QCD vacuum and its excitations

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February 2, 2024

Towards self-consistent mean field approach
to QCD vacuum and hadron phenomenology

Almost everywhere Abelian (anti-)self-dual homogeneous gluon field

G.V. Efimov, B. Galilo, A. Kalloniatis, V. Voronin, A. Nikolskii, V. Tainov

- Effective quantum action and self-consistent mean field as the vacuum of quantum field systems.
- Towards self-consistent mean field approach to QCD: statistical ensemble of almost everywhere homogeneous (anti-) self-dual Abelian gluon fields as QCD vacuum: confinement of dynamic and static quarks, chiral symmetry
- Electromagnetic field as a trigger of deconfinement, direct photon production
- Collective excitation modes and composite fields: nonlocal Kutkosky Model
- Collective excitations in QCD vacuum and QCD Bozonization
- Meson effective action: the mass spectrum of mesons
- Effects of strong, weak and electromagnetic interactions of mesons: decay constants and form factors.
- "Projection" to other approaches: FRG, DSE+BS, 4-dim. oscillator - harmonic confinement, AdS/QCD models

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Quantum effective action of QCD

- **Confinement of both static and dynamical quarks** $\longrightarrow$

  $$W(C) = \langle \mathrm{Tr}\ \mathrm{P}\ e^{i\int_C dz_\mu \hat{A}_\mu} \rangle$$
  $$S(x,y) = \langle \psi(y)\bar{\psi}(x) \rangle$$

- **Dynamical Breaking of chiral** $SU_L(N_f)\times SU_R(N_f)$ **symmetry** $\longrightarrow \langle \bar{\psi}(x)\psi(x) \rangle$

- $U_A(1)$ **Problem** $\longrightarrow \eta'$ (χ, Axial Anomaly)

- **Strong CP Problem** $\longrightarrow Z(\theta)$

- **Colorless Hadron Formation:** $\longrightarrow$ Effective action for colorless collective modes:
  hadron masses,
  form factors, scattering

  Light mesons, **Regge spectrum** of excited states of light hadrons,
  heavy-light hadrons, **heavy quarkonia**

- **Deconfinement, chiral symmetry restoration under "extreme" conditions**

QFT effective action and vacuum field configurations
Scalar field

In Euclidean functional integral for QFT with classical action

$$S_V = \int\limits_V d^d x L(x)$$

one has to define the functional space of integration:

$$Z_V = N\mathrm{reg}\int\limits_{\mathcal{F}_\phi} \mathrm{D}\phi \exp\{\mathrm{S_V}[\phi]\} \qquad \mathcal{F}_\phi = \left\{\phi : \lim_{\mathrm{V}\to\infty} \frac{1}{\mathrm{V}} \int_{\mathrm{V}} \mathrm{d^d x}\phi^2(\mathrm{x}) = \phi_0\right\}.$$

separation of the component φ_0 with extensive action:

$$\phi(x) = \varphi_0 + \varphi(x), \qquad \mathcal{F}_\varphi = \left\{\varphi : \lim_{V\to\infty} \frac{1}{V} \int_V d^d x \varphi^2(x) = 0\right\}.$$

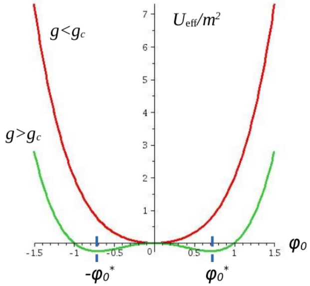
$$1 = \int\limits_{-\infty}^{+\infty} d\phi_0 \int\limits_{\mathcal{F}_\varphi} D\varphi \delta\left[\phi - \varphi_0 - \varphi\right] \delta\left[\phi_0^2 - V^{-1}\int_V d^d x \phi^2(x)\right]$$

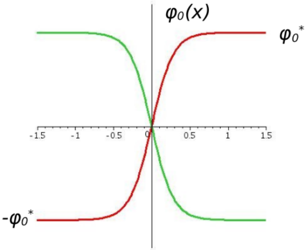
$$Z_V = N\mathrm{reg}\int\limits_{\mathcal{F}_{\varphi_0}} \mathrm{D}\varphi_0 \int\limits_{\mathcal{F}_\varphi} \mathrm{D}\varphi \exp\{\mathrm{S_V}[\varphi+\varphi_0]\} = \mathrm{N'}\int\limits_{\mathcal{F}_{\varphi_0}} \mathrm{D}\varphi_0 \exp\{-\mathrm{VU_{eff}}[\varphi_0]\}$$

$$U_{\mathrm{eff}}[\varphi_0] = -\lim_{V\to\infty} V^{-1} S^V_{\mathrm{eff}}[\varphi_0]$$

$$L = \frac{1}{2}\phi(x)\partial^2\phi(x) - m^2\phi^2(x) - \frac{g}{4}\phi^4(x), \quad x \in R^d, \quad m^2 > 0, \quad g > 0$$

Dynamical breaking of the discrete Z_2 symmetry $\phi \to -\phi$ in the strong coupling limit



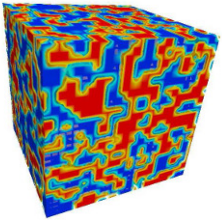


Minima $\pm\varphi_0^*(g)$ of the effective potential $U_{\rm eff}[\varphi_0]$ - quantum vacua in the strong coupling limit. Z_2 kink and anti-kink configurations - domain wall;

$$\bar{\varphi}_0(x) = \pm\varphi_0^*(g)\,\tanh\left((x-a)m_{\rm eff}(g)/\sqrt{2}\right)$$

Superposition of kinks and anti-kinks describes domain wall networks
TANMAY VACHASPATI, KINKS AND DOMAIN WALLS: An Introduction to Classical and Quantum Solitons, Cambridge University Press, 2006

Figure 6.6 The distribution of two phases (black and white) on a square lattice in two spatial dimensions. Domain walls lie at the interface of the black and white regions.



Lattice QCD confining configurations (P.J. Moran, D.B. Leinweber, arXiv:0805.4246v1 [hep-lat])

$$\lim_{V\to\infty} Z_V = N' \int\limits_{\mathcal{F}_{\varphi_0}} D\varphi_0 \exp\{-VU_{\rm eff}[\varphi_0]\}$$

$$\approx \lim_{V\to\infty} N'' \mathrm{reg} \int\limits_{\mathcal{F}_{\bar\varphi_0}} \mathrm{D}\mu[\bar\varphi_0] \int\limits_{\mathcal{F}_{\varphi}} \mathrm{D}\varphi \exp\{\mathrm{S_V}[\varphi+\bar\varphi_0] - \mathrm{S_V}[\bar\varphi_0]\}$$

Here $\mathcal{F}_{\bar\varphi_0}$ - statistical ensemble of domain wall networks which describe almost everywhere homogeneous mean field, and $\mathcal{F}_{\varphi}$ - small localized fluctuations φ with effective mass $m_{\rm eff}(g)$ in the presence of almost everywhere constant background vacuum field $\bar\varphi_0$.

