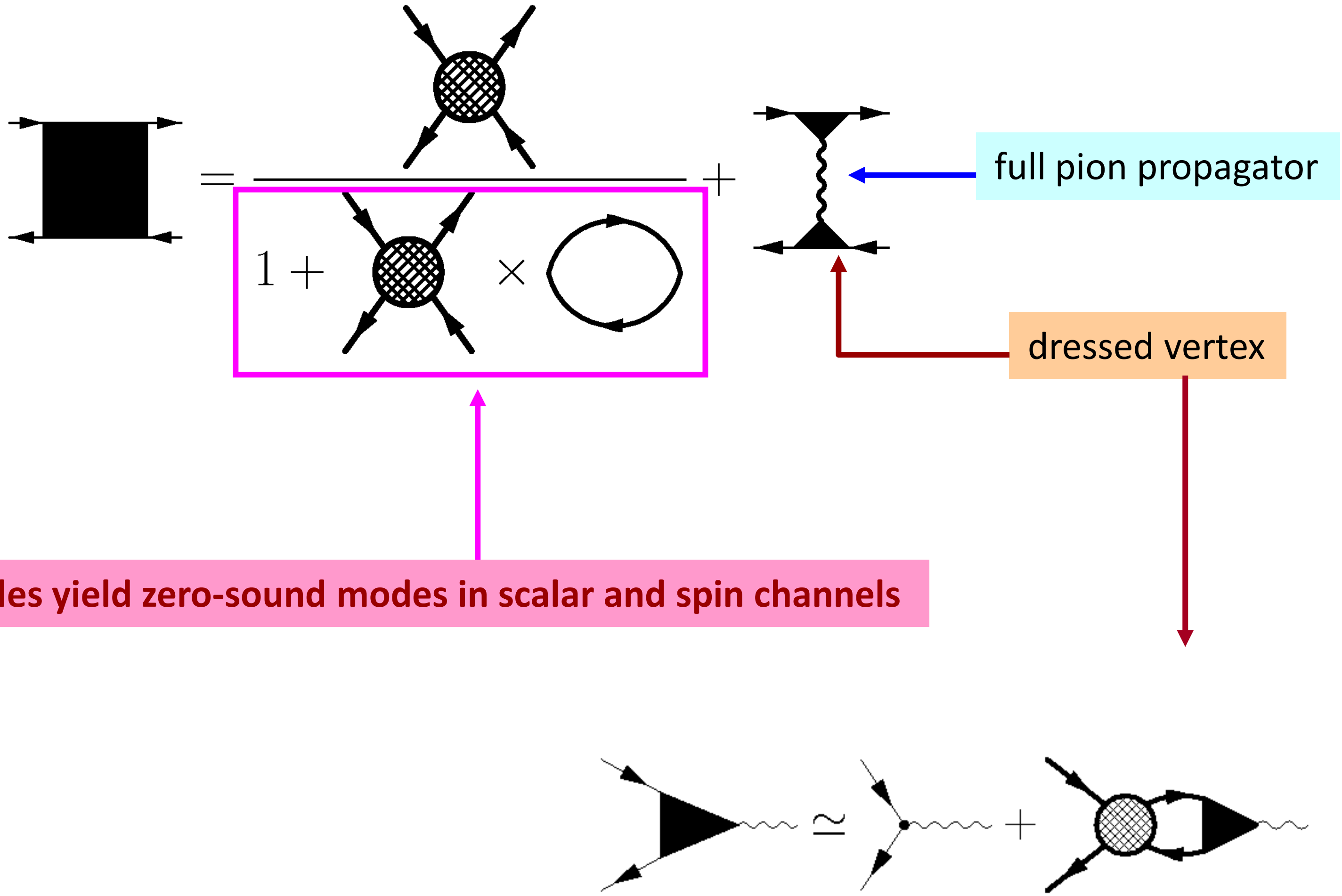


Part of the interaction involving Δ isobar is analogously constructed:

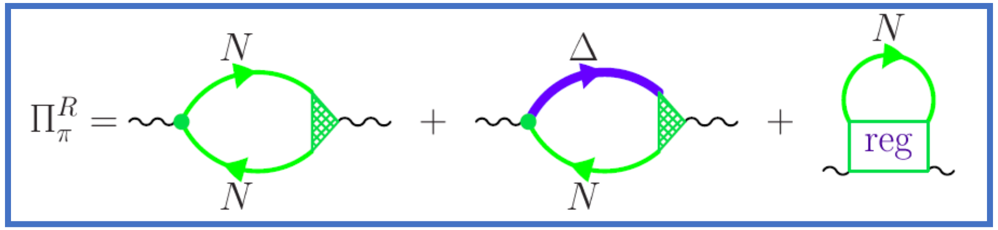


Resummed NN interaction

Graphically, the resummation is straightforward and yields:

