QCD matter on a lattice

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30 January, 2024

Outline:

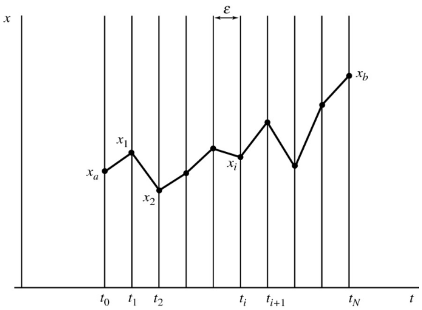
- Introduction
  - Statistical mechanics
  - QED as a gauge theory
  - Building gluodynamics and QCD
- Lattice gluodynamics
- Lattice QCD
- Numerical methods of lattice QCD
- Applications

Partition function

- Partition function:
  $Z = \sum_n e^{-\frac{E_n}{T}} = \sum_n \langle n|e^{-\frac{\hat{H}}{T}}|n\rangle = Tr\left[e^{-\frac{\hat{H}}{T}}\right]$
- Free energy:
  $F = -T\log Z = E - TS, \quad Z = e^{-\frac{F}{T}}$
- Probability to find a system at the n-th level:
  $P_n = \frac{e^{-\frac{E_n}{T}}}{Z}$
- $\langle O\rangle = \sum_n P_n \langle n|\hat{O}|n\rangle = \frac{1}{Z}\sum_n \langle n|O|n\rangle e^{-\frac{E_n}{T}}$
- Z contains an important information about system:
  - $\langle E\rangle = T^2 \frac{\partial log Z}{\partial T} = -T^2 \frac{\partial}{\partial T}\left(\frac{F}{T}\right)$
  - $p = -\frac{\partial F}{\partial V}$
  - $S = \frac{\partial T log Z}{\partial T} = -\frac{\partial F}{\partial T}$

Path integral formulation for partition function

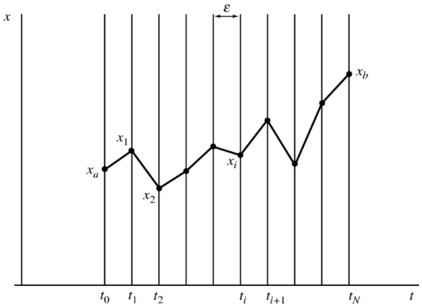
- $Z = Tr\left[e^{-\frac{\hat{H}}{T}}\right] = \sum_q \langle q|e^{-\frac{\hat{H}}{T}}|q\rangle = \int dq \langle q|e^{-\frac{\hat{H}}{T}}|q\rangle$
- Quantum evolution in time: $\langle q'|e^{-i\frac{\hat{H}}{\hbar}t}|q\rangle$, $q(0) = q$, $q(t) = q'$
- Z looks like quantum evolution in Euclidean time
  $t = -i\tau = -i\frac{1}{T}$, $q(0) = q$, $q(\tau = \frac{1}{T}) = q$
- $Z \sim \lim_{N\to\infty} \int \prod_{\tau=1}^{N} dq(\tau) e^{-S_E}$
  $S_E = \int_0^{1/T} d\tau (\frac{m\dot{q}(\tau)^2}{2} + V(q(\tau)))$, $\quad q(0) = q(\tau = \frac{1}{T}) = q$



N degrees of freedom

- $q_i(\tau)$, $i = 1..N$

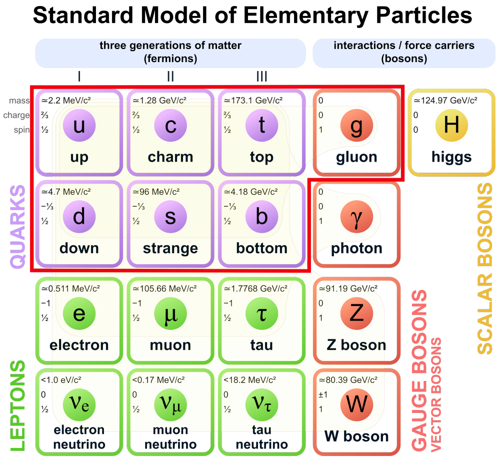
- $Z \sim \int \prod_\tau \prod_{i=1}^{N} dq_i(\tau) e^{-S_E}$
  $S_E = \int_0^{1/T} d\tau \left(\frac{m \sum_i \dot{q}_i(\tau)^2}{2} + V(q_i(\tau)) \right), \quad q_i(0) = q_i(\tau = \frac{1}{T}) = q_i$