QCD

$$L = \bar{\psi}_f(x)\left(i\not{D} - m_f\right)\psi_f(x) - \frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu}$$

$$t^a F^a_{\mu\nu} = i\left[D_\mu, D_\nu\right], \quad F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + if^{abc}A^b_\mu A^c_\nu.$$

Color $SU(3)$ $\quad \psi^U_f = U(x)\psi, \quad A^U_\mu = U(x)\left[t_a A^a_\mu + i\partial_\mu\right]U^{-1}(x)$

Flavour $SU(N_f)$: $\quad \psi_f \to U^{ff'}\psi_{f'}, \quad U = \exp\{i\theta^a t^a\}, \quad [t^a, t^b] = if^{abc}t^c$

Chiral symmetry:

$$\gamma_5\psi_\pm = \pm\psi_\pm.$$

$$P_\pm = \frac{1}{2}(I \pm \gamma_5), \quad P_+ + P_- = I, \quad P_+P_- = 0, \quad P_\pm^2 = P_\pm,$$

$$\psi_\pm = P_\pm\psi, \quad \psi = \psi_+ + \psi_-,$$

$$L = i\bar{\psi}^f_+ \not{D}\psi^f_+ + i\bar{\psi}^f_- \not{D}\psi^f_- - m_f(\bar{\psi}^f_+\psi^f_- + \bar{\psi}^f_-\psi^f_+)$$

For $m_f = 0$,

$$L = i\bar{\psi}^f_+ \not{D}\psi^f_+ + i\bar{\psi}^f_- \not{D}\psi^f_-.$$

Lagrangian is invarint with respect to $SU_{\mathrm{R}}(N_f) \times SU_{\mathrm{L}}(N_f)$:

$$\psi_+ \to e^{i\theta^a t^a}\psi_+, \qquad \psi_- \to e^{i\chi^b t^b}\psi_-,$$

θ^a and χ^a – independent parameters. Lagrangian is also invariant with respect to

$$\psi \to e^{i\gamma_5\alpha}\psi \quad (\psi_+ \to e^{i\alpha}\psi_+, \quad \psi_- \to e^{-i\alpha}\psi_-)$$

that is axial $U_{\mathrm{A}}(1)$ symmetry.

Faddeev-Popov quantzation

$$Z = N \int\limits_{\mathcal{F}} DA \exp\{-S[A]\}, \quad L = -\frac{1}{4} F^a_{\mu\nu} F^a_{\mu\nu}$$

$$\mathcal{F} = \{A : S[A] = \int\limits_{-\infty}^{\infty} d^4x F^a_{\mu\nu}(x) F^a_{\mu\nu}(x) < \infty\}, \qquad A^U_\mu = U(x) \left[t_a A^a_\mu + i\partial_\mu \right] U^{-1}(x).$$

$\mathcal{F}$ - fields with finite action. Functional space $\mathcal{F}$ contains gauge copies

$$\partial_\mu A_\mu = 0, \qquad 1 = \Phi[A] \int\limits_{\Omega} D\omega \delta[\partial_\mu A^\omega_\mu],$$

$$Z = N \int\limits_{\mathcal{F}} DA \Phi[A] \int\limits_{\Omega} D\omega \delta[\partial_\mu A^\omega_\mu] \exp\{-S[A]\} = N \int\limits_{\Omega} D\omega \int\limits_{\mathcal{F}} DA \Phi[A] \delta[\partial_\mu A^\omega_\mu] \exp\{-S[A]\}.$$

$$Z = N \int\limits_{\Omega} D\omega \int\limits_{\mathcal{F}} DA \Phi[A] \delta[\partial_\mu A_\mu] \exp\{-S[A]\} = N' \int\limits_{\mathcal{F}} DA \Phi[A] \delta[\partial_\mu A_\mu] \exp\{-S[A]\},$$

$$\Phi^{-1}[A] = \int\limits_{\Omega} D\omega \delta[\partial_\mu A^\omega_\mu], \quad \partial_\mu A^{a,\omega}_\mu = \partial_\mu A^a_\mu + \partial_\mu \left(f^{abc} A^b_\mu \omega^c + \partial_\mu \omega^a \right) = \partial_\mu A^a_\mu + \partial_\mu D^{ab}_\mu \omega^b$$

$$D^{ac}_\mu = \partial_\mu \delta^{ac} + f^{abc} A^b_\mu, \quad \Phi^{-1}[A] = \int\limits_{\Omega} D\omega \delta[\partial_\mu D_\mu \omega] = \frac{1}{\det\left[\partial_\mu D_\mu\right]}.$$

$$Z = N' \int\limits_{\mathcal{F}} DA \delta[\partial_\mu A_\mu] \det\left[\partial_\mu D_\mu\right] \exp\{-S[A]\}$$

Gribov problem!

QCD effective action and vacuum gluon configurations

In Euclidean functional integral for YM theory one has to allow the gluon condensates to be nonzero:

$$Z = N \int\limits_{\mathcal{F}_B} DA \int\limits_{\Psi} D\psi D\bar{\psi} \exp\{-S[A,\psi,\bar{\psi}]\}$$

$$\mathcal{F}_B = \left\{ A : \lim_{V\to\infty} \frac{1}{V} \int_V d^4x g^2 F^a_{\mu\nu}(x) F^a_{\mu\nu}(x) = B^2 \right\}.$$

B.V. Galilo and S.N. ,
Phys. Rev. D84 (2011) 094017

L. D. Faddeev,
[arXiv:0911.1013 [math-ph]]

H. Leutwyler,
Nucl. Phys. B 179 (1981) 129

$A^a_\mu = B^a_\mu + Q^a_\mu$, background gauge fixing condition $D(B)Q = 0$:

$$1 = \int\limits_{\mathcal{B}} DB\Phi[A,B] \int\limits_{\mathcal{Q}} DQ \int\limits_{\Omega} D\omega \delta[A^\omega - Q^\omega - B^\omega]\delta[D(B^\omega)Q^\omega]$$

Q^a_μ – local (perturbative) fluctuations of gluon field with zero gluon condensate: $Q \in \mathcal{Q}$;
B^a_μ are long range field configurations with nonzero condensate: $B \in \mathcal{B}$.

$$Z = N' \int\limits_{\mathcal{B}} DB \int\limits_{\mathcal{Q}} DQ \int\limits_{\Psi} D\psi D\bar{\psi} \det[D(B)D(B+Q)]\delta[D(B)Q] \exp\{-S[B+Q,\psi,\bar{\psi}]\}$$

Particular features of background fields B have yet to be identified by the dynamics of fluctuations:

$$Z = N' \int\limits_{\mathcal{B}} DB \int\limits_{\Psi} D\psi D\bar\psi \int\limits_{\mathcal{Q}} DQ \det[D(B)D(B+Q)]\delta[D(B)Q] \exp\{-S[B+Q,\psi,\bar\psi]\}$$
$$= N' \int\limits_{\mathcal{B}} DB \exp\{-S_{\rm eff}[B]\}$$

Global minima of $S_{\rm eff}[B]$ – field configurations that are dominant in the limit $V \to \infty$.

$$Z = N'' \int\limits_{\mathcal{B}_{\rm vac}} D\sigma_B \int\limits_{\Psi} D\psi D\bar\psi \int\limits_{\mathcal{Q}} DQ \det[D(B)D(B+Q)]\delta[D(B)Q] \exp\{-S[B+Q,\psi,\bar\psi]\}$$

Background (vacuum) fields $\mathcal{B}_{\rm vac}$ have to be treated nonperturbatively, while the rest of fields are fluctuations - perturbation expansions.

QCD vacuum as a medium characterized by certain condensates,
quarks and gluons - elementary coloured excitations (confined),
mesons and baryons - collective colorless excitations

L.D. Faddeev,
arXiv:1509.06186 [hep-th], Theor.Math.Phys. 181 (2014), arXiv:0911.1013[hep-th]

In general, **division of the gauge field potential into the (nonperturbative) background and fluctuation parts** is a complicated task.

