Crust of compact stars
Lecture 1.

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Ioffe Institute

Many-particle systems: from condensed matter to quarks and stars

January 29 – February 03, 2024

BLTR, JINR, Dubna, Russia

Crust of compact stars

Selection of topics and plan

"Everything should be made as simple as possible, but not simpler"

Attributed to Albert Einstein

According to Robinson [*Nature* **557**, 30 (2018)], it can be a compressed version of lines from a 1933 lecture by Einstein:
"*It can scarcely be denied that the supreme goal of all theory is to make the irreducible basic elements as simple and as few as possible without having to surrender the adequate representation of a single datum of experience*."

My preference in these lectures: *models, which can be solved analytically*

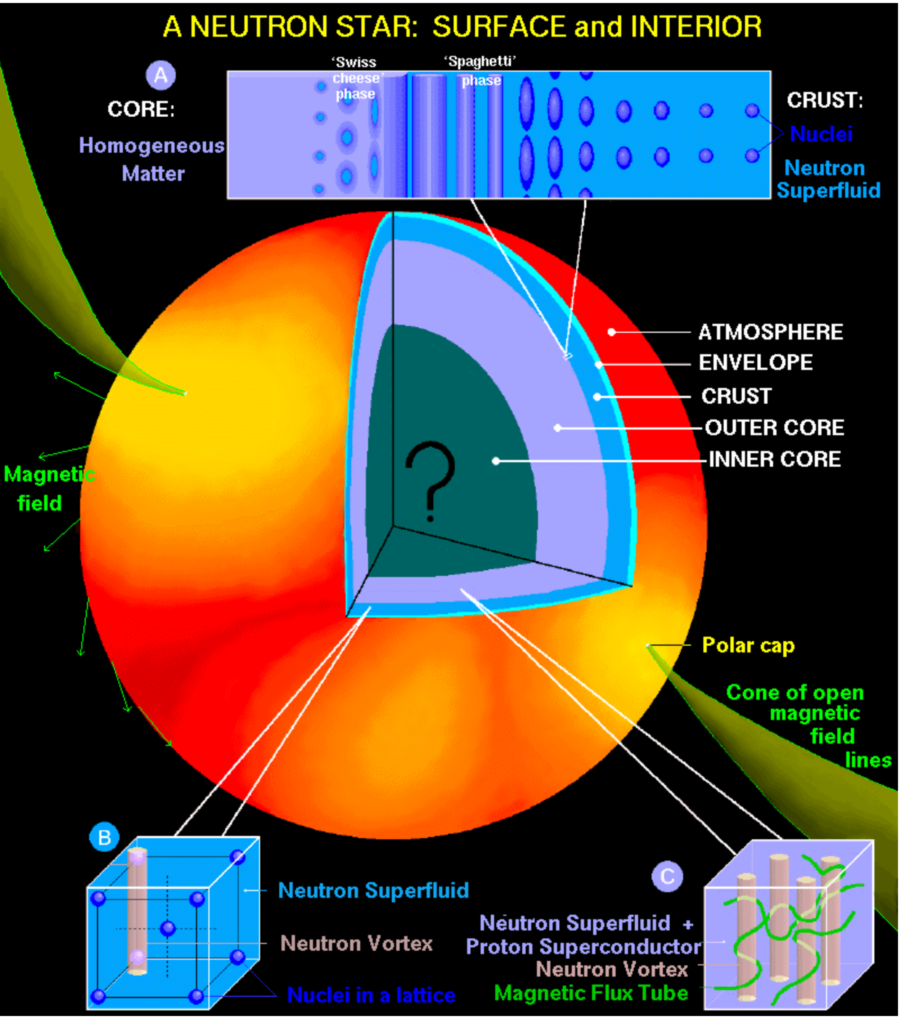
Lecture 1: Introduction and basic models of outer crust at T=0

Lecture 2: Outer crust: thermodynamics and elasticity

Lecture 3: Inner crust

Lecture 4: Aren't crustal models simpler, than it is possible?
+ M(R) not dealing with crust

Neutron star structure



$$\rho \sim 10^{15}\ \mathrm{g/cm^3}$$

$$T \lesssim 10^9\ \mathrm{K}$$

$$B \sim 10^{12}\ \mathrm{G}$$

$$g \sim 10^{14}\ \mathrm{cm/s^2}$$

$$R \sim 2R_{\mathrm{g}} = 4GM/c^2$$

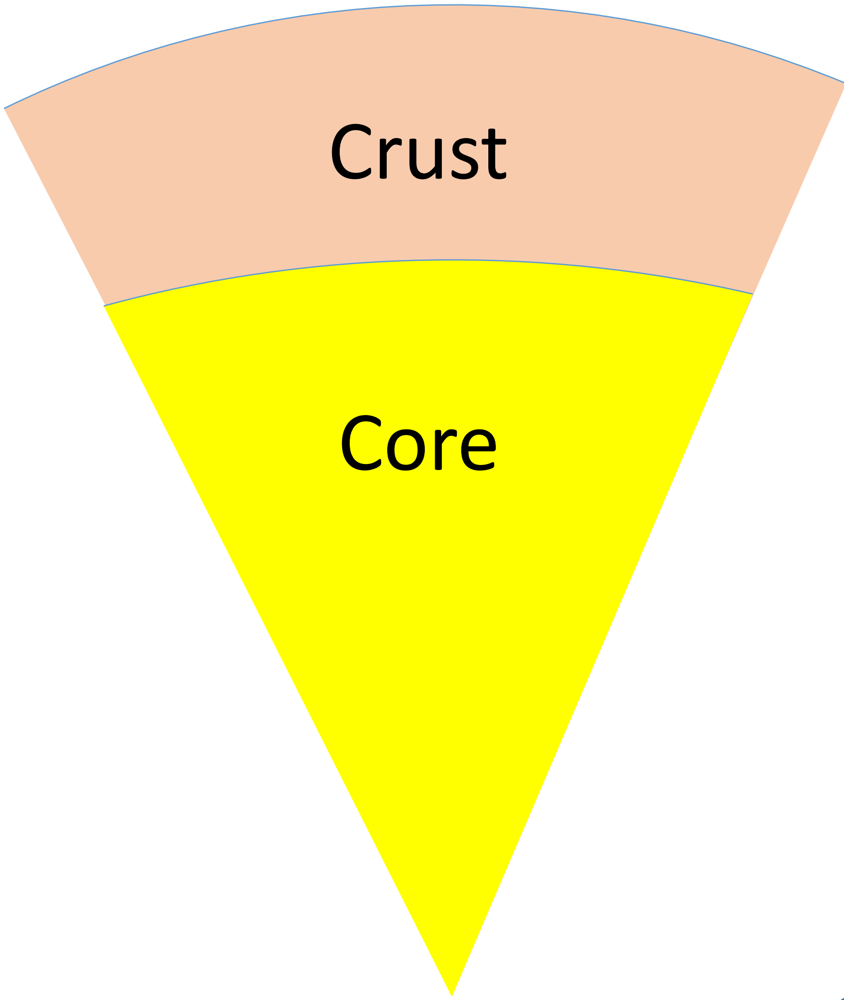
$$T_{\mathrm{cp}} \sim 10^9\ \mathrm{K}$$

$$T_{\mathrm{cn}} \sim 10^8\ \mathrm{K}$$

$$R \sim 10 - 14\ \mathrm{km},\ M \sim 1.4 M_{\odot}$$

- Neutron star are extreme objects
- They are observed
- Observations are affected by crust

Neutron star structure



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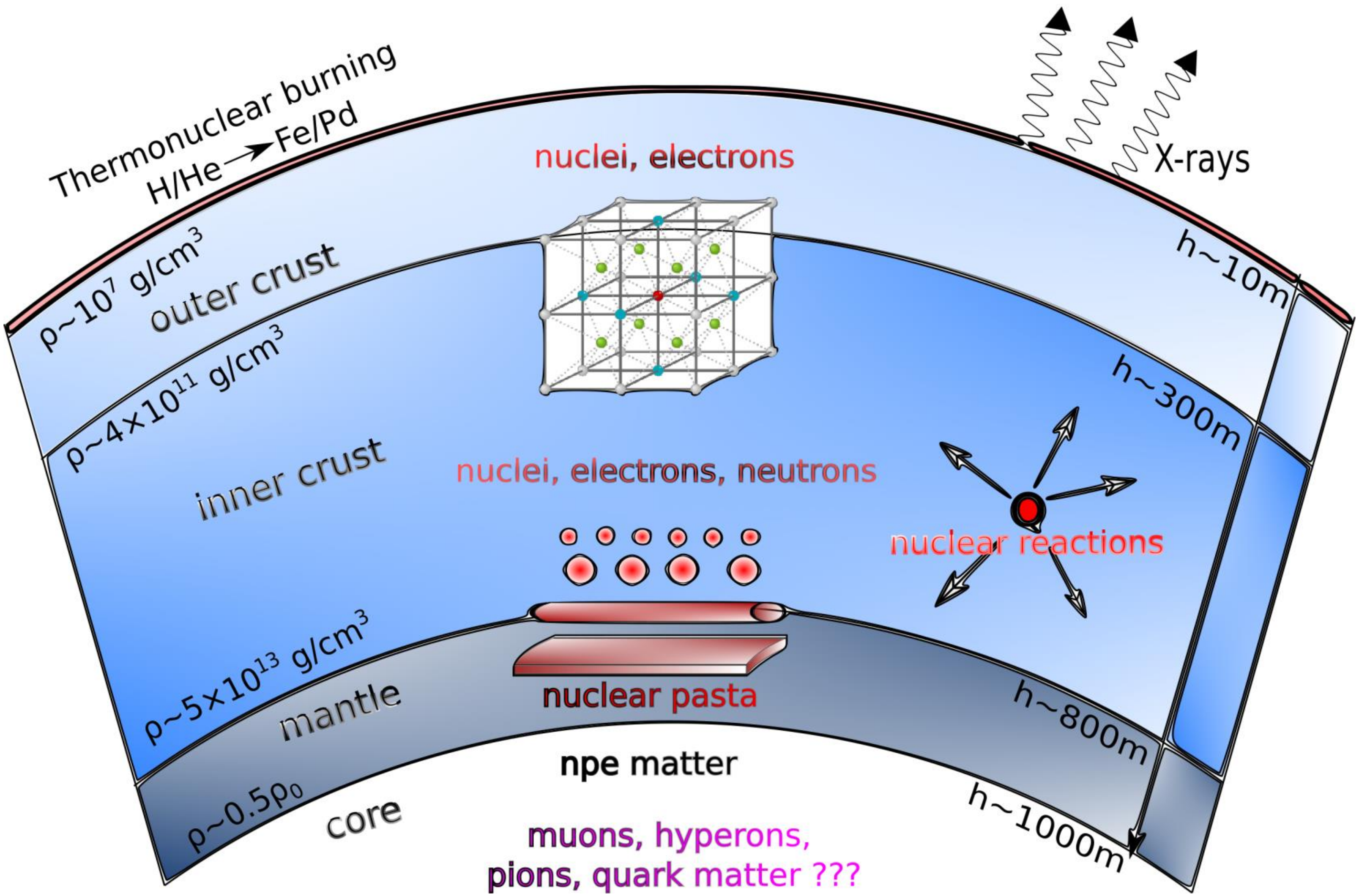
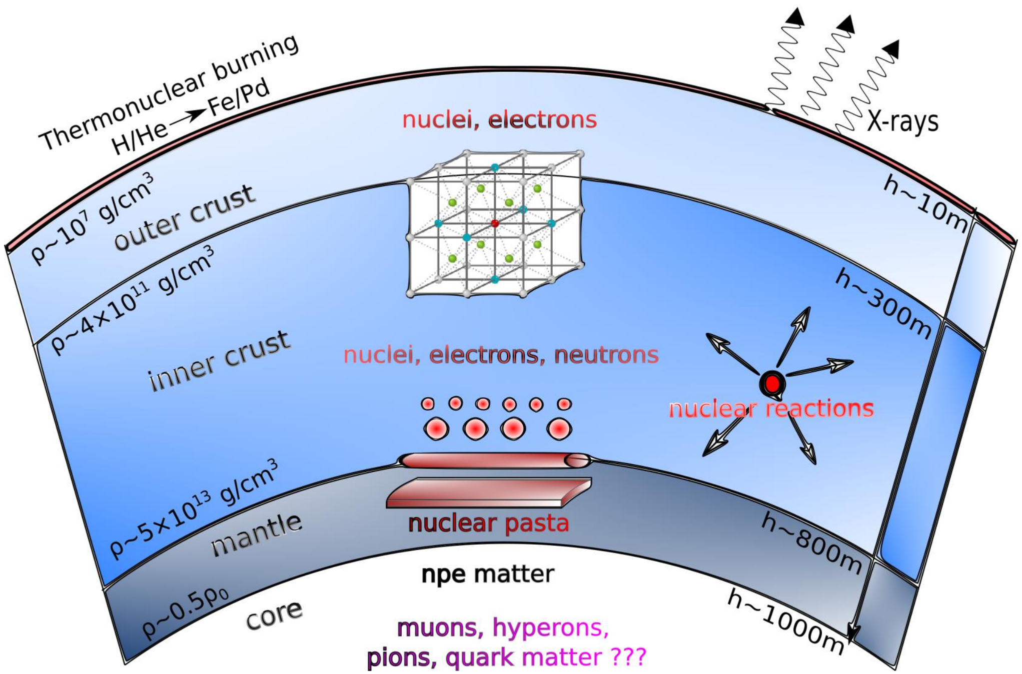
$R \sim 10 - 14\ \mathrm{km},\ M \sim 1.4 M_{\odot}$