Partition function for φ^4-theory

- Field theory:
  - $q_i(\tau) \to \varphi(\vec{x}, \tau)$
  - $i \to \vec{x}$
  - $\sum_i \to \int d^3x$
- $S = \int dt d^3x (\frac{1}{2}(\nabla_t \varphi)^2 - \frac{1}{2}(\vec{\nabla}\varphi)^2 - \frac{m^2}{2}\varphi^2 - \frac{\lambda}{4!}\varphi^4)$
- Mechanics:
  $Z \sim \int \prod_\tau \prod_{i=1}^N dq_i(\tau) e^{-S_E}$ $S_E = \int_0^{1/T} d\tau (\frac{m \sum_i \dot{q}_i(\tau)^2}{2} + V(q_i(\tau)))$,
  $q_i(0) = q_i(\tau = \frac{1}{T}) = q_i$
- Field theory:
  $Z \sim \int \prod_\tau \prod_{\vec{x}} d\varphi(\tau, \vec{x}) e^{-S_E}$
  $S_E = \int d\tau d^3x (\frac{1}{2}(\nabla_\tau \varphi)^2 + \frac{1}{2}(\vec{\nabla}\varphi)^2 + \frac{m^2}{2}\varphi^2 + \frac{\lambda}{4!}\varphi^4)$,
  $\varphi(\vec{x}, 0) = \varphi(\vec{x}, \tau = \frac{1}{T}) = \varphi(\vec{x})$

Elementary particles



Building QED

- Interaction of charged particles
- Gauge transformation:
  $\psi(x) \to S(x)\psi(x)$, $S(x) = e^{if} \in U(1)$
- Covariant derivative:
  $\partial_\mu \psi \to S\partial_\mu \psi + (\partial_\mu S)\psi = S(\partial_\mu + (S^{-1}\partial_\mu S))\psi$
  $A_\mu \to A_\mu - i\partial_\mu S S^{-1} = A_\mu + \partial_\mu f$
  $\partial_\mu \to D_\mu = \partial_\mu + iA_\mu$, $D_\mu \psi \to S D_\mu \psi$
- $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu,\ F_{\mu\nu} \to F_{\mu\nu}$
  electric field: $F_{0x} = E_x$, $F_{0y} = E_y, F_{0z} = E_z$
  magnetic field: $F_{xy} = -H_z$, $F_{yz} = H_x, F_{xz} = -H_y$
- QED action: $S = \int d^4x \big[-\frac{1}{4e^2} F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\gamma^\mu D_\mu - m)\psi\big]$
- Coupling constant: $\alpha_{em} = \frac{e^2}{4\pi\hbar c} \simeq \frac{1}{137} \ll 1$

Maxwell equations

$$divE = 4\pi\rho$$
$$divH = 0$$
$$rotE = -\frac{1}{c}\frac{\partial H}{\partial t}$$
$$rotH = \frac{4\pi}{c}j + \frac{1}{c}\frac{\partial E}{\partial t}$$

- Maxwell equations are linear

Building QCD

- New quantum number: $\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix}$
- Interactions of particles with the color
- Gauge transformation: $\psi(x) \to S(x)\psi(x), \quad S(x) \in SU(3)$
- Covariant derivative:
  $\partial_\mu \psi \to S\partial_\mu \psi + (\partial_\mu S)\psi = S(\partial_\mu + (S^{-1}\partial_\mu S))\psi$
  $\hat{A}_\mu \to S\hat{A}_\mu S^{-1} - i\partial_\mu S S^{-1} \quad \hat{A}_\mu = t^a A^a_\mu,\ a = 1...8$
  $\partial_\mu \to \hat{D}_\mu = \partial_\mu + i\hat{A}_\mu,\ \hat{D}_\mu \psi \to S\hat{D}_\mu \psi$

Generators of $SU(3)$

$$\hat{A}_\mu = t^a A^a_\mu, \quad t^a = \frac{\lambda^a}{2}, a = 1...8$$

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}$$

$$\lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

Building QCD

- $\hat{F}_{\mu\nu} = t^a F^a_{\mu\nu} = \partial_\mu \hat{A}_\nu - \partial_\nu \hat{A}_\mu + [\hat{A}_\mu, \hat{A}_\nu], \ \hat{F}_{\mu\nu} \to S^{-1} \hat{F}_{\mu\nu} S$
  chromo-electric field: $F^a_{0x} = E^a_x$, $F^a_{0y} = E^a_y, F^a_{0z} = E^a_z$
  chromo-magnetic field: $F^a_{xy} = -H^a_z$, $F^a_{yz} = H^a_x, F^a_{xz} = -H^a_y$
- QCD action: $S = \int d^4x \big[-\frac{1}{2g^2} Tr \hat{F}_{\mu\nu} \hat{F}^{\mu\nu} + \bar{\psi}(i\gamma^\mu \hat{D}_\mu - m)\psi \big]$
- Coupling constant: $\alpha_s = \frac{g^2}{4\pi\hbar c} \sim 1$