Specifically, a separation of the Abelian part from the general gauge field requires application of the special gauge field parametrization suggested by L.D. Faddeev, A. J. Niemi (2007); K.-I. Kondo, T. Shinohara, T. Murakami(2008); Y.M. Cho (1980, 1981); L.Prokhorov, S.V. Shabanov (1989,1999)
The Abelian part $\hat{V}_\mu(x)$ of the gauge field $\hat{A}_\mu(x)$ is separated manifestly,

$$\begin{aligned}
\hat{A}_\mu(x) &= \hat{V}_\mu(x) + \hat{X}_\mu(x),\ \hat{V}_\mu(x) = \hat{B}_\mu(x) + \hat{C}_\mu(x), \\
\hat{B}_\mu(x) &= [n^a A^a_\mu(x)]\hat{n}(x) = B_\mu(x)\hat{n}(x), \\
\hat{C}_\mu(x) &= g^{-1}\partial_\mu \hat{n}(x) \times \hat{n}(x), \\
\hat{X}_\mu(x) &= g^{-1}\hat{n}(x) \times \left(\partial_\mu \hat{n}(x) + g\hat{A}_\mu(x) \times \hat{n}(x)\right),
\end{aligned}$$

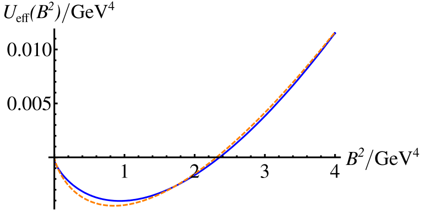
where $\hat{A}_\mu(x) = A^a_\mu(x)t^a$, $\hat{n}(x) = n_a(x)t^a$, $n^a n^a = 1$, and

$$\partial_\mu \hat{n} \times \hat{n} = if^{abc}\partial_\mu n^a n^b t^c,\ [t^a, t^b] = if^{abc}t^c.$$

$$[\hat{V}_\mu(x), \hat{V}_\nu(x)] = 0$$

Homogeneous Abelian (anti-)self-dual fields are of particular interest.

$$B_\mu = -\frac{1}{2} n B_{\mu\nu} x_\nu,\ \tilde{B}_{\mu\nu} = \pm B_{\mu\nu}$$
$$n = T^3 \ \cos\xi + T^8 \ \sin\xi.$$

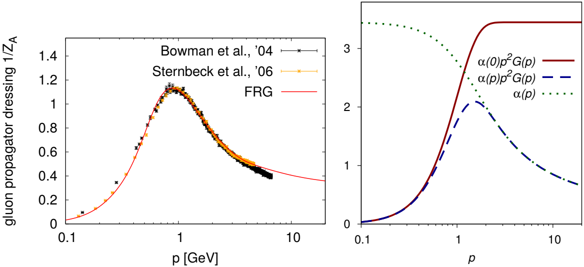


H. Pagels, and E. Tomboulis, Nucl. Phys. B 143 (1978) 485
P. Minkowski, Nucl. Phys. B177 (1981) 203
H. Leutwyler, Nucl. Phys. B 179 (1981) 129

A. Eichhorn, H. Gies and J.M. Pawlowski, Phys. Rev. D 83, 045014 (2011)

Discrete symmetries of the effective action – Weyl reflections & CP

(SN, Kalloniatis, B. Galilo, V. Voronin, PRD (2001, 2011, 2015); K.-I. Kondo *et al* Phys. Rep. (2015); Y.M. Cho *et al* PRD (2012))



A signature of self-consistency:

Functional RG (used for evaluation of U_{eff}) *vs* mean field propagators

Analytical properties of the quark and gluon propagators

$$G\left(z^2\right) \sim \frac{e^{-Bz^2}}{z^2}, \quad \tilde{G}\left(p^2\right) \sim \frac{1}{p^2}\left(1 - e^{-p^2/B}\right)$$

$\Rightarrow$ dynamical color confinement

H. Leutwyler, Phys. Lett. B 96 (1980) 154

Analytical properties of polarization diagram

$\Rightarrow$ confinement

A.I. Milstein, Yu. Pinelis, Phys. Lett. B 137 (1984)

$\Rightarrow$ Regge mass spectrum of mesons

G.V. Efimov, and S.N., Phys. Rev. D 51 (1995)

(Anti-)self-duality $\Rightarrow$ Chiral symmetry

H. Leutwyler, Phys. Lett. B 96 (1980) 154

A.I. Milshtein, Yu.F. Pinelis, Z.Phys. C27 (1985).
A.G. Grozin, Yu.F. Pinelis, Z.Phys. C33 (1987)

Gluon condensates and domain wall network

Pure gluodynamics - Ginzburg-Landau approach:

$$\mathcal{L}_{\rm eff} = -\frac{1}{4\Lambda^2}\left(D^{ab}_{\nu}F^{b}_{\rho\mu}D^{ac}_{\nu}F^{c}_{\rho\mu} + D^{ab}_{\mu}F^{b}_{\mu\nu}D^{ac}_{\rho}F^{c}_{\rho\nu}\right) - U_{\rm eff}$$

$$U_{\rm eff} = \frac{\Lambda^4}{12}{\rm Tr}\left(C_1F^2 + \frac{4}{3}C_2F^4 - \frac{16}{9}C_3F^6\right),$$

B.V. Galilo, S.N. , Phys. Part. Nucl. Lett., 8 (2011) 67

D. P. George, A. Ram, J. E. Thompson and R. R. Volkas, Phys. Rev. D 87, 105009 (2013) [arXiv:1203.1048 [hep-th]]

where

$$D^{ab}_{\mu} = \delta^{ab}\partial_{\mu} - iA^{ab}_{\mu} = \partial_{\mu} - iA^{c}_{\mu}(T^c)^{ab},$$

$$F^{a}_{\mu\nu} = \partial_{\mu}A^{a}_{\nu} - \partial_{\nu}A^{a}_{\mu} - if^{abc}A^{b}_{\mu}A^{c}_{\nu},$$

$$F_{\mu\nu} = F^{a}_{\mu\nu}T^a, \quad T^{a}_{bc} = -if^{abc}$$

$$C_1 > 0,\ C_2 > 0,\ C_3 > 0.$$

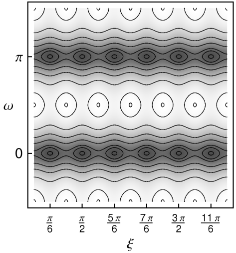
U_{eff} possesses degenerate discrete minima:

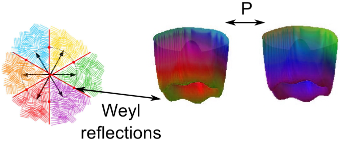
$$B_\mu = -\frac{1}{2} n_k B_{\mu\nu} x_\nu,\ \tilde{B}_{\mu\nu} = \pm B_{\mu\nu},$$

matrix n_k belongs to the Cartan subalgebra of $su(3)$

$$n_k = T^3\ \cos\left(\xi_k\right) + T^8\ \sin\left(\xi_k\right), \quad \xi_k = \frac{2k+1}{6}\pi,\ k = 0, 1, \ldots, 5,$$

$$\vec{E}\vec{H} = B^2 \cos(\omega)$$





Discrete minima related by Weyl reflections and CP $\leftrightarrow$ **kinks and solitons**

L.D. Faddev

[In "Diakonov, D. (ed.): Subtleties in quantum field theory", arXiv:1003.4854 [hep-th]]:

"Quantum equations could have soliton solutions, which are absent in the classical limit. In particular, it is not completely crazy idea that quantum Yang-Mills equations have soliton-like solutions due to the dimensional transmutation."