Typically: Crust encodes information from the core. We need to know the crust to decode information from core.

Neutron star crust



Why we want to know about the crust?



N. N. Shchechilin ©

- **Composition**
  - Equilibrium
  - Nonequilibrium
- **Equation of state**
  - T=0
  - Thermal properties
  - State of matter (solid/liquid)
- **Dynamical properties One/two liquid hydro (magneto) dynamics**
- **Transport properties (kinetic coefficients)**
- **Elasticity, strength**
- ...

D.G. Yakovlev, HEA2017(?)
Crust as Cinderella of NS

Why???

These properties affect observations, and thus they are required for adequate interpretation of observations

Typically: the main mystery of NSs is the core. The crustal properties should be known accurately to avoid biases for the core properties

Why we want to know about the crust?

We can not calculate anything without
COMPOSITION

We can not build a crust model without
Equation of state
(crust is not uniform: its density increases with depth increase)

We can not analyze glitches without
Dynamical properties

We can not analyze any thermal and magnetic evolution
without transport properties

We can not analyze starquakes without
Elasticity

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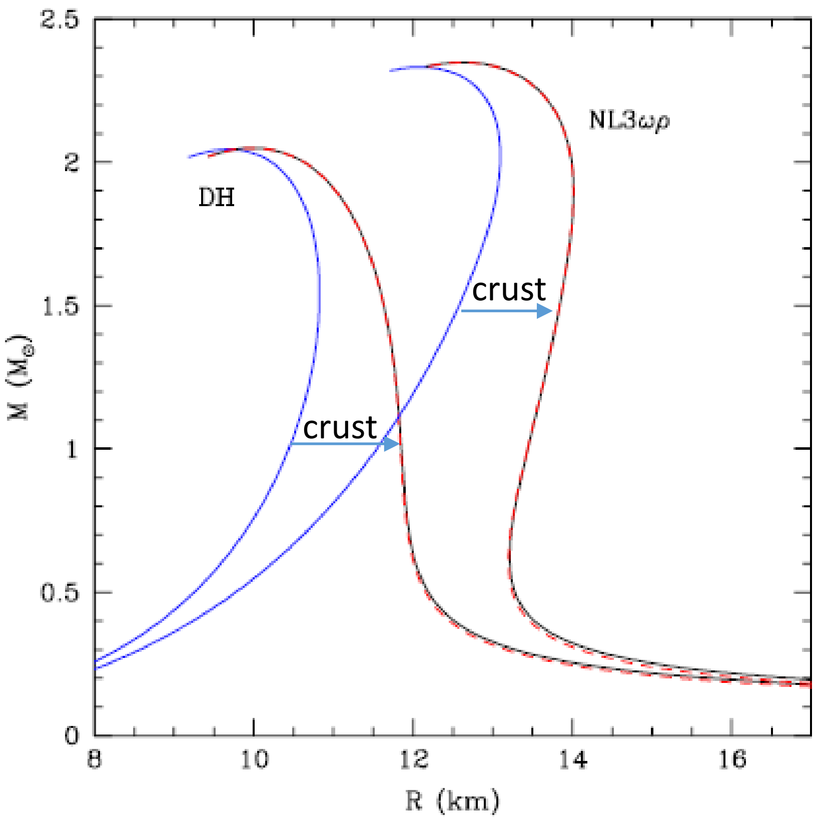
The above mentioned processes reveal themselves in observations

Why we want to know about the crust?

We can not build a crust model without
Equation of state
(crust does not have uniform density)



From Zdunik et al., A&A 599 (2017), A119

(see the last lecture for bypass)

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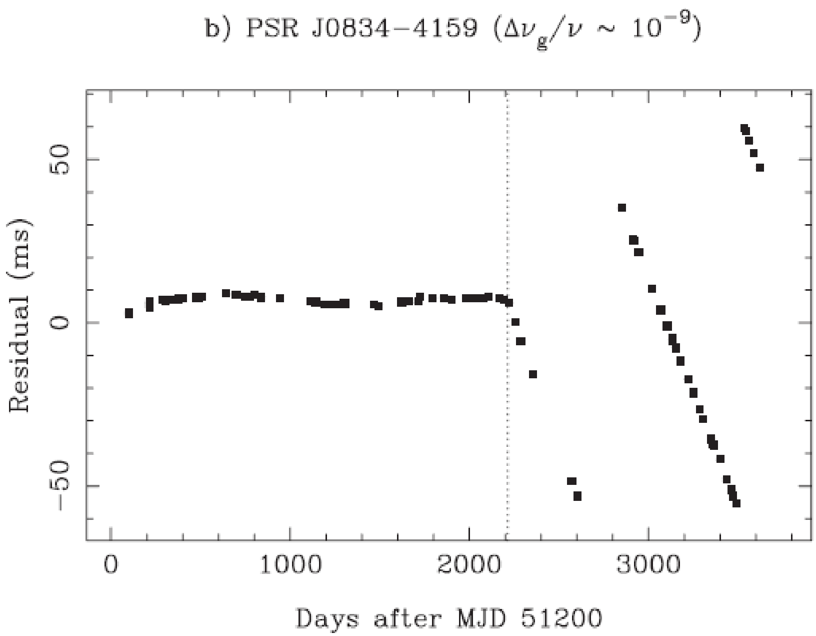
Why we want to know about the crust?

We can not analyze glitches without Dynamical properties



From Yu et al., MNRAS 429 (2013), 688

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Pulsar are good clocks, but glitches happen sometimes…

$$I \sim M\,R^2 \approx 10^{45}\ \mathrm{g\,cm^2}$$

Observed spin down

Conserved (until glitch): pinned vortexes

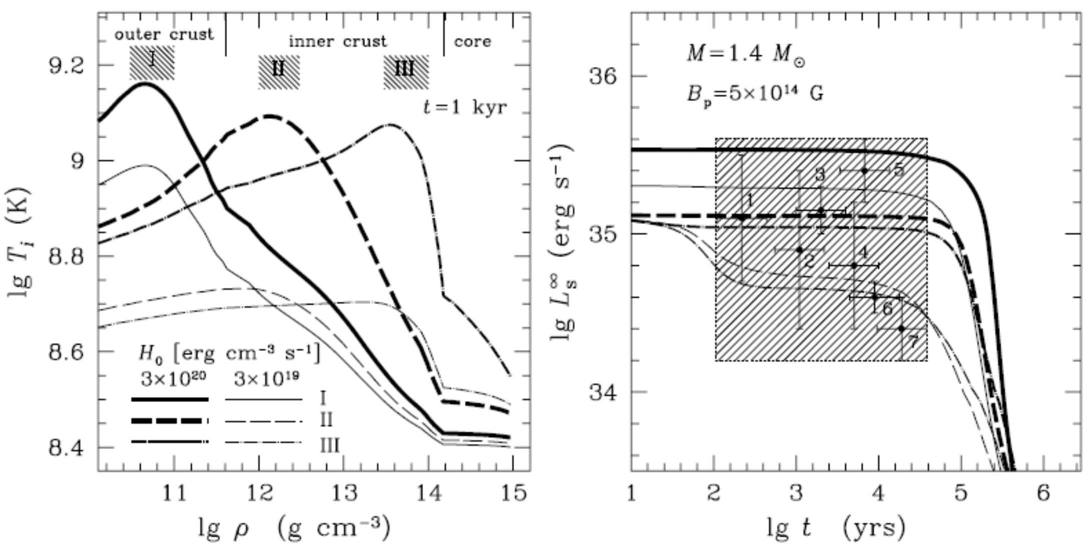
$$M = M_{\mathrm{nofm}} + M_{\mathrm{sf}}$$

Most of the glitch models are based on the crust superfluidity (+elastisity)

Why we want to know about the crust?