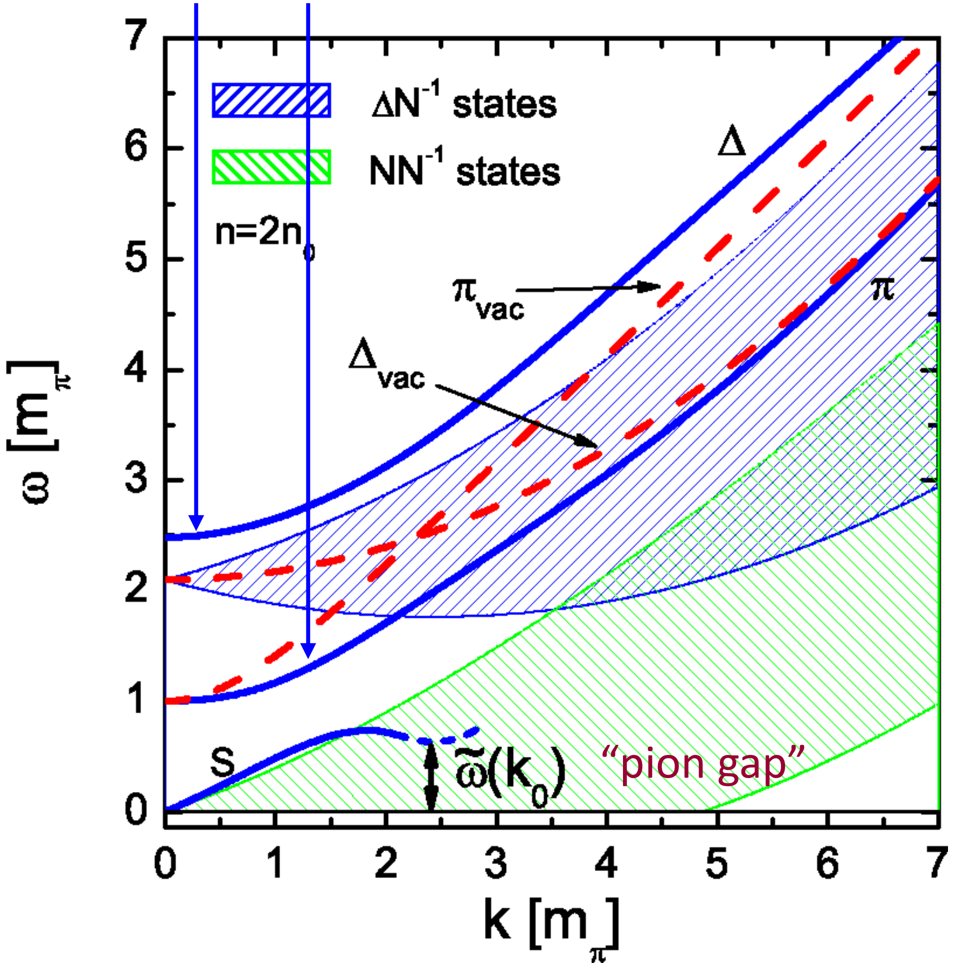


Pion modes in nuclear medium



$$A_\pi(\omega,\boldsymbol{k}) \approx \sum_{i=\pi,\Delta} \frac{2\,\pi\,\delta(\omega-\omega_i(\boldsymbol{k}))}{\left(2\omega - \frac{\partial \Pi^R}{\partial\omega}\right)\Big|_{\omega=\omega_i(\boldsymbol{k})}} + \frac{2\,\beta\,k\,\omega}{\widetilde{\omega}^4(k)+\beta^2\,k^2\,\omega^2}\,\theta(\omega < v_F\,k)$$

quasi-particle modes



pion propagator has a complex pole

$$D^{-1}(\omega,k) \simeq D^{-1}(0,k) + i\beta\,\omega$$

$$\beta = m_N^{*2}\,k\,f_{\pi NN}^2/\pi$$

$$\widetilde{\omega}^2(k) = -D^{-1}(0,k)$$

$$\omega \propto -i\widetilde{\omega}^2(k_{\rm min})/\beta$$

when $\widetilde{\omega}^2(k_{\rm min} \simeq p_{\rm F}) < 0$

→ instability → pion condensation

[A.B.Migdal et al, Phys. Rept. 192 (1990) 179]