Search for solutions of the effective nonlinear equations, derived from the Ginzburg-Landau-like effective action.

Field strength: $\mathrm{Tr}(F^2(x))$

Topological charge density: $\mathrm{Tr}\tilde{F}(x)F(x)\rangle$

Color "orientation": $n(x) \leftrightarrow \xi(x)$

Domain wall network

ξ, $\langle g^2F^2\rangle \longrightarrow$ vacuum values

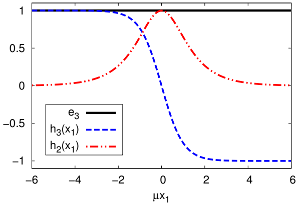
$$\mathcal{L}_{\rm eff} = -\frac{1}{2}\Lambda^2 b_{\rm vac}^2 \partial_\mu\omega\partial_\mu\omega - b_{\rm vac}^4\Lambda^4\left(C_2+3C_3b_{\rm vac}^2\right)\sin^2\omega,$$

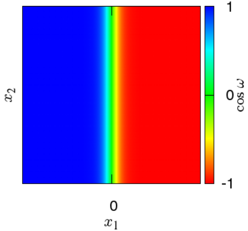
leads to sine-Gordon equation

$$\partial^2\omega = m_\omega^2\sin 2\omega, \quad m_\omega^2 = b_{\rm vac}^2\Lambda^2\left(C_2+3C_3b_{\rm vac}^2\right),$$

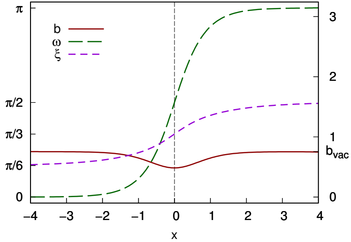
and the standard kink solution

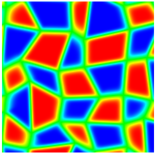
$$\omega(x_\nu) = 2\ \mathrm{arctg}\left(\exp(\mu x_\nu)\right)$$

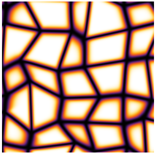




"Domain wall involving the topological charge density (in units of $\langle g^2 F^2 \rangle$), $su(3)$ angle ξ and the background action density simultaneously"







The general kink configuration can be parametrized as

$$\zeta(\mu_i, \eta^i_\nu x_\nu - q^i) = \frac{2}{\pi} \arctan \exp(\mu_i(\eta^i_\nu x_\nu - q^i)).$$

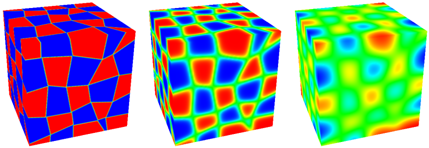
A single lump in two, three and four dimensions is given by

$$\omega(x) = \pi \prod_{i=1}^{k} \zeta(\mu_i, \eta^i_\nu x_\nu - q^i).$$

for $k = 4, 6, 8$, respectively. The general kink network is then given by the additive superposition of lumps

$$\omega = \pi \sum_{j=1}^{\infty} \prod_{i=1}^{k} \zeta(\mu_{ij}, \eta^{ij}_\nu x_\nu - q^{ij})$$

S.N., V.E. Voronin, Eur.Phys.J. A51 (2015) 4



$\langle F^2 \rangle = B^2$
$\langle |F\widetilde{F}| \rangle = B^2$

$\langle F^2 \rangle = B^2$
$\langle |F\widetilde{F}| \rangle \ll B^2$

"Phase transitions and heterophase fluctuations" V. I. Yukalov, Phys. Rep. 208, 396 (1991)

What could stabilize a finite mean size of the domains?

Lower dimensional defects?

Quark (quasi-)zero modes?

Domain bulk - harmonic confinement

Elementary color charged excitations - fluctuations, eigenmodes decay in all four directions.

Eigenvalue problem for scalar field in $\mathbb{R}^4$: *H. Leutwyler, Nucl. Phys. B 179 (1981);*

$$B_\mu = B_{\mu\nu}x_\nu, \widetilde{B}_{\mu\nu} = \pm B_{\mu\nu}, B_{\mu\alpha}B_{\nu\alpha} = B^2\delta_{\mu\nu}.$$

$$-\left(\partial_\mu - iB_\mu\right)^2 G = \delta \quad \longrightarrow \quad G\left(x-y\right) \sim \frac{e^{-B(x-y)^2/4}}{(x-y)^2}$$

$$-\left(\partial_\mu - i\breve{B}_\mu\right)^2 \Phi = \lambda\Phi \quad \longrightarrow \quad \left[\beta_\pm^+\beta_\pm + \gamma_+^+\gamma_+ + 1\right]\Phi = \frac{\lambda}{4B}\Phi,$$

$$\beta_\pm = \frac{1}{2}(\alpha_1 \mp i\alpha_2), \quad \gamma_\pm = \frac{1}{2}(\alpha_3 \mp i\alpha_4), \quad \alpha_\mu = \frac{1}{\sqrt{B}}x_\mu + \partial_\mu,$$

$$\beta_\pm^+ = \frac{1}{2}(\alpha_1^+ \pm i\alpha_2^+), \quad \gamma_\pm^+ = \frac{1}{2}(\alpha_3^+ \pm i\alpha_4^+), \quad \alpha_\mu^+ = \frac{1}{\sqrt{B}}x_\mu - \partial_\mu.$$

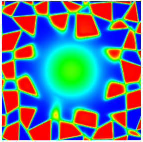
The eigenfunctions and eigenvalues - 4-dim. harmonic oscillator

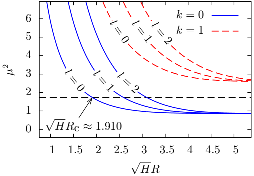
$$\Phi_{nmkl}(x) = \frac{1}{\pi^2\sqrt{n!m!k!l!}}\left(\beta_+^+\right)^k\left(\beta_-^+\right)^l\left(\gamma_+^+\right)^n\left(\gamma_-^+\right)^m\Phi_{0000}, \; \Phi_{0000} = e^{-\frac{1}{2}Bx^2}$$

$$\lambda_r = 4B(r+1), \; r = k+n \text{ self-dual field}, \; r = l+n \text{ anti-self-dual field}$$

Domain wall junctions - deconfinement

S.N. , V.E. Voronin, Eur.Phys.J. A51 (2015) 4





The color charged scalar field inside junction:

$$-\left(\partial_\mu - i\breve{B}_\mu\right)^2 \Phi = 0, \quad \Phi(x) = 0, \quad x \in \partial\mathcal{T}, \quad \mathcal{T} = \left\{x_1^2 + x_2^2 < R^2, \ (x_3, x_4) \in \mathrm{R}^2\right\}$$