We can not analyze thermal and magnetic evolution without transport properties

Magnetars (NSs with strong magnetic field, ~10^{14} G) are hotter, than other NSs



From Kaminker et al., MNRAS 395 (2009), 2257

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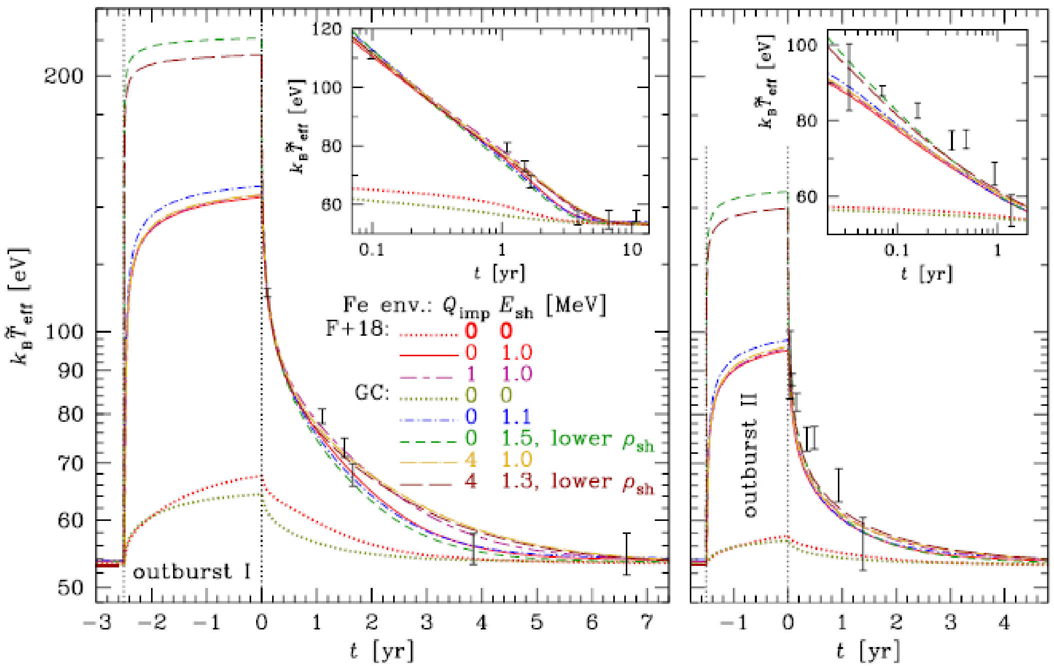
Magnetars are heated by dissipation of the magnetic field energy (likely in the crust, but ...)

(thermal conductivity, specific heat, magnetic field dissipation, neutrino emission are required)

Why we want to know about the crust?

We can not analyze thermal and magnetic evolution without transport properties

Realtime cooling for some NSs: cooling after accretion



From Potekhin et al., MNRAS 522(2023), 4830

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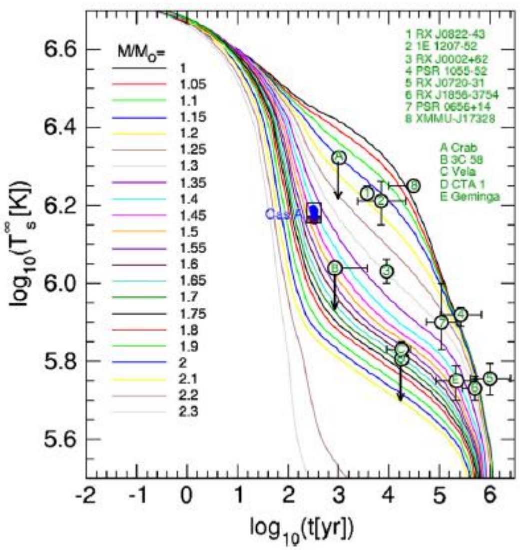
(thermal conductivity, specific heat, heating power (reaction), neutrino emission are required)

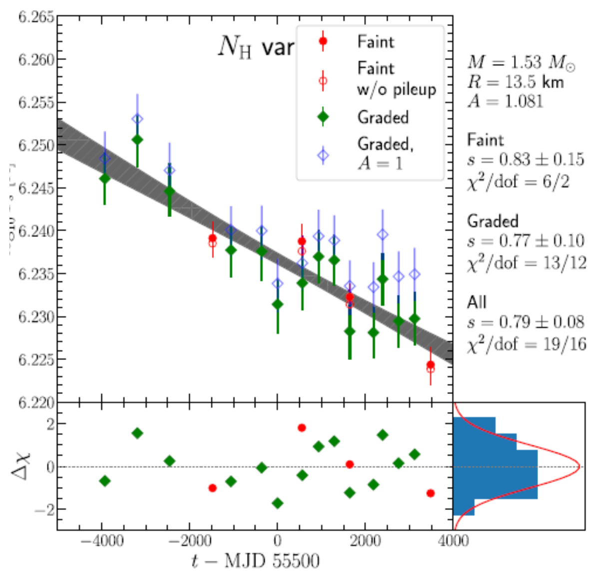
Why we want to know about the crust?

We can not analyze thermal and magnetic evolution without transport properties

Cooling of isolated NS (realtime for CasA!!!)



From Grigorian et al. (2018)



From Shternin et al. (2023)

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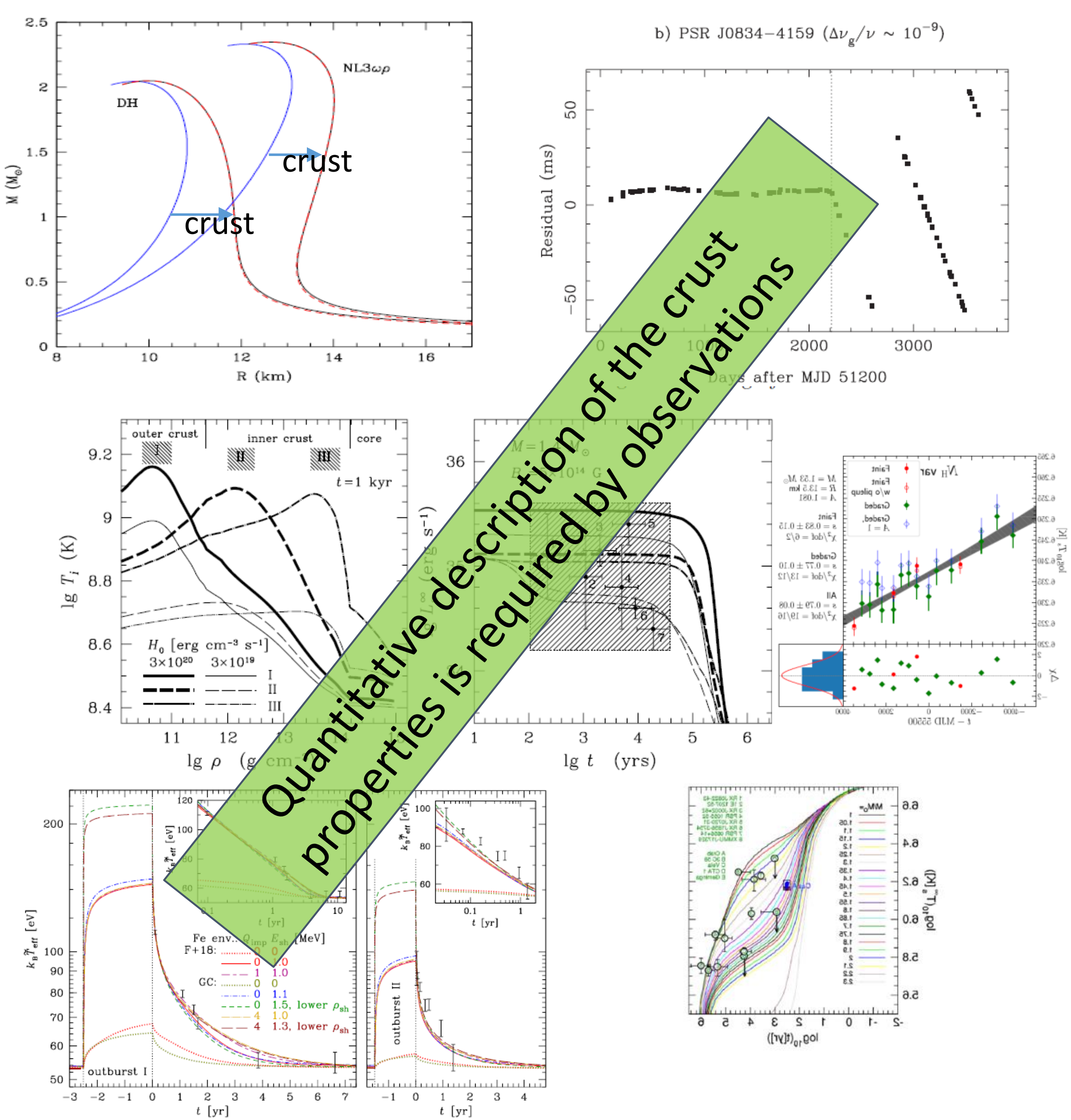
(thermal conductivity, specific heat, neutrino emission are required + CORE)

Why we want to know about the crust?



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What we need to know to study the crust?

- Nuclear physics
- Statistical physics (many-particle system)
- Solid state physics
  - Transport theory
  - Theory of elasticity
- Superfluidity
- Magnetohydrodynamics
- …

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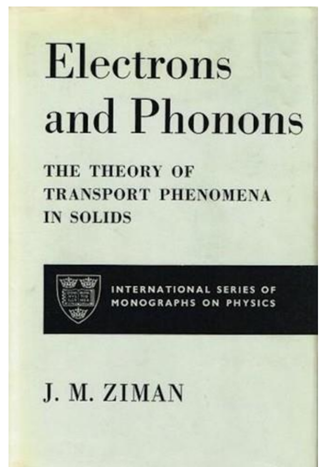
Seems to be extremely complicated…..

"Everything should be made as simple as possible, but no simpler"

Attributed to Albert Einstein

What we need to know to study the crust?

- Nuclear physics
- Statistical physics (many-particle system)
- Solid state physics
  - Transport theory
  - Theory of elasticity
- Superfluidity
- Magnetohydrodynamics
- …



Seems to be extremely complicated:
we need simplified models

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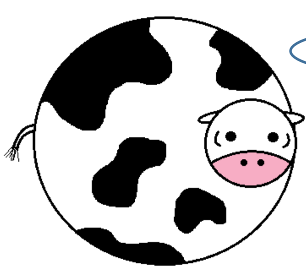

Some problems can be solved analytically!

Outer crust

➢ Nuclear physics

Atomic mass tables to extract nuclei masses

Mostly experimentally (AME2020)!

Additional experimentally efforts:

- ❑ Joint Institute for Nuclear Research (JINR, Dubna, Russia)
- ❑ Facility for Rare Isotope Beams (FRIB, USA)
- ❑ Radioactive Isotope Beam Factory (RIBF, Japan)
- ❑ ….

Note!!!

Atomic mass table reports atomic masses, not nuclear masses

$$M_{\rm atom} = M_{\rm nuc} + Zm_e + E_{\rm e,\ bind}$$

A bit different for different atomic mass tables

➢ **Composition**

<u>Common introduction:</u>

Outer crust is composed of degenerate electrons and fully ionized nuclei

(similarly for white dwarf cores)

Not enough for quantitative description!!!

That are the nuclei types???