Maxwell equations in QCD

$$
\begin{aligned}
divE^a &= 4\pi\rho^a + f_1(E, H, ...) \\
divH^a &= 0 + f_2(E, H, ...) \\
rotE^a &= -\frac{1}{c}\frac{\partial H^a}{\partial t} + f_3(E, H, ...) \\
rotH^a &= \frac{4\pi}{c} j^a + \frac{1}{c}\frac{\partial E^a}{\partial t} + f_4(E, H, ...)
\end{aligned}
$$

- Maxwell equations for QCD are nonlinear

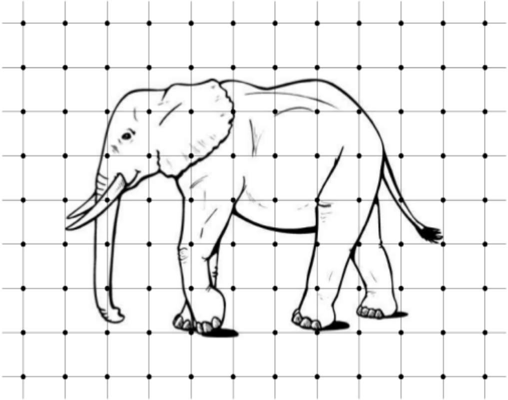
Quantum chomodynamics(QCD)

- Degrees of freedom: Quarks q, gluons A
- QCD Lagrangian

$$L = -\frac{1}{4}\sum_{a=1}^{8} F_a^{\mu\nu} F_{\mu\nu}^a + \sum_{f=u,d,s,\ldots} \bar{q}_f (i\gamma^\mu \partial_\mu - m) q_f + g \sum_{f=1}^{N_f} \bar{q}_f \gamma^\mu \hat{A}_\mu q_f$$

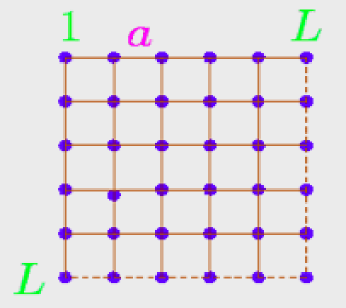
- Nonlinear equation of motion with $\alpha_s \sim 1$
- **The most complicated physical theory**
- QCD Lagrangian is well known but the calculations are not possible
  - *In particular:* Confinement from QCD lagrangian is a millenium problem
- Reliable results can be obtained on modern supercomputers

Lattice simulation of QCD



- Allows to study strongly interacting nonlinear systems
- Based on the first principles of quantum field theory
- Most effective approach due to supercomputers and algorithms

Lattice set up



- Lattice coordinates $x_\mu = a(n_1, n_2, n_3, n_4)$
- $n_i \in [0, L_s - 1]$, $i = 1, 2, 3 \quad n_4 \in [0, L_t - 1]$
- a-lattice spacing, the size of lattice $L_s^3 \times L_t$
- $\varphi(x)$ "live" at the lattice nodes
- Impose periodic boundary conditions for spatial directions
- Bosons: Impose periodic boundary condition for τ-direction
- Fermions: Impose antiperiodic boundary condition for τ-direction

Lattice φ^4-theory

- Derivatives on the lattice ($\hat{\mu}$ unit vector in μ direction)

$$\Delta^f_\mu f(x) = \frac{1}{a}(f(x+a\hat{\mu}) - f(x))|_{a\to 0} = f'(x)$$
$$\Delta^b_\mu f(x) = \frac{1}{a}(f(x) - f(x-a\hat{\mu}))|_{a\to 0} = f'(x)$$

- Lattice action for φ-theory

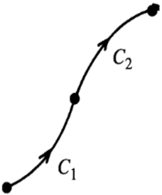
$$S_l = \frac{1}{2}\sum_{\mu=1,2,3,4}\sum_x a^4(\Delta^f_\mu\varphi)^2 + \frac{m^2}{2}\sum_x a^4\varphi^2 + \frac{\lambda}{4!}\sum_x a^4\varphi^4 =$$
$$= \frac{1}{2}\sum_{\mu=1,2,3,4}\sum_x a^2(\varphi(x+a\hat{\mu}) - \varphi(x))^2 + \frac{m^2}{2}\sum_x a^4\varphi^2 + \frac{\lambda}{4!}\sum_x a^4\varphi^4$$