The solutions are quasi-particle excitations

$$\phi^a(x) = \sum_{lk} \int_{-\infty}^{+\infty} \frac{dp_3}{2\pi} \frac{1}{\sqrt{2\omega_{alk}}} \left[a^+_{akl}(p_3) e^{ix_0\omega_{akl} - ip_3x_3} + b_{akl}(p_3) e^{-ix_0\omega_{akl} + ip_3x_3}\right] e^{il\vartheta} \phi_{alk}(r),$$

$$\phi^{a\dagger}(x) = \sum_{lk} \int_{-\infty}^{+\infty} \frac{dp_3}{2\pi} \frac{1}{\sqrt{2\omega_{alk}}} \left[b^+_{akl}(p_3) e^{-ix_0\omega_{akl} + ip_3x_3} + a_{akl}(p_3) e^{ix_0\omega_{akl} - ip_3x_3}\right] e^{-il\vartheta} \phi_{alk}(r),$$

$$p_0^2 = p_3^2 + \mu_{akl}^2, \quad p_0 = \pm\omega_{akl}(p_3), \quad \omega_{akl} = \sqrt{p_3^2 + \mu_{akl}^2},$$

$$k = 0, 1, \dots, \infty, \quad l \in \mathbb{Z},$$

Topology driven disorder

In general near the boundaries $\mathrm{div}\vec{H} \neq 0, \quad \mathrm{div}\vec{E} \neq 0$
The description of the domain walls as well as separation of the Abelian part in the general network in terms of the vector potential requires application of the gauge field parametrization suggested by L.D. Faddeev, A. J. Niemi (2007); K.-I. Kondo, T. Shinohara, T. Murakami(2008); Y.M. Cho (1980, 1981); L.Prokhorov, S.V. Shabanov (1989,1999)
The Abelian part $\hat{V}_\mu(x)$ of the gauge field $\hat{A}_\mu(x)$ is separated manifestly,

$$\begin{aligned}
\hat{A}_\mu(x) &= \hat{V}_\mu(x) + \hat{X}_\mu(x), \, \hat{V}_\mu(x) = \hat{B}_\mu(x) + \hat{C}_\mu(x), \\
\hat{B}_\mu(x) &= [n^a A^a_\mu(x)]\hat{n}(x) = B_\mu(x)\hat{n}(x), \\
\hat{C}_\mu(x) &= g^{-1}\partial_\mu \hat{n}(x) \times \hat{n}(x), \\
\hat{X}_\mu(x) &= g^{-1}\hat{n}(x) \times \left(\partial_\mu \hat{n}(x) + g\hat{A}_\mu(x) \times \hat{n}(x)\right),
\end{aligned}$$

where $\hat{A}_\mu(x) = A^a_\mu(x)t^a$, $\hat{n}(x) = n_a(x)t^a$, $n^a n^a = 1$, and

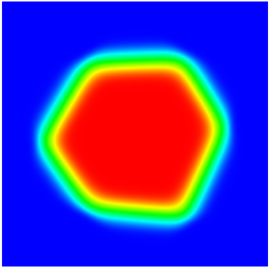
$$\partial_\mu \hat{n} \times \hat{n} = i f^{abc} \partial_\mu n^a n^b t^c, \; [t^a, t^b] = i f^{abc} t^c.$$

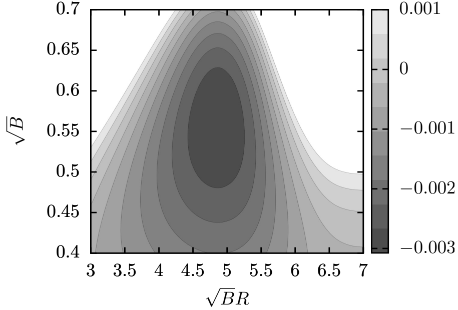
$$[\hat{V}_\mu(x), \hat{V}_\nu(x)] = 0$$

Both the color and space orientation of the field can become frustrated at the junction location and, thus, develop the singularities in the vector potential. Potentially singularities cover the whole range of defects – vortex-like, monopole/dyon-like and instanton-like defects.

Energy driven disorder

Finite size effects in the free energy density for Abelian (anti-)self-dual gluon field in $SU(3)$ gluodynamics.



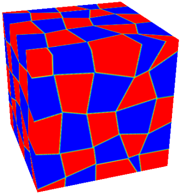


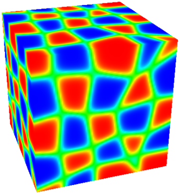
Minimum of free energy density at finite values of the **field strength and size of the domain**.

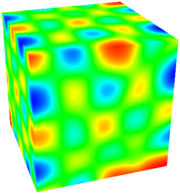
$$U = V_R F(B,R) = U^{\rm cl} + \delta U$$
$$= \frac{\pi^2 B^2 R^4}{2g^2} - {\rm Tr}\ln\left[-\breve{D}^2\right] + \frac{1}{2}{\rm Tr}\ln\left[-\breve{D}^2\delta_{\mu\nu} + 2i\breve{B}_{\mu\nu}\right]' - {\rm Tr}\ln\left[\hat{\not{D}}\right]' + \delta U_0$$

Normalization $U|_{B=0} = 0$.

V.Voronin & SN Phys.Rev.D 95 (2017) 7, 074038; Phys.Rev.D 103 (2021)







$\langle F^2 \rangle = B^2$
$\langle |F\widetilde{F}| \rangle = B^2$

Confinement - colorless hadrons

$\langle F^2 \rangle = B^2$
$\langle |F\widetilde{F}| \rangle \ll B^2$

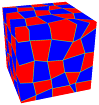
Deconfinement - color charged quasi-particles

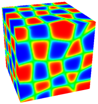
"Phase transitions and heterophase fluctuations",
V. I. Yukalov, Phys. Rep. 208, 396 (1991)

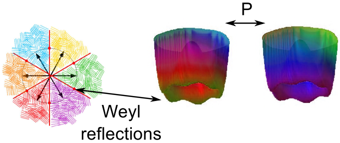
An ensemble of almost everywhere (in R^4) homogeneous Abelian (anti-)self-dual gluon fields

$$\langle : g^2 F^2 : \rangle \neq 0, \qquad \chi = \int d^4x \langle Q(x)Q(0)\rangle \neq 0, \qquad \langle Q(x)\rangle = 0$$

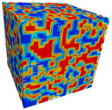
Topological charge density $Q(x) = \frac{g^2}{32\pi^2} F^a_{\mu\nu}(x)\tilde{F}^a_{\mu\nu}(x)$







Domain wall network (S.N., V.E. Voronin, EPJA (2015); A. Kalloniatis, S.N., PRD (2001))



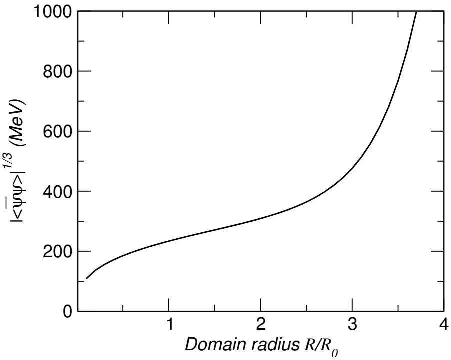
Lattice confining configuration (P.J. Moran, D.B. Leinweber, arXiv:0805.4246v1 [hep-lat])