- <u>Equilibrium</u>

Optimal energy density

- <u>Nonequilibrium</u>

Formation history + nuclei reaction

Outer crust

- Nuclear physics

  Atomic mass tables to extract nuclei masses

- Statistical physics to calculate thermodynamically quantities at given
  - Composition $\{Z_i, A_i, x_i\}$
  - Baryon number density n_b
  - Temperature T

- Thermodynamics to determine equilibrium composition and state

- **Composition**

  <u>Common introduction:</u>
  Outer crust is composed of degenerate electrons and fully ionized nuclei
  (similarly for white dwarf cores)

  - <u>Equilibrium</u>

  Optimal energy density

- **Equation of state**
  - T=0
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Thermodynamics and statistical physics

Calculate partition function => determine thermodynamics

Thermodynamic function	Variables	Partition functions	How to calculate
Entropy S	E, V, N_j	$\Gamma = \sum_s 1$	$S = \ln \Gamma$
Helmholtz free energy $F = E - TS$	T, V, N_j	$Z = \sum_s \mathrm{e}^{-E_s/T}$	$F = -T \ln Z$
Gibbs free energy $\Phi = F + PV = \sum \mu_j N_j$	T, P, N_j	$Y = \sum_{s,V} \mathrm{e}^{-\frac{E_s + PV}{T}}$	$\Phi = -T \ln Y$
Grand potential $\Omega = F - \Phi = -PV$	T, V, μ_j	$\tilde{Z} = \sum_{s,V} \mathrm{e}^{-\frac{E_s - \mu_j N_j}{T}}$	$\Omega = -T \ln \tilde{Z}$

Thermodynamics of the outer crust

(school topic: theoretical description of interacting fermion systems)

$$F = F_{\mathrm{e}} + F_{\mathrm{i}} + F_{\mathrm{ie}} + \sum_i M_i N_i$$

Nuclei mass
(nuclear physics)

Model system

Ideal degenerate gas of electrons

Electrons: relativistic noninteracting fermions

On the electron rest mass

It is natural to include it into the electron free energy because electrons are relativistic

Model system

Nuclei on the **uniform** neutralizing background of electrons

Nuclei: point like massive particles

Electrons: (just) background

Strong Coulomb interaction, but statistics (fermions or bosons) is not important for astrophysical applications

Physical motivation (D.A. Baiko): If nuclei wave functions overlap, it leads to fusion

Quantum effects: important, but associated with collective effects

Corrections

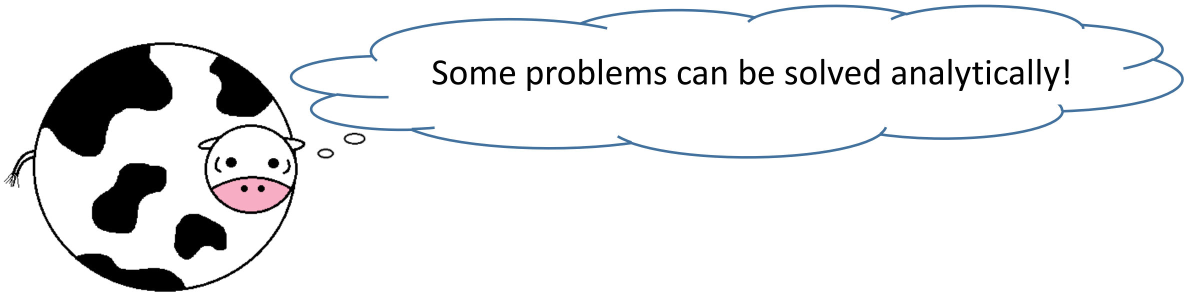
- No uniform electron background (screening)

Liquid:
Potekhin & Chabrier (2000)
Crystall: Baiko (2002)

The corrections, as predicted today, affects melting temperature. BUT it can be byproduct of different approaches, applied to calculate these corrections….

- ….

Ideal degenerate gas of electrons



(remind lection by E.E. Kolomeitsev)

Ideal degenerate gas of electrons

The grand potential is very useful to quantify thermodynamics

Variables: T, V, μ_j

State: occupation numbers $\boldsymbol{j} = (n_1,\ n_2, n_3,\ ,\ldots,\ n_n)$

Partition function:

$$\tilde{Z} = \sum_{\boldsymbol{j}} \mathrm{e}^{-\dfrac{E_{\boldsymbol{j}} - \mu N_{\boldsymbol{j}}}{T}} = \prod_\alpha \left(1 + \mathrm{e}^{-\dfrac{\epsilon_\alpha - \mu n_\alpha}{T}}\right)$$

$$E_{\boldsymbol{j}} = \sum_\alpha n_\alpha \epsilon_\alpha, \quad N_{\boldsymbol{j}} = \sum_\alpha n_\alpha$$

Grand potential

$$\Omega = -T \ln \tilde{Z} = -T \sum_\alpha \ln \left(1 + \mathrm{e}^{-\dfrac{\epsilon_\alpha - \mu}{T}}\right)$$

$$\sum_\alpha \to \frac{2V}{(2\pi\hbar)^3} \int \mathrm{d}^3 p \qquad \epsilon_\alpha = \sqrt{m_e^2 c^4 + p_\alpha^2 c^2}$$

$$n = \frac{2}{(2\pi\hbar)^3} \int \frac{\mathrm{d}^3 p}{1 + \mathrm{e}^{(\epsilon_\alpha - \mu)/T}}, \qquad E = \frac{2V}{(2\pi\hbar)^3} \int \frac{\epsilon_\alpha \mathrm{d}^3 p}{1 + \mathrm{e}^{(\epsilon_\alpha - \mu)/T}} = V\,\epsilon(n, T)$$

$$\mu(n, T)$$

Ideal degenerate gas of electrons

Grand potential is good, but energy is 'common' for (general) intuition

$$n = \frac{2}{(2\pi\hbar)^3}\int \frac{\mathrm{d}^3p}{1+\mathrm{e}^{(\epsilon_\alpha-\mu)/T}}, \qquad E = \frac{2V}{(2\pi\hbar)^3}\int \frac{\epsilon_\alpha \mathrm{d}^3p}{1+\mathrm{e}^{(\epsilon_\alpha-\mu)/T}} = V\epsilon(n,T)$$

Fermi momentum $$p_\mathrm{F} = \hbar(3\pi^2 n)^{1/3}$$

Relativistic parameter $$x = p_\mathrm{F}/mc \approx 1.009\left(\frac{\rho}{10^6\mathrm{g/cm}^3}\frac{\tilde{Z}}{\tilde{A}}\right)^{1/3}$$

Electrons are ultra relativistic in the crust

$$T = 0: \qquad \mu = \sqrt{m_\mathrm{e}^2c^4 + p_\mathrm{F}^2c^2}$$

$$T_\mathrm{F} = \mu - mc^2 \approx 5.9\times 10^9\ \mathrm{K}\left(\sqrt{1+x^2}-1\right)$$

$$E = P_0V\left[x\,(1+x^2)^{1/2}\,(2x+1) - \ln(x+(1+x^2)^{1/2})\right],$$

$$P_0 = \frac{m^4c^5}{8\pi^2\hbar^3} \approx 1.8\times 10^{23}\,\frac{\mathrm{dyn}}{\mathrm{cm}^2}$$

$$x \ll 1: \quad P \propto n^{5/3}$$
$$x \gg 1: \quad P \propto n^{4/3}$$

Ideal degenerate gas of electrons

Finite temperature corrections

$$n = \frac{2}{(2\pi\hbar)^3}\int \frac{\mathrm{d}^3p}{1+\mathrm{e}^{(\epsilon_\alpha-\mu)/T}}, \qquad E = \frac{2V}{(2\pi\hbar)^3}\int \frac{\epsilon_\alpha \mathrm{d}^3p}{1+\mathrm{e}^{(\epsilon_\alpha-\mu)/T}} = V\epsilon(n,T)$$

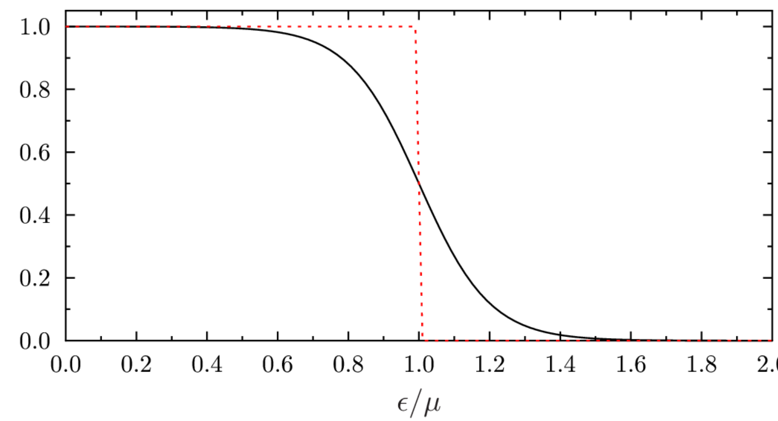
Fermi momentum $p_\mathrm{F} = \hbar(3\pi^2 n)^{1/3}$

Relativistic parameter $x = p_\mathrm{F}/mc \approx 1.009\left(\frac{\rho}{10^6\mathrm{g/cm}^3}\frac{\tilde{Z}}{\tilde{A}}\right)$

Electrons are ultra relativistic in the crust

$0 < T \ll T_F$: Sommerfeld expansion

$$\int_{mc^2}^{\infty} \frac{g(\epsilon)}{\exp\left(\frac{\epsilon-\mu}{T}\right)+1}\mathrm{d}\epsilon = \int_{mc^2}^{\mu} g(\epsilon)\mathrm{d}\epsilon + \frac{\pi^2}{6}T^2 g'(\mu) + \dots$$



$$\mu = \mu_0 + \delta\mu(T), \quad \mu_0 = \sqrt{m_\mathrm{e}^2 c^4 + p_\mathrm{F}^2 c^2}$$

Fixed density

$$\delta\mu = -\frac{\pi^2}{6}T^2\frac{p_F^2c^2+\mu_0^2}{c^2\mu_0 p_F^2} < 0$$

Ideal degenerate gas of electrons

Finite temperature corrections

$$n = \frac{2}{(2\pi\hbar)^3}\int \frac{\mathrm{d}^3 p}{1+\mathrm{e}^{(\epsilon_\alpha-\mu)/T}}, \qquad E = \frac{2V}{(2\pi\hbar)^3}\int \frac{\epsilon_\alpha \mathrm{d}^3 p}{1+\mathrm{e}^{(\epsilon_\alpha-\mu)/T}} = V\epsilon(n,T)$$

Fermi momentum $$p_{\mathrm{F}} = \hbar(3\pi^2 n)^{1/3}$$

$0 < T \ll T_F$: Sommerfeld expansion

$$\int_{mc^2}^{\infty} \frac{g(\epsilon)}{\exp\left(\frac{\epsilon-\mu}{T}\right)+1}\mathrm{d}\epsilon = $$

$$= \int_{mc^2}^{\mu} g(\epsilon)\mathrm{d}\epsilon + \frac{\pi^2}{6}T^2 g'(\mu) + \ldots$$

$$\delta\mu = -\frac{\pi^2}{6}T^2\frac{p_F^2c^2+\mu_0^2}{c^2\mu_0 p_F^2} < 0$$

$$\delta P = \frac{p_F}{18\mu_0}\left(2\mu_0^2 - p_F^2\right)T^2 > 0$$

$$S = \frac{1}{3}VTp_F\mu_0 \propto T, \quad C_V = T\left.\frac{\partial S}{\partial T}\right|_V = S$$

$$\frac{S}{N} = \frac{C_V}{N} \propto \frac{T}{T_F}$$

Practical formulae for partially degenerate electrons: Chabrier&Potekhin (1998)