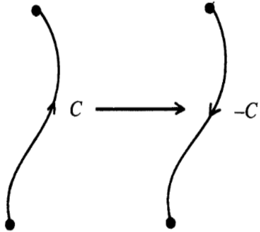
- $Z_l \sim \int \prod_\tau \prod_{\vec{x}} d\varphi(\tau,\vec{x}) e^{-S_l}$
- Continuum limit $a \to 0$: $Z_l \to Z_{\varphi^4}$

Building lattice gluodynamics

- Suppose one has a gauge theory
- $\psi(x) \to S(x)\psi(x)$
- How one can build lattice derivative?
  $\Delta^f_\mu \psi(x) = \frac{1}{a}(\psi(x)(x + a\hat{\mu}) - \psi(x))$
  $\Delta^f_\mu \psi(x) \to \frac{1}{a}(S(x + a\hat{\mu})\psi(x + a\hat{\mu}) - S(x)\psi(x))$
- We would like to have
  $\Delta^f_\mu \psi(x) \to S(x)\Delta^f_\mu \psi(x)$
  $\Delta^f_\mu \psi(x)|_{a\to 0} = D_\mu \psi$
- Parallel transporter $U(C)$:
  $\Delta^f_\mu \psi(x) = \frac{1}{a}(U(x, x + a\hat{\mu})f(x + a\hat{\mu}) - f(x))$

Parallel transport in gauge theory





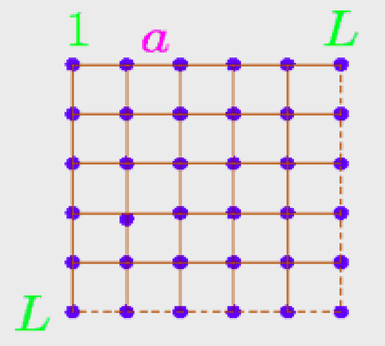
$$\psi(y) = U(C_{yx})\psi(x), \quad U(C_{xy}) \in SU(3)$$

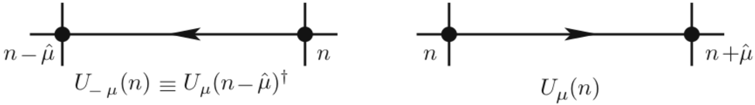
- $U(0) = 1$
- $U(C_2 * C_1) = U(C_2) \cdot U(C_1)$
- $U(-C) = U(C)^{-1}$
- $\psi'(y) = S(y)\psi(y), \quad \psi'(x) = S(x)\psi(x)$
  $\psi^+(y)U(C_{yx})\psi(x)$ is gauge invariant
  $U(C_{yx}) \to S(y)U(C_{yx})S^{-1}(x)$

Parallel transport in gauge theory

- Covariant derivative:
  $D^l_\mu \psi(x) dx^\mu = U(C_{x,x+dx})\psi(x+dx) - \psi(x)$
  $D^l_\mu \psi(x)|_{a\to 0} = (\partial_\mu + iA_\mu)\psi(x)$
- $U(C_{x+dx,x}) = 1 - i\hat{A}_\mu dx^\mu = 1 - it^a A^a_\mu dx^\mu,$
- $dU(C_s) = (-iA_\mu dx^\mu)U(C_s)$
  $U(C) = P\exp\left(-i\int_C dx^\mu A_\mu\right) =$
  $U(x_N, x_{N-1})...U(x_2, x_1)U(x_1, x_0)$

Building lattice gluodynamics

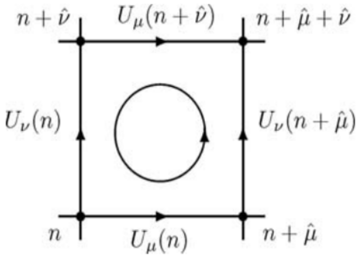




- Lattice spacing-a
- Degrees of freedom:

  $U_\mu(n) = P\exp\left(-i\int_C dx^\mu \hat{A}_\mu\right)\Big|_{a\to 0} = e^{ia\hat{A}_\mu(n)}$

Building lattice gluodynamics



- $U_{\mu\nu}(x) = U_\nu^{-1}(n)U_\mu^{-1}(x+\hat{\nu})U_\nu(x+\hat{\mu})U_\mu(n)$
- $U_{\mu\nu}(n)|_{a\to 0} = \exp\big(ia^2\hat{F}_{\mu\nu}\big)$
- $S_l = \frac{2}{g^2}\sum_n\sum_{\mu<\nu} ReTr[1-U_{\mu\nu}(n)]|_{a\to 0} =$
  $\frac{a^4}{2g^2}\sum_n\sum_{\mu\nu} Tr\hat{F}^2_{\mu\nu} \to \frac{1}{2g^2}\int d^4x Tr\hat{F}^2_{\mu\nu}$