Testing the model: characteristics of the domain wall network ensemble

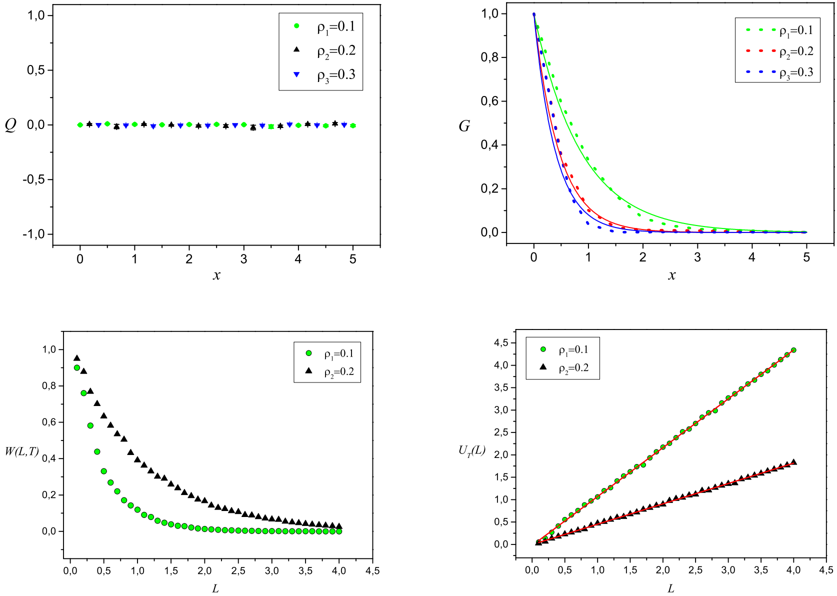
Spherical domains

A.C. Kalloniatis and S.N. , Phys. Rev. D 64 (2001); Phys. Rev. D 69 (2004); Phys. Rev. D 71 (2005); Phys. Rev. D 73 (2006), Eur.Phys.J. A51 (2015), arXiv:1603.01447 [hep-ph] (2016)

Area law
Spontaneous chiral symmetry breaking
$U_A(1)$ is broken by anomaly
There is no strong CP violation



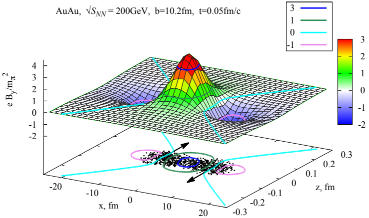
Pure glue, domains - tubes with two finite dimensions (mean topological charge, two-point correlator of top. charge density, Wilson loop and static potential)

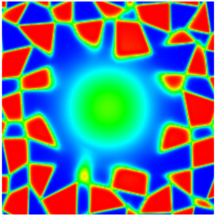


"Polarization" of QCD vacuum by the strong electromagnetic fields

- **Relativistic heavy ion collisions - strong electromagnetic fields**

V. Skokov, A. Y. Illarionov and V. Toneev, Int. J. Mod. Phys. A **24** *(2009) 5925*
V. Voronyuk, V. D. Toneev, W. Cassing, E. L. Bratkovskaya,
V. P. Konchakovski and S. A. Voloshin, Phys. Rev C 84 (2011)





Strong electro-magnetic field plays catalyzing role for deconfinement and anisotropies!

One-loop quark contribution to the effective potential in the background of Abelian self-dual $SU(3)$ gauge field and electromagnetic field

$$(i\not{D}-m)S(x,y)=-\delta(x-y)$$
$$\not{D}=\gamma_\mu D_\mu,\quad D_\mu=\partial_\mu-iG_\mu,\quad \{\gamma_\mu,\gamma_\nu\}=-2\delta_{\mu\nu}$$

$$G_\mu=\hat{B}_\mu+A_\mu,$$
$$G_\mu=-\frac{1}{2}G_{\mu\nu}x_\nu,\quad G_{\mu\nu}=F_{\mu\nu}+nB_{\mu\nu},\quad \tilde{G}_{\mu\nu}=\frac{1}{2}\varepsilon_{\mu\nu\alpha\beta}G_{\alpha\beta}$$

$$G_{ij}=\varepsilon_{ijk}\mathcal{H}_k,\quad G_{4k}=\mathcal{E}_k,$$
$$\vec{\mathcal{H}}=n\vec{B}+\vec{H},\quad \vec{\mathcal{E}}=\pm n\vec{B}+\vec{E},$$
$$\vec{\mathcal{H}}\vec{\mathcal{E}}=\frac{1}{4}G_{\mu\nu}\tilde{G}_{\mu\nu}=\mathcal{Q},\quad \frac{1}{2}(\vec{\mathcal{H}}^2+\vec{\mathcal{E}}^2)=\frac{1}{4}G_{\mu\nu}G_{\mu\nu}=\mathcal{R}.$$

$$\begin{aligned}\mathcal{Z}(G) &= \mathcal{Z}^{-1}(0)\int D\psi D\bar{\psi}\exp\left\{\int d^4x\bar{\psi}(x)\left(i\not{D}-m\right)\psi(x)\right\}\\ &= \frac{\det(i\not{D}-m)}{\det(i\not{\partial}-m)}=\exp\left\{-VU_{\rm eff}(G)\right\}\end{aligned}$$

$$U_{\rm eff}(G)=-\frac{1}{V}\mathrm{Tr}\ln\left(\frac{i\not{D}-m}{i\not{\partial}-m}\right)$$

$$\mathrm{Tr}\ln\left(\frac{m-i\not{D}}{m-i\not\partial}\right) = -\mathrm{Tr}\int_0^\infty \frac{d\alpha}{\alpha}\left[e^{-\alpha(m-i\not{D})} - e^{-\alpha(m-i\not\partial)}\right]$$

$$= -\mathrm{Tr}\int_m^\infty dm\left[\frac{1}{m-i\not{D}} - \frac{1}{m-i\not\partial}\right]$$

$$U_{\rm eff}(G) = \mathrm{Tr}\int_m^\infty dm\,[S(x,x) - S_0(x,x)]\,.$$

$$S(x,y) = \frac{1}{m-i\not{D}_x}\delta(x-y) = (m+i\not{D}_x)H(x,y)$$

$$H(x,y) = \frac{1}{m^2+\not{D}^2}\delta(x-y).$$

$$\not{D}^2 = D_\mu D_\nu\left(\frac{1}{2}\{\gamma_\mu,\gamma_\nu\} + \frac{1}{2}[\gamma_\mu,\gamma_\nu]\right) = -D^2 + \frac{i}{2}[D_\mu,D_\nu]\sigma_{\mu\nu} = -D^2 + \frac{1}{2}G_{\mu\nu}\sigma_{\mu\nu}.$$

$$\not{D}^2 = -D^2 + \frac{1}{2}\sigma_{\mu\nu}G_{\mu\nu} = -D^2 + \frac{1}{2}(\sigma G),$$

$$G_{\mu\nu} = i[D_\mu,D_\nu], \quad \sigma_{\mu\nu} = \frac{1}{2i}[\gamma_\mu,\gamma_\nu].$$

$$\begin{aligned}(\sigma G)P_{\pm} &= \frac{1}{2}(\sigma G) \pm \frac{1}{2}[-\frac{1}{2}\varepsilon_{\alpha\beta\mu\nu}\sigma_{\mu\nu}G_{\alpha\beta}] \\ &= \frac{1}{2}(\sigma G) \mp \frac{1}{2}(\sigma\tilde{G})\end{aligned}$$

$$(\sigma G)P_{\pm} = \frac{1}{2}\sigma_{\mu\nu}[G_{\mu\nu} \mp \tilde{G}_{\mu\nu}]$$