Ideal degenerate gas of electrons

Magnetic field

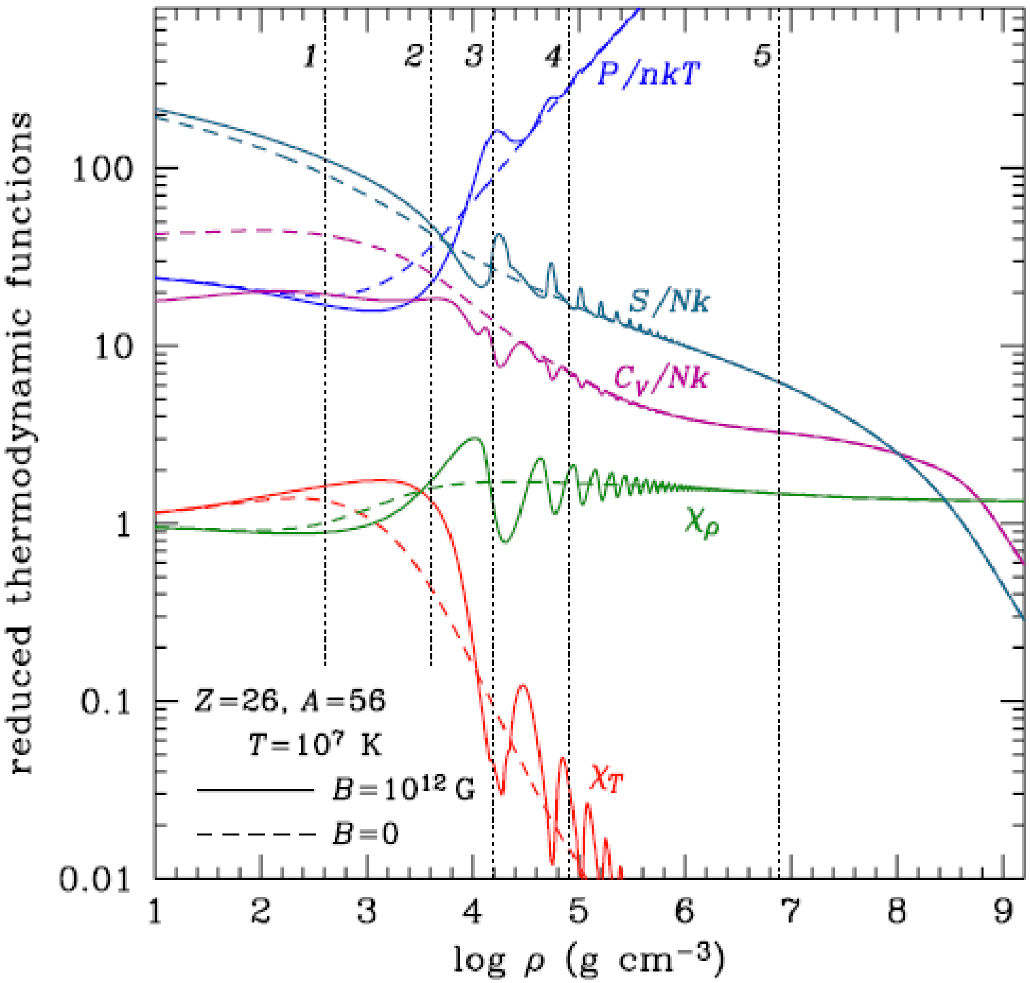


Fig. 6. Reduced thermodynamic functions $P/n_i k_B T$, $S/N_i k_B$, $C_V/N_i k_B$, χ_ρ, and χ_T for a fully-ionized nonmagnetic (dashed lines) and magnetized ($B = 10^{12}$ K, solid lines) iron plasma at $T = 10^7$ K. The vertical dotted lines mark the densities at which (1) $T_F^{(0)} = T$; (2) $T_F = T$; (3) $\rho = \rho_B$; (4) $\Gamma = \Gamma_m$; and (5) $T_p = T$.

States: Landau-Rabi

Can be solved (semi-) analytically.

Simple explicit equations for:

- Very high B (one occupied level)
- Low B (nonmagnetized theory is reproduced)

Potekhin&Chabrier [A&A 550, A43 (2013)]

Thermodynamics of the outer crust

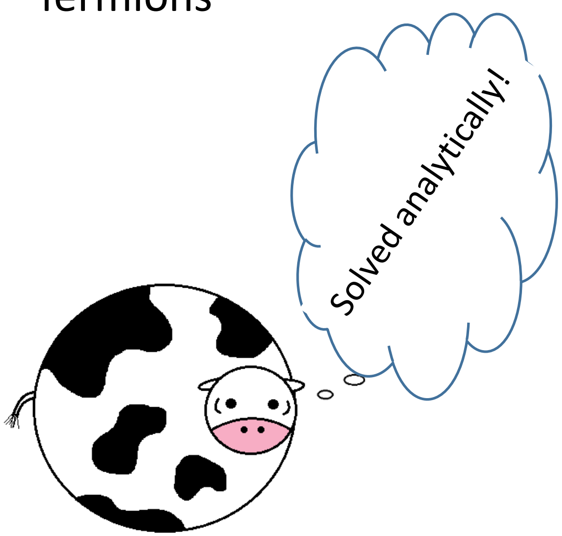
(school topic: theoretical description of interacting fermion systems)

$$F = F_{\mathrm{e}} + F_{\mathrm{i}} + F_{\mathrm{ie}} + \sum_i M_i N_i$$

Nuclei mass (nuclear physics)

<u>Model system</u>
Ideal degenerate gas of electrons

<u>Electrons:</u> relativistic noninterracting fermions



<u>Model system</u>
Nuclei on the **uniform** neutralizing background of electrons

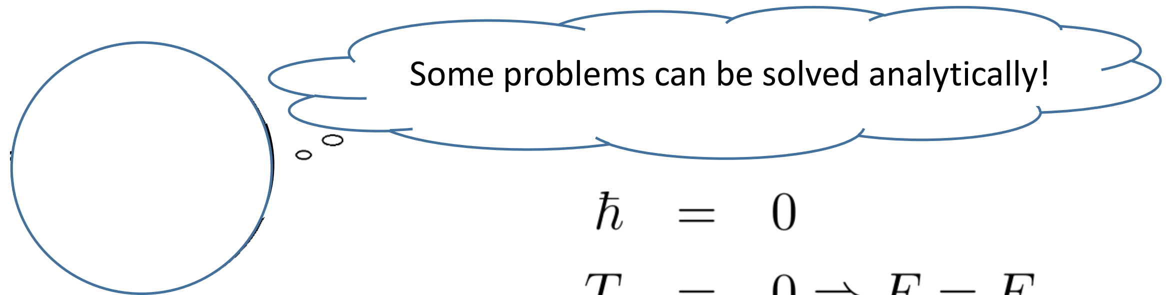
<u>Nuclei:</u> point like massive particles
<u>Electrons:</u> (just) background

Strong Coulomb interaction, but statistics (fermions or bosons) is not important for astrophysical applications

Physical motivation (D.A. Baiko):
If nuclei wave functions overlap, it leads to fusion

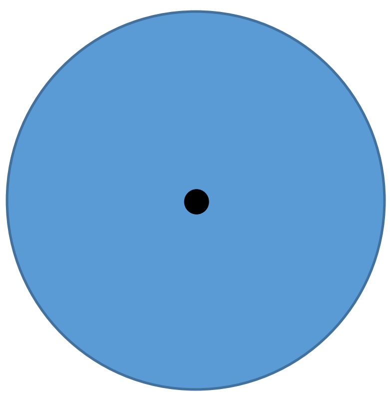
<u>Quantum effects:</u> important, but associated with collective effects

Thermodynamics of nuclei in outer crust
The ion sphere model (Wigner-Seitz approximation)



$$\hbar = 0$$
$$T = 0 \Rightarrow F = E$$

Salpeter (1954):



$$\frac{E_{\rm i}}{N} = \mathcal{E}_{ee} + \mathcal{E}_{ie} = -\frac{9}{10}\frac{Z^2e^2}{a}, \quad a = \left(\frac{3}{4\pi n}\right)^{1/3}$$

$$\mathcal{E}_{ie} = \int \rho_e \phi_i {\rm d}V = -4\pi Z^2 e^2 \int_0^a r n_e {\rm d}r = -\frac{3}{2}\frac{Z^2e^2}{a}$$

$$\rho_e = -e\, n_e = -\frac{3\,Ze}{4\pi a^3}$$
$$\phi_i = \frac{Ze}{r}$$

<u>Simple electrostatic problem</u>
Point-like charge (Ze) at the center of uniformly charged sphere of radius a

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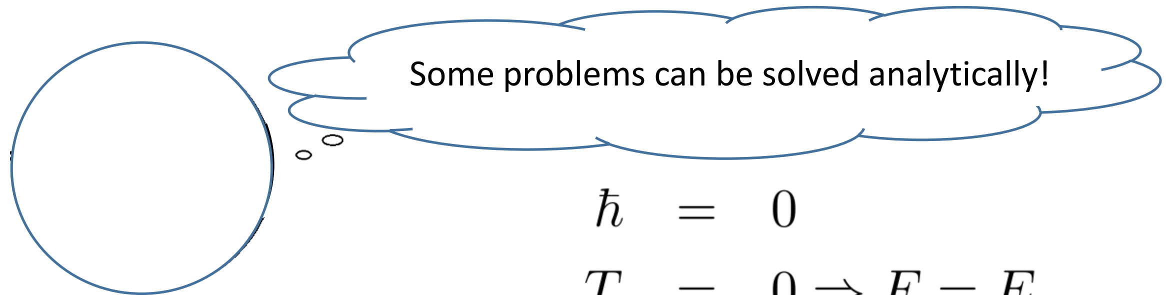

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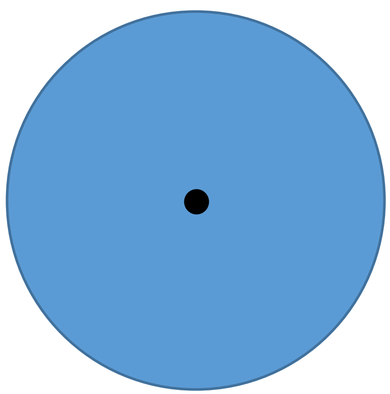


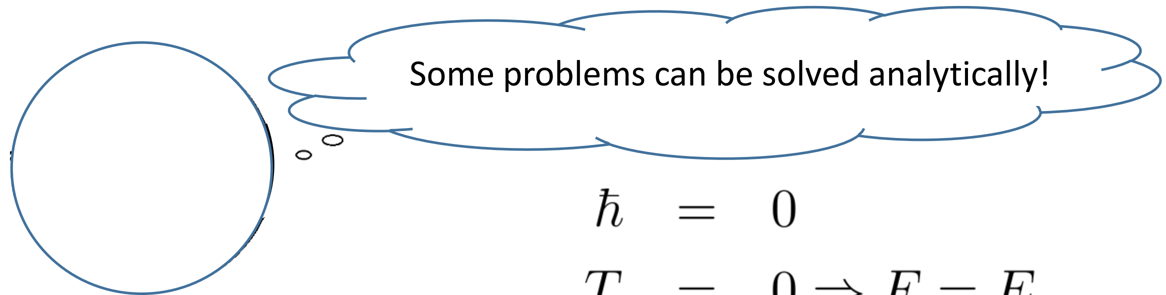
<u>Simple electrostatic problem</u>
Point-like charge (Ze) at the center of uniformly charged sphere of radius a

$$\frac{E_{\mathrm{i}}}{N} = \mathcal{E}_{ee} + \mathcal{E}_{ie} = -\frac{9}{10}\frac{Z^2e^2}{a}, \quad a = \left(\frac{3}{4\pi n}\right)^{1/3}$$

$$\mathcal{E}_{ee} = \int \frac{E_e^2}{8\pi}\mathrm{d}V = \frac{Z^2e^2}{2}\left(\int_a^\infty \frac{r^2\mathrm{d}r}{r^4} + \int_0^a \frac{r^4}{a}\mathrm{d}r\right)$$
$$= \frac{Z^2e^2}{2}\left(\frac{1}{a} + \frac{1}{5a}\right) = \frac{3}{5}\frac{Z^2e^2}{a}$$