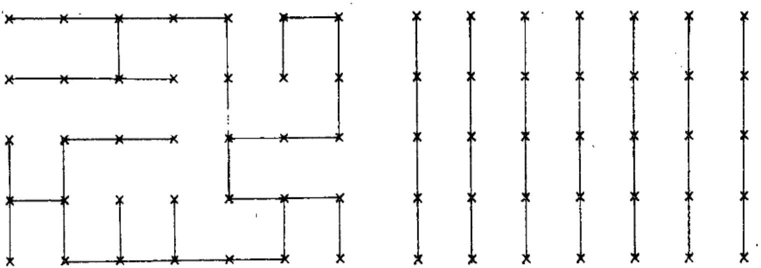
Building lattice gluodynamics

- $S_l = \frac{\beta}{3} \sum_n \sum_{\mu<\nu} ReTr[1 - U_{\mu\nu}(n)]$
- Inverse coupling constant: $\beta = \frac{6}{g^2}$
- Partition function of gluodynamics
  $Z_l = \int \prod_{n,\mu} dU_\mu(n) e^{-S_l}$
- Gauge theory: $U(n)_\mu \to S(x + a\mu) U_\mu(n) S^{-1}(x)$
- One can prove that $Z_l|_{a\to 0} \to Z_{gluodynamics}$

Haar measure

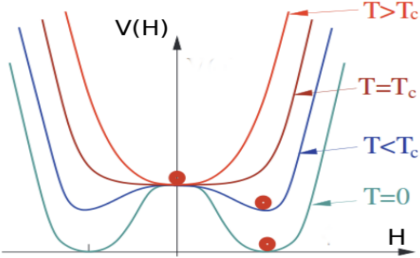
- $\int_G dU f(U) = \int_G dU f(VU) = \int_G dU f(UV)$
- $\int_G dU = 1$
- $\int_G dU f(U) = \int_G dU f(U^{-1})$

Gauge fixing



- $U'_\mu(n) = S(n+\mu)U_\mu(n)S^{-1}(n) = 1$
- Temporal gauge: $U_4(n) = 1, \quad A_4 = 0$

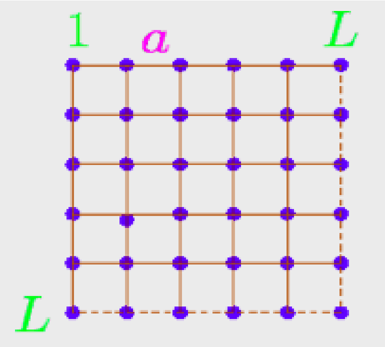
Phase transition in gluodynamics



Experience from φ^4-theory

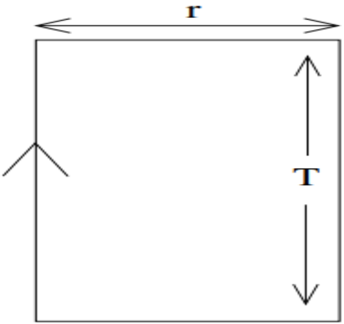
- Local order parameter: $\langle\varphi\rangle$
- Low temperature phase: $\langle\varphi\rangle \neq 0$
- High temperature phase: $\langle\varphi\rangle = 0$

Phase transition in gluodynamics



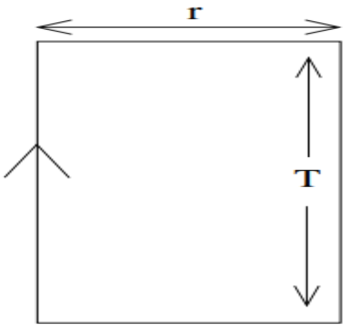
- $S_l = \frac{\beta}{3} \sum_n \sum_{\mu<\nu} ReTr[1 - U_{\mu\nu}(n)]$
- Try local order parameter: $\langle U_{ij} \rangle$, but $\int_G dU U_{ij} = 0$

Wilson loop



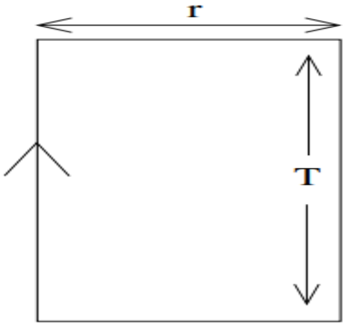
- Wilson loop
  $W(C) = TrP\exp\left(i\int_{\mathrm{C}} dx^{\mu}\hat{A}_{\mu}\right) = Tr\prod_{C} U_{\mu}(x)$
- Nonlocal gauge invariant object
- Low temperature phase: $\langle W(C)\rangle \sim e^{-\sigma S}$
- High temperature phase: $\langle W(C)\rangle \sim e^{-\kappa P}$