$$\begin{aligned}\exp\left\{-\frac{1}{2}(\sigma G)\right\} &= P_{+}\cosh(s|\vec{\mathcal{E}} - \vec{\mathcal{H}}|) + P_{-}\cosh(s|\vec{\mathcal{E}} + \vec{\mathcal{H}}|) \\ &- \frac{1}{2}\sigma_{\mu\nu}[G_{\mu\nu} - \tilde{G}_{\mu\nu}]\frac{\sinh(s|\vec{\mathcal{E}} - \vec{\mathcal{H}}|)}{|\vec{\mathcal{E}} - \vec{\mathcal{H}}|} \\ &- \frac{1}{2}\sigma_{\mu\nu}[G_{\mu\nu} + \tilde{G}_{\mu\nu}]\frac{\sinh(s|\vec{\mathcal{E}} + \vec{\mathcal{H}}|)}{|\vec{\mathcal{E}} + \vec{\mathcal{H}}|}\end{aligned}$$

$$e^{sD^2}\delta(x-y) = ?$$

$$\exp\{sD^2\}\delta(x-y) = \int \delta a_\mu P_\beta \exp\left\{ -\int\limits_0^1 d\beta a^2(\beta) + 2\sqrt{s}\int_0^1 d\beta a_\mu(\beta) D_\mu \right\} \delta(x-y)$$

$$a(\beta) = a_0 + \sum_{n=1}^{\infty} \left[a_n \cos(2\pi n\beta) + b_n \sin(2\pi n\beta)\right] = a_0 + \bar{a}(\beta)$$

$$\left\{1, \sqrt{2}\cos(2\pi n\beta), \quad \sqrt{2}\sin(2\pi n\beta)\right\}_{n=1}^{\infty}$$

$$\int\limits_0^1 d\beta \cos(2\pi n\beta)\cos(2\pi m\beta) = \int\limits_0^1 d\beta \sin(2\pi n\beta)\sin(2\pi m\beta) = \frac{1}{2}\delta_{nm}$$

$$D_\mu = \partial_\mu - iG_\mu(x) = \partial_\mu + \frac{i}{2}G_{\mu\nu}x_\nu$$

$$\exp\{sD^2\}\delta(x-y) = \prod_{n=1}^{\infty}\int \frac{d^4a_n d^4b_n d^4a_0}{\pi^2(2\pi)^2(2\pi)^2} \exp\left\{-a_0^2 - \frac{1}{2}a_n^2 - \frac{1}{2}b_n^2\right\}$$

$$\times \exp\left\{ i\sqrt{s}\int\limits_0^1 d\beta \left[a_0 + \bar{a}(\beta)\right]_\mu G_{\mu\nu}\left[x_\nu + 2\sqrt{s}\int\limits_\beta^1 d\beta'(a_0 + \bar{a}(\beta'))_\nu\right]\right\} \delta(x - y + 2\sqrt{s}a_0)$$

$$\exp\{sD^2\}\delta(x-y) = \prod_{n=1}^{\infty}\int \frac{d^4a_n d^4b_n}{(2\pi)^2(2\pi)^2}\frac{1}{16\pi^2 s^2}\exp\left\{-\frac{(x-y)^2}{4s} - \frac{1}{2}a_n^2 - \frac{1}{2}b_n^2\right\}$$

$$\times \exp\left\{i\sqrt{s}\int\limits_0^1 d\beta \left[\frac{y-x}{2\sqrt{s}} + \bar{a}(\beta)\right]_\mu G_{\mu\nu}\left[x_\nu + 2\sqrt{s}\int\limits_\beta^1 d\beta' \left(\frac{y-x}{2\sqrt{s}} + \bar{a}(\beta')\right)_\nu\right]\right\}$$

$$\exp\{sD^2\}\delta(x-y) = \frac{1}{16\pi^2 s^2}\exp\left\{-\frac{(x-y)^2}{4s} - \frac{i}{2}x_\mu G_{\mu\nu}y_\nu\right\}\prod_{n=1}^{\infty}\int \frac{d^4a_n d^4b_n}{(2\pi)^2(2\pi)^2}e^{-\frac{1}{2}a_n^2 - \frac{1}{2}b_n^2}$$

$$\times \exp\left\{2i\sqrt{s}\frac{1}{\pi n}b_n^\nu G_\nu(x-y) + 2i\sqrt{s}\frac{1}{2\pi n}a_n^\mu G_{\mu\nu}b_n^\nu\right\}$$

$$
\begin{aligned}
\exp\{sD^2\}\delta(x-y) &= \frac{1}{16\pi^2 s^2}\exp\left\{-\frac{(x-y)^2}{4s} - \frac{i}{2}x_\mu G_{\mu\nu}y_\nu\right\}\prod_{n=1}^{\infty}\int\frac{d^4 b_n}{(2\pi)^2(2\pi)^2}e^{-\frac{1}{2}b_n^2} \\
&\times \exp\left\{2i\sqrt{s}\frac{1}{\pi n}b_n^\nu G_\nu(x-y) - \frac{s^2}{2\pi^2 n^2}G_{\mu\nu}G_{\mu\alpha}b_n^\nu b_n^\alpha\right\} \\
&= \frac{1}{16\pi^2 s^2}\exp\left\{-\frac{(x-y)^2}{4s} - \frac{i}{2}x_\mu G_{\mu\nu}y_\nu\right\}\prod_{n=1}^{\infty}\int\frac{d^4 b_n}{(2\pi)^2(2\pi)^2} \\
&\times \exp\left\{-\frac{1}{2}b_n^\nu O_{\nu\alpha}(n)b_n^\alpha + 2i\sqrt{s}\frac{1}{\pi n}b_n^\nu G_\nu(x-y)\right\} \\
O_{\nu\alpha}(n) &= \left[\delta_{\nu\alpha} + \frac{s^2}{\pi^2 n^2}G_{\mu\nu}G_{\mu\alpha}\right]
\end{aligned}
$$

$$\exp\{sD^2\}\delta(x-y) = \frac{1}{4\pi^2 s^2}\exp\left\{-\frac{(x-y)^2}{4s} - \frac{i}{2}x_\mu G_{\mu\nu} y_\nu\right\}$$
$$\times \prod_{n=1}^{\infty} [\det O(n)]^{-1/2} \exp\left\{-\frac{2s}{\pi^2 n^2} G_\mu(x-y) O^{-1}_{\mu\nu}(n) G_\nu(x-y)\right\}$$

$$\det O(n) = \det\left[I - \frac{s^2}{\pi^2 n^2} G^2\right], \quad G^2_{\mu\nu} = G_{\mu\rho} G_{\rho\nu}$$

$$\det O(n) = \left\{\frac{1}{\pi^4 n^4}\left[(\vec{\mathcal{E}}\vec{\mathcal{H}})^2 s^4 + (\vec{\mathcal{E}}^2 + \vec{\mathcal{H}}^2)\pi^2 n^2 s^2 + \pi^4 n^4\right]\right\}^2$$

$$\mathcal{Q} = (\vec{\mathcal{E}}\vec{\mathcal{H}}), \quad \mathcal{R} = \frac{1}{2}(\vec{\mathcal{E}}^2 + \vec{\mathcal{H}}^2), \quad \sigma_\pm = \frac{\mathcal{R}}{\mathcal{Q}}\left(1 \pm \sqrt{1 - \frac{\mathcal{Q}^2}{\mathcal{R}^2}}\right), \quad \sigma_+\sigma_- = 1.$$