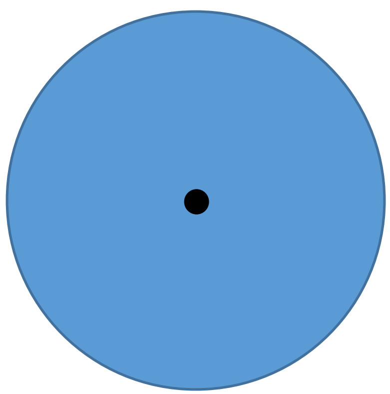
$$E_e = \begin{cases} \frac{Z^2e^2}{r^2} & r > a \\ \frac{Z^2e^2}{a^2}\frac{r}{a} & r \leq a \end{cases}$$

Thermodynamics of nuclei in outer crust
The ion sphere model (Wigner-Seitz approximation)



$$\begin{aligned} \hbar &= 0 \\ T &= 0 \Rightarrow F = E \end{aligned}$$

Salpeter (1954):



$$\frac{E_{\rm i}}{N} = \mathcal{E}_{ee} + \mathcal{E}_{ie} = -\frac{9}{10}\frac{Z^2 e^2}{a}, \quad a = \left(\frac{3}{4\pi n}\right)^{1/3}$$

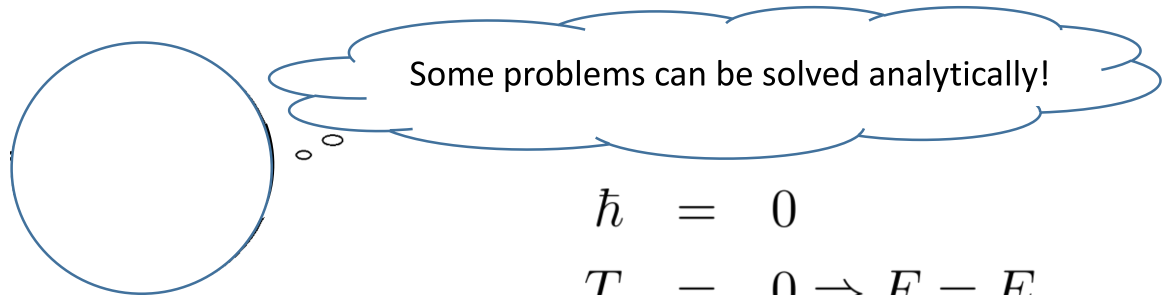
$$\epsilon_i = \frac{E_i}{V} = -\frac{9}{10}\frac{Z^2 e^2}{a} n$$

$$P_{\rm i} = -\frac{\partial E_{\rm i}}{\partial V} = \frac{E_{\rm i}}{3V} = -\frac{3}{10}\frac{Z^2 e^2}{a} n \propto n^{4/3}$$

<u>Simple electrostatic problem</u>

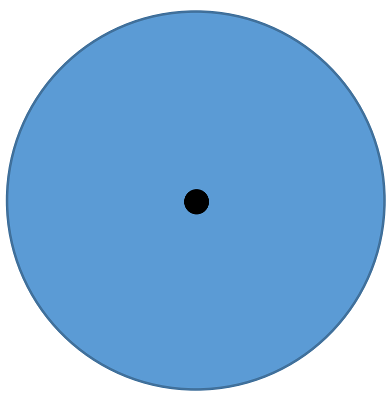
Neutral system of point-like charge (Ze) at the center of uniformly charged sphere of radius a

Thermodynamics of nuclei in outer crust
The ion sphere model (Wigner-Seitz approximation)



$$\hbar = 0$$
$$T = 0 \Rightarrow F = E$$

Salpeter (1954):



Simple electrostatic problem

Neutral system of point-like charge (Ze) at the center of uniformly charged sphere of radius a

$$\epsilon_i = \frac{E_i}{V} = -\frac{9}{10}\frac{Z^2e^2}{a}n$$

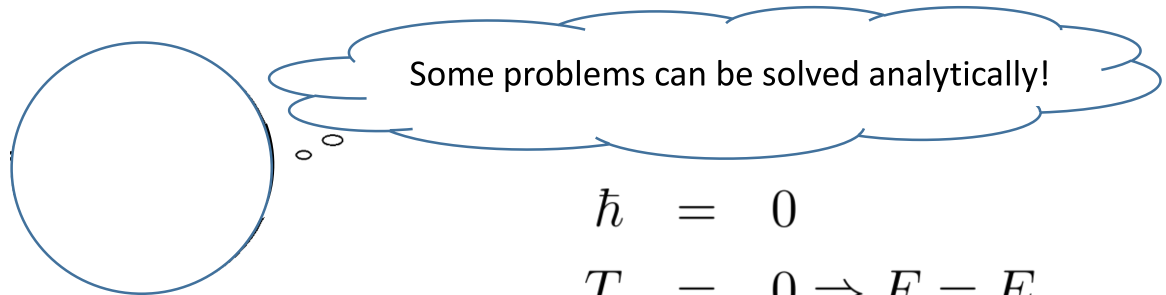
$$P_{\rm i} = -\frac{\partial E_{\rm i}}{\partial V} = \frac{E_{\rm i}}{3V} = -\frac{3}{10}\frac{Z^2e^2}{a}n \propto n^{4/3}$$

- Negative pressure?????

Don't forget about electrons!!!

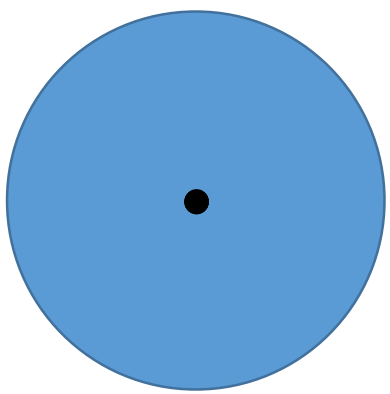
$$P_{\rm i} \approx -4.6\times10^{-3} Z^{2/3} P_{\rm e} \quad (\rho \gg 10^6 \ {\rm g/cm^3})$$

Thermodynamics of nuclei in outer crust
Perfect one component crystal



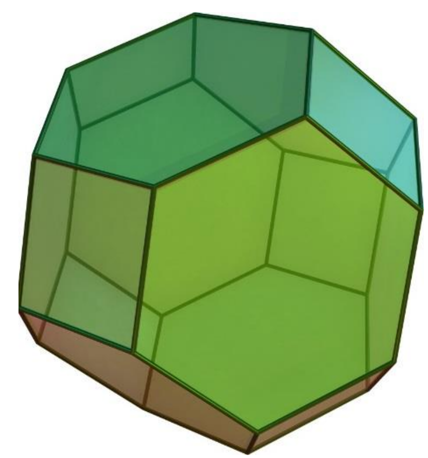
$$\hbar = 0$$
$$T = 0 \Rightarrow F = E$$

Salpeter (1954): Ion sphere



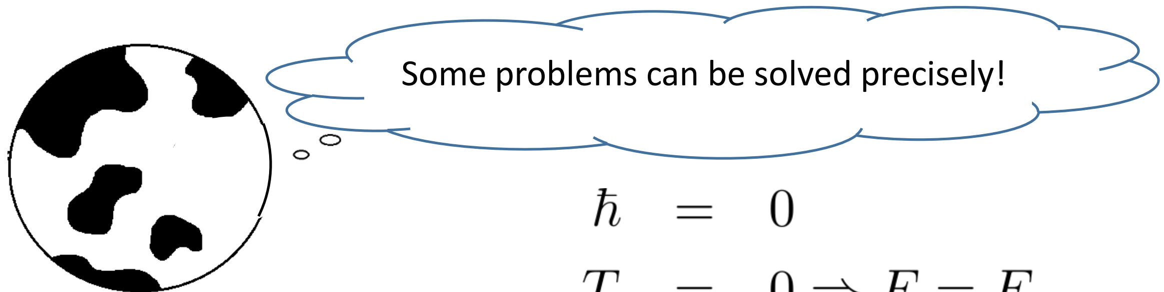
Simple electrostatic problem
Point-like charge (Ze) at the center of uniformly charged sphere of radius a

Nuclei (ions) form a perfect regular lattice
On uniform neutralizing background



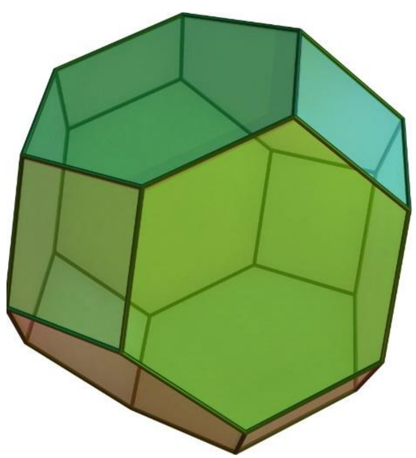
Energy per nuclei
=
energy of the cell

Thermodynamics of nuclei in outer crust
Perfect one component crystal



$$\hbar = 0$$
$$T = 0 \Rightarrow F = E$$

Nuclei (ions) are the same and form a perfect regular lattice on uniform neutralizing background (Coulomb crystal)



Energy per nuclei
=
energy of the cell

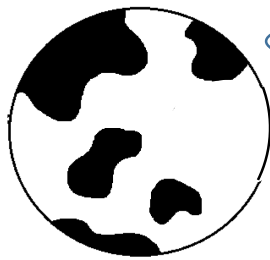
Dimensional analysis

We have:

	Ze	Charge
	a	Distance
Energy scale	$\frac{Z^2e^2}{a}$	
Density	$n = \frac{3}{4\pi a^3}$	
Energy density	$\epsilon = \zeta \frac{Z^2e^2}{a} n$	

We need to calculate a constant!