Wilson loop



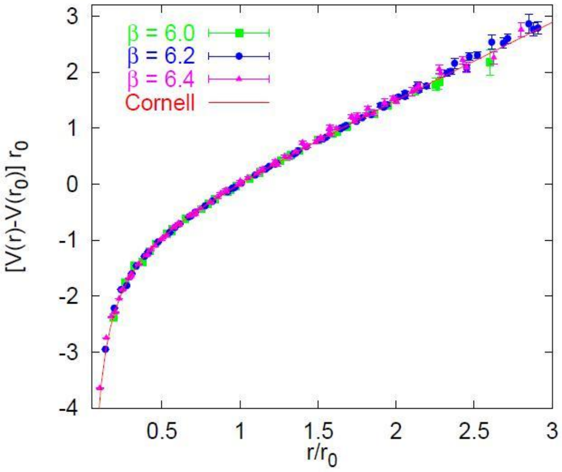
- Gauge $U_i(x) = 1$
- $W(C) = Tr P \exp\left(i \int_0^T d\tau \hat{A}_4(0,\tau)\right) P \exp\left(-i \int_0^T d\tau \hat{A}_4(r,\tau)\right)$
- Experience from QED: $S_{int} = \int d\tau d^3x J^\mu A_\mu$
- $J^\mu = \delta^3(\vec{x})\delta_{\mu 4} - \delta^3(\vec{x} - \vec{r})\delta_{\mu 4}$
  $S_{int} = \int d\tau A_4(0,\tau) - \int d\tau A_4(\vec{r},\tau)$

Wilson loop



- $\langle W \rangle = \langle e^{-S_{int}} \rangle \sim e^{-TV(r)}$
- $V(r) = -\lim_{T\to\infty} \frac{1}{T} \log\langle W(C)\rangle$
- Confinement phase(low temperature):
  $V(r) = -\lim_{T\to\infty} \frac{1}{T} \log e^{-\sigma r T} = \sigma r$

Confinement potential



Polyakov line

φ^4-theory

- $V(\varphi) = -\frac{m^2}{2}\varphi^2 + \frac{\lambda}{4!}\varphi^4$
- Order parameter: $\langle\varphi\rangle$
- Z_2-symmetry: $\varphi \to (\pm 1)\varphi$
- $V(\varphi)$ is invariant but not the $\langle\varphi\rangle$
- Low temperature phase Z_2 is broken
- High temperature phase Z_2 is restored

Polyakov line

Gluodynamics

- $S_l = \frac{\beta}{3} \sum_n \sum_{\mu<\nu} ReTr[1 - U_{\mu\nu}(n)]$

- Polyakov line: $\langle P(\vec{x}) \rangle = TrP \exp\left(i \int_0^T dx^4 \hat{A}_4(\vec{x}, x^4)\right)$

- It is gauge invariant because periodic boundary conditions

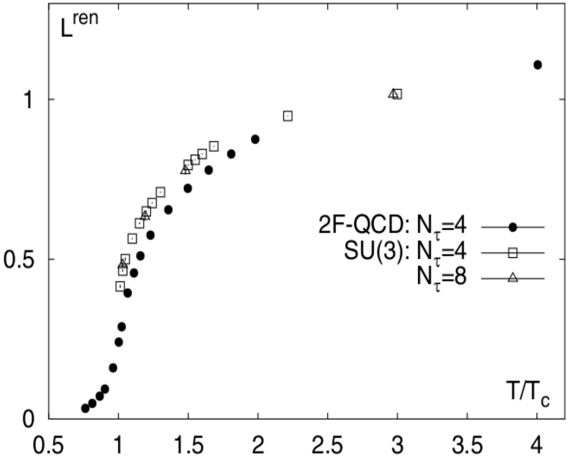
- Z_3 symmetry: $U \to e^{2\pi k/3i} U$, $k = 0, 1, 2$
- S_l is invariant but not the $\langle P(\vec{x}) \rangle$

- $P = e^{-F_Q/T}$

- Low temperature phase: $\langle P(\vec{x}) \rangle = 0$, $F_Q = \infty$, i.e. Z_3 is restored

- High temperature phase $\langle P(\vec{x}) \rangle \neq 0$, $F_Q = finite$, , i.e. Z_3 is broken

Polyakov line



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Building lattice QCD

- 4-dimensional lattice: $L_s \times L_s \times L_s \times L_t = L_s^3 \times L_t$
- Lattice spacing–a
- $S = \frac{\beta}{3} \sum_n \sum_{\mu<\nu} ReTr[1 - U_{\mu\nu}(n)] + \bar{\psi}(\hat{D}(U) + m)\psi$
- $Z_l = \int \prod dU d\bar{\psi} d\psi e^{-S_l} =$
  $\int \prod dU e^{-S_G(U)} \prod_{i=u,d,s...} \det(\hat{D}_i(U) + m_i) =$
  $\int \prod dU e^{-S_{eff}(U)}$

Lattice simulation of QCD

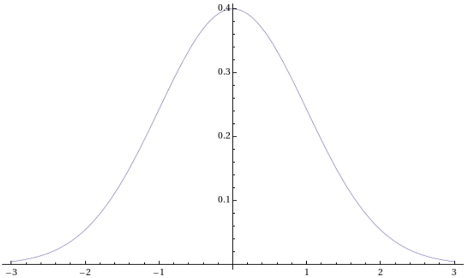
- We study QCD in thermodynamic equilibrium
- The system is in the finite volume
- Calculation of the partition function
  $Z \sim \int DU e^{-S_G(U)} \prod_{i=u,d,s...} \det\left(\hat{D}_i(U) + m_i\right) = \int DU e^{-S_{eff}(U)}$
- Monte Carlo calculation of the integral
- Carry out continuum extrapolation $a \to 0$
- Uncertainties (discretization and finite volume effects) can be systematically reduced
- The first principles based approach. No assumptions!
- Parameters: g^2 and masses of quarks

Modern lattice simulation of QCD

$$Z_l \sim \int DU e^{-S_{eff}(U)}$$

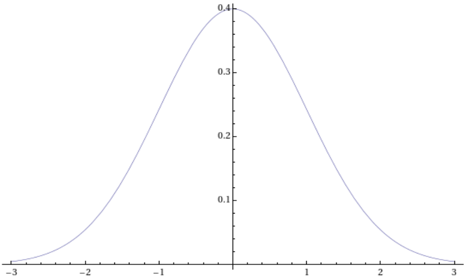
- Lattices
  - 96×48^3
  - Variables: $96 \cdot 48^3 \cdot 4 \cdot 8 \sim 300 \cdot 10^6$
  - Matrices: $100 \cdot 10^6 \times 100 \cdot 10^6$
- Dynamical u, d, s, c–quarks
- Physical masses of u, d, s, c–quarks
- Lattice spacing $a \sim 0.05\,\text{fm}$

Monte Carlo method



- We calculate the integral: $I = \int_{+\infty}^{-\infty} dx \frac{e^{-x^2/2}}{\sqrt{2\pi}} = \int_{+\infty}^{-\infty} dx f(x) = 1$
- Generate the sequence of random numbers: $(x_1, x_2, x_3, ...x_N)$ in the region $x \in [-c, c]$
- $I_N = \frac{2c}{N} \sum_{i=1}^{N} f(x_i)$
- $\lim_{N\to\infty} I_N = I$
- $I_{10} = 0.8836, \quad I_{100} = 1.0708, \quad I_{1000} = 0.9807, \quad I_{10000} = 0.9983,$ $I_{100000} = 1.0018$

Monte Carlo method



- We calculate the integral: $I = \int_{+\infty}^{-\infty} dx \frac{e^{-x^2/2}}{\sqrt{2\pi}} = \int_{+\infty}^{-\infty} dx f(x) = 1$
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- $I_{10} = 0.8836, \quad I_{100} = 1.0708, \quad I_{1000} = 0.9807, \quad I_{10000} = 0.9983, \quad I_{100000} = 1.0018$
- Not very effective!

Metropolis algorithm

Calculation of the $\int dx e^{-S(x)}, \quad S(x) = \frac{x^2}{2}$

- The first approximation $x_0 = 0$
- Choose randomly $\Delta x \in [-c, c]$
- $x' = x_k + \Delta x$
- Metropolis algorithm(accept/reject procedure):
  $\Delta S = S(x') - S(x_k)$. If $\Delta S < 0, \quad S(x') < S(x_k)$, then $x_{k+1} = x'$. Else, x' is accepted with probability: $e^{-\Delta S}$.
- In practice: generate a random number $r \in [0, 1]$. If $r < e^{-\Delta S}$, then $x_{k+1} = x'$, else $x_{k+1} = x_k$.