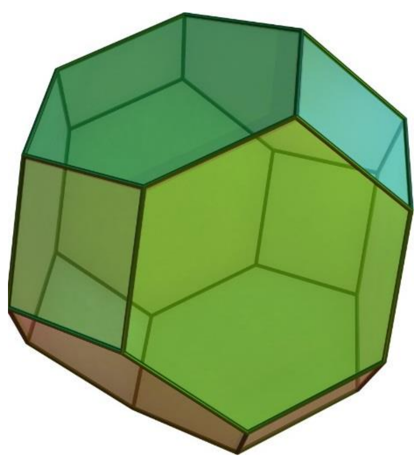
Thermodynamics of nuclei in outer crust
Perfect one component crystal



Some problems can be solved precisely!

$$\hbar = 0$$
$$T = 0 \Rightarrow F = E$$

Nuclei (ions) form a perfect regular lattice
On uniform neutralizing background



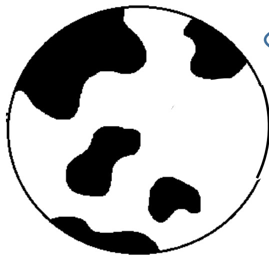
Energy per nuclei
=
energy of the cell

Lattice sums

$$\begin{aligned} \epsilon V &= \sum_i \sum_{j>i} \frac{Z_i Z_j e^2}{|\boldsymbol{R}_i - \boldsymbol{R}_j|} \\ &- n_\mathrm{e} \sum_i \int \frac{Z_i e^2}{|\boldsymbol{R}_i - \boldsymbol{r}|} \mathrm{d}^3\boldsymbol{r} \\ &+ \frac{n_\mathrm{e}^2}{2} \int\int \frac{e^2}{|\boldsymbol{r} - \boldsymbol{r}'|} \mathrm{d}^3\boldsymbol{r}\, \mathrm{d}^3\boldsymbol{r}' \end{aligned}$$

- All terms are divergent, if treated separately…
- If combined, convergence is conditional

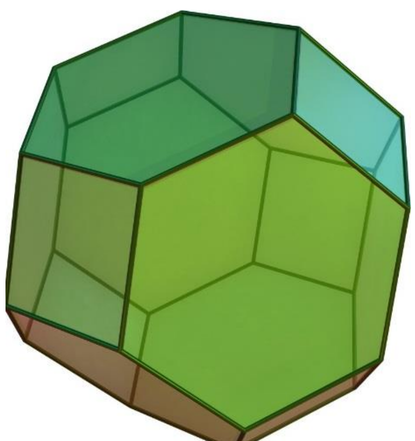
Thermodynamics of nuclei in outer crust
Perfect one component crystal



Some problems can be solved precisely!

$$
\begin{aligned}
\hbar &= 0 \\
T &= 0 \Rightarrow F = E
\end{aligned}
$$

Nuclei (ions) form a perfect regular lattice
On uniform neutralizing background



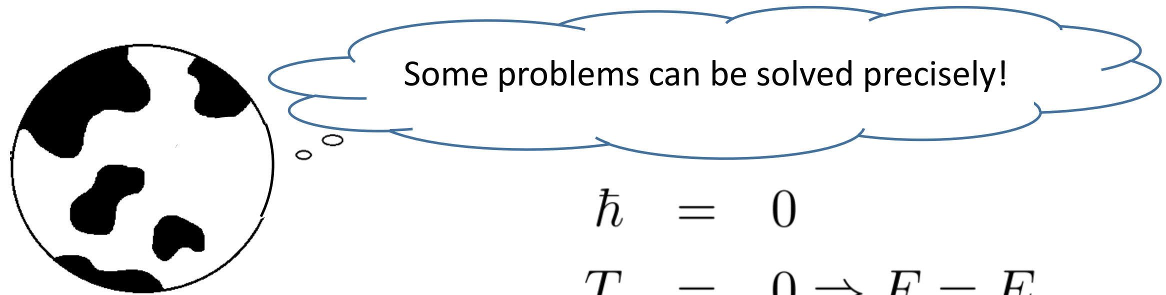
Energy per nuclei
=
energy of the cell

Lattice sums

$$
\begin{aligned}
\epsilon V &= \sum_i \sum_{j>i} \frac{Z_i Z_j e^2}{|\boldsymbol{R}_i - \boldsymbol{R}_j|} \\
&- n_\mathrm{e} \sum_i \int \frac{Z_i e^2}{|\boldsymbol{R}_i - \boldsymbol{r}|} \mathrm{d}^3\boldsymbol{r} \\
&+ \frac{n_\mathrm{e}^2}{2} \int\int \frac{e^2}{|\boldsymbol{r} - \boldsymbol{r}'|} \mathrm{d}^3\boldsymbol{r}\, \mathrm{d}^3\boldsymbol{r}'
\end{aligned}
$$

Physically reasonable thermodynamic limit corresponds to summation, ordered by the cells

Thermodynamics of nuclei in outer crust
Perfect one component crystal



$$\hbar = 0$$
$$T = 0 \Rightarrow F = E$$

Lattice sums

Physically reasonable thermodynamic limit corresponds to summation, ordered by the cells, but convergence is very slow

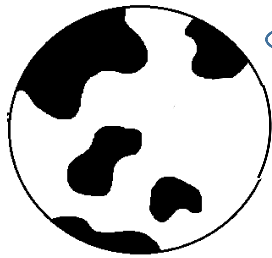
Ewald summation (physical interpretation):

- Substitute point-like nuclei to charge clouds with Gaussian distribution
- Calculate energy in reciprocal space
- Include correction, which accounts that nuclei are point-like

$$U_{\mathrm{b}}(\mathbf{r}) = \frac{1}{\pi v} \sum_{\mathbf{h},j}{}' \frac{q_j}{|\mathbf{h}|^2} \exp\left[-\frac{\pi^2 |\mathbf{h}|^2}{\eta^2} + 2\pi i \mathbf{h}(\mathbf{b}_j - \mathbf{r})\right] +$$
$$+ \sum_{i,j}{}' q_j \frac{\operatorname{erfc}(\eta\,|\mathbf{R}_i + \mathbf{b}_j - \mathbf{r}|)}{|\mathbf{R}_i + \mathbf{b}_j - \mathbf{r}|} - \left\{\frac{2\eta q_j}{\sqrt{\pi}}\right\}_{\mathbf{r}=\mathbf{b}_j},$$

Kholopov, Physics-Uspehi, 47 (2004), 965

Thermodynamics of nuclei in outer crust
Perfect one component crystal



Some problems can be solved precisely!

$$\hbar = 0$$
$$T = 0 \Rightarrow F = E$$

<u>Dimensional analysis</u>

We have:

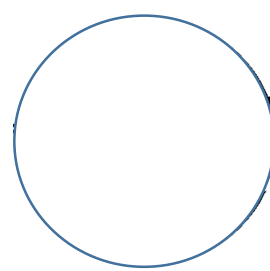
	Ze	Charge
	a	Distance
Energy scale	$\frac{Z^2e^2}{a}$	
Density	$n = \frac{3}{4\pi a^3}$	
Energy density	$\epsilon = \zeta \frac{Z^2e^2}{a} n$	

We need to calculate a constant!

Baiko et al. (2001):

$$\zeta = \begin{cases} -0.895929255682 & \text{bcc} \\ -0.895873615195 & \text{fcc} \\ -0.895838120459 & \text{hcp} \end{cases}$$

Ion sphere model:



$$\zeta = -0.9$$

Still valid

$$P_{\rm i} \approx -4.6 \times 10^{-3} Z^{2/3} P_{\rm e}$$

Ground state composition of outer crust
One component approximation

AME16

BSK27

Z_1	A_1	Z_2	A_2	x_r	$\bar{n}_1^{\max}$	$\bar{n}_2^{\min}$	$P_{1\to2}$	$\mu_e^{1\to2}$	$\mu_{1\to2}$	ξ_1	$z_1/z_{\rm drip}$
26	56	28	62	1.57	4.92×10^{-9}	5.06×10^{-9}	3.35×10^{-10}	0.966	930.6	6.93×10^{-7}	0.0207
28	62	28	64	5.01	1.63×10^{-7}	1.68×10^{-7}	4.34×10^{-8}	2.50	931.3	8.92×10^{-5}	0.0985
28	64	28	66	8.42	8.01×10^{-7}	8.26×10^{-7}	3.56×10^{-7}	4.16	932.0	6.47×10^{-4}	0.177
28	66	36	86	8.61	8.83×10^{-7}	9.00×10^{-7}	3.89×10^{-7}	6.21	932.1	6.89×10^{-5}	0.181
36	86	34	84	11.0	1.87×10^{-6}	1.93×10^{-6}	1.04×10^{-6}	5.13	932.6	1.35×10^{-3}	0.234
34	84	32	82	16.8	6.83×10^{-6}	7.08×10^{-6}	5.62×10^{-6}	7.84	933.7	4.83×10^{-4}	0.357
32	82	30	80	22.3	1.68×10^{-5}	1.74×10^{-5}	1.78×10^{-5}	10.5	934.8	2.52×10^{-2}	0.473
30	80	28	78	28.2	3.51×10^{-5}	3.66×10^{-5}	4.53×10^{-5}	13.3	935.8	5.69×10^{-2}	0.591
28	78	44	126	34.7	6.85×10^{-5}	7.12×10^{-5}	1.05×10^{-4}	24.4	937.0	1.23×10^{-1}	0.717
44	126	42	124	36.8	8.37×10^{-5}	8.62×10^{-5}	1.29×10^{-4}	16.9	937.3	5.15×10^{-2}	0.752
42	124	40	122	42.1	1.29×10^{-4}	1.34×10^{-4}	2.23×10^{-4}	19.4	938.2	1.92×10^{-1}	0.847
40	122	38	120	44.7	1.60×10^{-4}	1.66×10^{-4}	2.84×10^{-4}	20.7	938.6	1.27×10^{-1}	0.893
38	120	38	122	49.8	2.29×10^{-4}	2.33×10^{-4}	4.38×10^{-4}	24.2	939.4	3.19×10^{-1}	0.980
38	122	38	124	50.9	2.49×10^{-4}	2.53×10^{-4}	4.78×10^{-4}	24.7	939.5	8.22×10^{-2}	0.998
38	124	–	–	51.1	2.55×10^{-4}	–	4.83×10^{-4}	24.8	939.6	1.12×10^{-2}	1.00

Equilibrium state is determined by thermodynamics

Pressure is continuous => minimize the Gibbs energy over *Z,A*

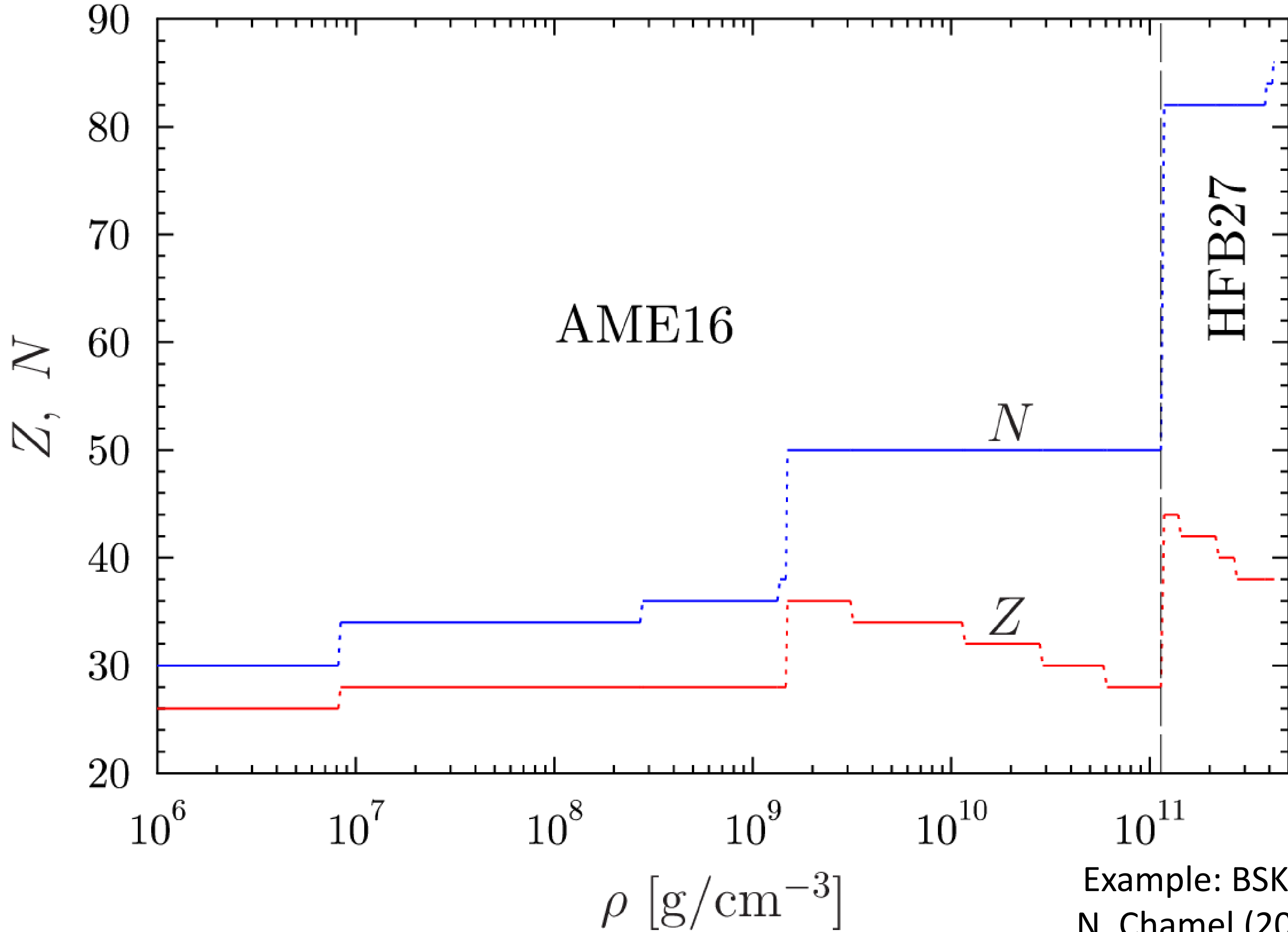
$$\mathfrak{g}(Z, A, P)$$

Density jumps at

$$\mathfrak{g}(Z_1, A_1, P_{12}) = \mathfrak{g}(Z_2, A_2, P_{12})$$

Example: BSK27, N. Chamel (2020)

Ground state composition of outer crust
One component approximation



Example: BSK27,
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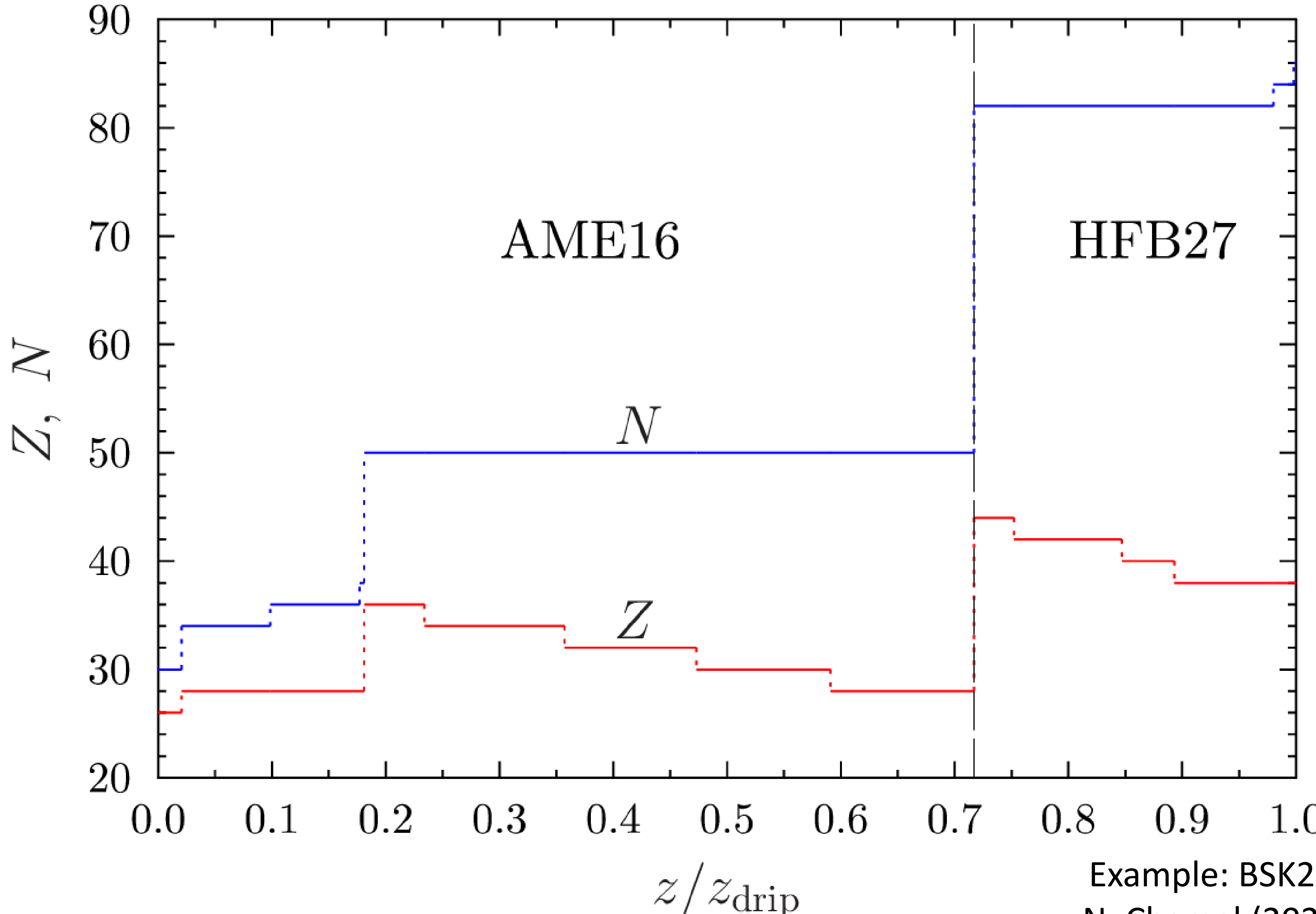
Ground state composition of outer crust
One component approximation



Example: BSK27,
N. Chamel (2020)

Ground state composition of outer crust

One component approximation

Outer crust ends at the neutron drip

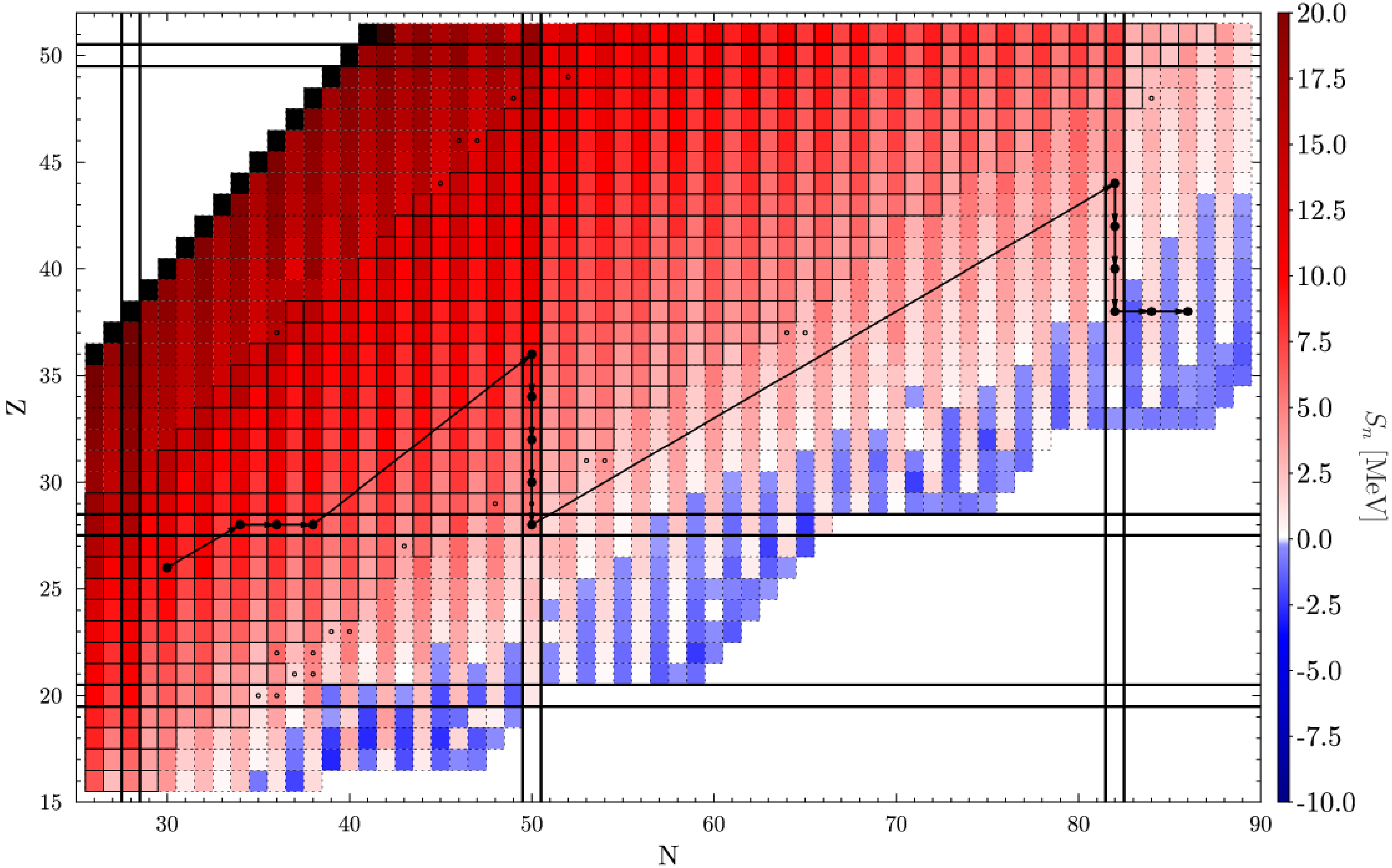


Figure is based on the table by N. Chamel (2020) for AME16 (boxed)+BSK27.
New experimentally measured nuclei in AME20 are shown by small dots

Crust of compact stars Summary of lecture 1.

- NS crust affect a wide set of observations and it's properties should be predicted quantitatively
- Properties of NS crust is a result of a complicated interplay of different physics (as in terrestrial conditions)
- Studies of NS crust are based on models, and some of them allows analytical solutions
  - Thermodynamics of degenerate electrons
  - Classical ions at zero-temperature
- Outer crust composition: approaching the drip line

Lecture 2:

- Finite temperature thermodynamics of ions from ab-initio simulations (for OCP)
- Crust elasticity (analytical)