Metropolis algorithm

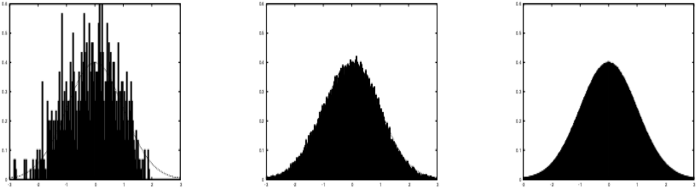
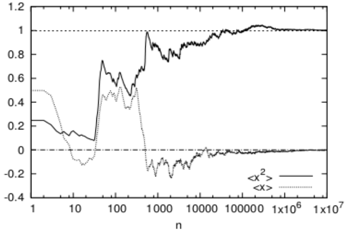
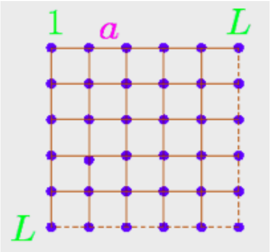


Figure 2: The distribution of $x^{(1)}, x^{(2)}, \cdots, x^{(n)}$, for $n = 10^3, 10^5$ and 10^7, and $\frac{e^{-x^2/2}}{\sqrt{2\pi}}$.



from arXiv:1808.08490

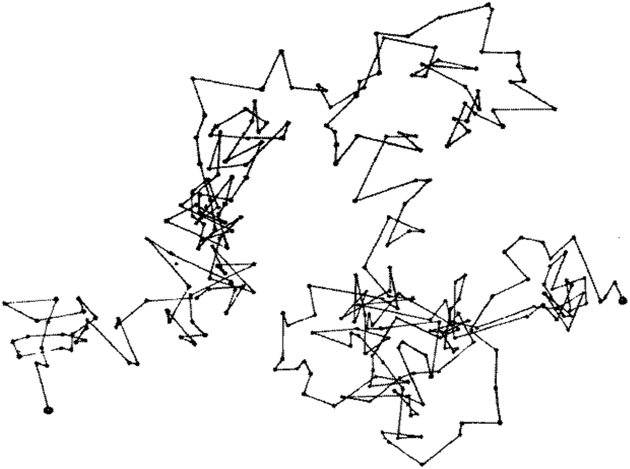
Metropolis algorithm for gluodynamics



- We calculate $Z = \int \prod dU e^{-S_l}$
  $S_l = \frac{\beta}{3} \sum_n \sum_{\mu<\nu} ReTr[1 - U_{\mu\nu}(n)]$
- The first approximation: $\{U_0\} = \hat{1}$
- Choose randomly $V \in SU(3)$
- $U' = VU_k$
- Metropolis algorithm(accept/reject procedure):
  $\Delta S_l = S_l(U') - S_l(U_k)$. If $\Delta S < 0$, then accept U'. Else, U' is accepted with probability: $e^{-\Delta S_l}$(quantum fluctuations).

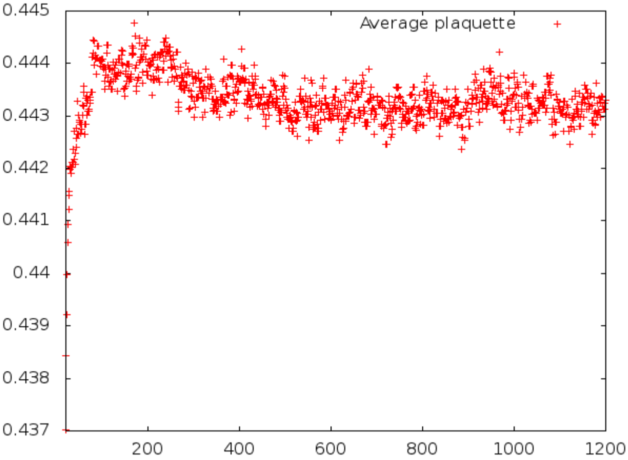
Hybrid Monte Carlo method (HMC)

- HMC method is brownian motion
- accept/reject procedure:
  with probability $p \sim e^{-\Delta S}$, $\quad \Delta S = S_{eff}(U_{new}) - S_{eff}(U_{old})$



Hybrid Monte Carlo method

- For sufficiently large number of steps the distribution is $\sim \exp\left(-\mathbf{S_{eff}}(\mathbf{U})\right)$



Applications

- Spectroscopy
- Matrix elements and correlations functions
- Thermodynamic properties of QCD
- Transport properties of QCD
- Phase transitions
- Nuclear physics
- Properties of QCD under extreme conditions (magnetic field, baryon density, relativistic rotation,...)
- Topological properties
- ...