$$\begin{aligned}
\det O(n) &= \left\{\frac{1}{\pi^4 n^4}\left[\mathcal{Q}^2 s^4 + 2\mathcal{R}\pi^2 n^2 s^2 + \pi^4 n^4\right]\right\}^2 \\
&= \left\{\frac{\mathcal{Q}^2}{\pi^4 n^4}\left[s^4 + 2\frac{\mathcal{R}}{\mathcal{Q}^2}\pi^2 n^2 s^2 + \frac{\pi^4 n^4}{\mathcal{Q}^2}\right]\right\}^2 \\
&= \left\{\frac{\mathcal{Q}^2}{\pi^4 n^4}\left[s^2 + \pi^2 n^2 \sigma_+ \frac{1}{\mathcal{Q}}\right]\left[s^2 + \pi^2 n^2 \sigma_- \frac{1}{\mathcal{Q}}\right]\right\}^2
\end{aligned}$$

$$\begin{aligned}
\left[\prod_{n=0}^{\infty} \det O(n)\right]^{-1/2} &= \left\{\prod_{n=0}^{\infty}\left[1 + \frac{\mathcal{Q}s^2\sigma_-}{\pi^2 n^2}\right]\left[1 + \frac{\mathcal{Q}s^2\sigma_+}{\pi^2 n^2}\right]\right\}^{-1} \\
&= \frac{s^2\mathcal{Q}}{\sinh(s\sqrt{\mathcal{Q}\sigma_-})\sinh(s\sqrt{\mathcal{Q}\sigma_+})}
\end{aligned}$$

$$\prod_{n=1}^{\infty} \exp\left\{-\frac{2s}{\pi^2 n^2} G_\mu(x-y) O^{-1}_{\mu\nu}(n) G_\nu(x-y)\right\}$$

$$= \exp\left\{-\sum_{n=1}^{\infty}\frac{2s}{\pi^2 n^2} G_\mu(x-y) O^{-1}_{\mu\nu}(n) G_\nu(x-y)\right\} = ?$$

$$\frac{2s}{\pi^2 n^2} G_\mu(z) O^{-1}_{\mu\nu}(n) G_\nu(z) = \frac{s\pi^2 n^2 G_\mu(z) G_\mu(z)}{\pi^4 n^4 + 2s^2\pi^2 n^2 \mathcal{R} + \mathcal{Q}^2 s^4} + \frac{1}{2} \frac{s^3 z^2 \mathcal{Q}^2}{\pi^4 n^4 + 2s^2\pi^2 n^2 \mathcal{R} + \mathcal{Q}^2 s^4}$$

$$\frac{s\pi^2 n^2}{\pi^4 n^4 + 2s^2\pi^2 n^2 \mathcal{R} + \mathcal{Q}^2 s^4} = \frac{s\pi^2 n^2}{\mathcal{Q}^2 \left[\frac{\pi^4 n^4}{\mathcal{Q}^2} + 2s^2\pi^2 n^2 \frac{\mathcal{R}}{\mathcal{Q}^2} + s^4\right]}$$

$$= \frac{s\pi^2 n^2}{\mathcal{Q}^2 \left(s^2 + \frac{\sigma_+}{\mathcal{Q}}\pi^2 n^2\right)\left(s^2 + \frac{\sigma_-}{\mathcal{Q}}\pi^2 n^2\right)} = \frac{s\pi^2 n^2}{\left(\frac{s^2 \mathcal{Q}}{\sigma_-} + \pi^2 n^2\right)\left(\frac{s^2 \mathcal{Q}}{\sigma_+} + \pi^2 n^2\right)}$$

$$= \frac{\sigma_- s}{(\sigma_- - \sigma_+)} \frac{1}{\pi^2 n^2 + s^2 \mathcal{Q}\sigma_-} + \frac{\sigma_+ s}{(\sigma_+ - \sigma_-)} \frac{1}{\pi^2 n^2 + s^2 \mathcal{Q}\sigma_+}$$

$$\sum_{n=1}^{\infty} \frac{1}{\pi^2 n^2 + \alpha^2} = \frac{\coth(\alpha)}{2\alpha} - \frac{1}{2\alpha^2}$$

$$\sum_{n=1}^{\infty} \frac{s\pi^2 n^2}{\pi^4 n^4 + 2s^2\pi^2 n^2 \mathcal{R} + \mathcal{Q}^2 s^4} = \frac{\sigma_- s}{2(\sigma_- - \sigma_+)} \left[\frac{\coth(s\sqrt{\mathcal{Q}\sigma_-})}{s\sqrt{\mathcal{Q}\sigma_-}} - \frac{1}{s^2 \mathcal{Q}\sigma_-} \right]$$

$$+ \frac{\sigma_+ s}{2(\sigma_+ - \sigma_-)} \left[\frac{\coth(s\sqrt{\mathcal{Q}\sigma_+})}{s\sqrt{\mathcal{Q}\sigma_+}} - \frac{1}{s^2 \mathcal{Q}\sigma_+} \right]$$

$$= \frac{1}{2\sqrt{\mathcal{Q}}(\sigma_- - \sigma_+)} \left[\sqrt{\sigma_-} \coth(s\sqrt{\mathcal{Q}\sigma_-}) - \sqrt{\sigma_+} \coth(s\sqrt{\mathcal{Q}\sigma_+}) \right]$$

$$\frac{s^3 \mathcal{Q}^2}{\pi^4 n^4 + 2s^2\pi^2 n^2 \mathcal{R} + \mathcal{Q}^2 s^4} = ?$$

$$\frac{s^3 \mathcal{Q}^2}{\pi^4 n^4 + 2s^2\pi^2 n^2 \mathcal{R} + \mathcal{Q}^2 s^4} = \frac{s^3}{\frac{\pi^4 n^4}{\mathcal{Q}^2} + 2s^2\pi^2 n^2 \frac{\mathcal{R}}{\mathcal{Q}^2} + s^4} = \frac{s^3}{\left(s^2 + \frac{\pi^2 n^2 \sigma_+}{\mathcal{Q}}\right)\left(s^2 + \frac{\pi^2 n^2 \sigma_-}{\mathcal{Q}}\right)}$$

$$= \frac{s\mathcal{Q}}{(\sigma_- - \sigma_+)} \left[\frac{1}{\pi^2 n^2 + \frac{s^2 \mathcal{Q}}{\sigma_-}} - \frac{1}{\pi^2 n^2 + \frac{s^2 \mathcal{Q}}{\sigma_+}} \right]$$

$$\sum_{n=1}^{\infty} \frac{1}{2} \frac{s^3 \mathcal{Q}^2}{\pi^4 n^4 + 2 s^2 \pi^2 n^2 \mathcal{R} + \mathcal{Q}^2 s^4} = \frac{1}{2} \frac{s\mathcal{Q}}{(\sigma_- - \sigma_+)} \sum_{n=1}^{\infty} \left[\frac{1}{\pi^2 n^2 + \frac{s^2 \mathcal{Q}}{\sigma_-}} - \frac{1}{\pi^2 n^2 + \frac{s^2 \mathcal{Q}}{\sigma_+}} \right]$$

$$= \frac{1}{4} \frac{s\mathcal{Q}}{(\sigma_- - \sigma_+)} \left[\frac{\coth\left(s\sqrt{\mathcal{Q}\sigma_+}\right)}{s\sqrt{\mathcal{Q}\sigma_+}} - \frac{\coth\left(s\sqrt{\mathcal{Q}\sigma_-}\right)}{s\sqrt{\mathcal{Q}\sigma_-}} - \frac{1}{s^2 \mathcal{Q}\sigma_+} + \frac{1}{s^2 \mathcal{Q}\sigma_-} \right]$$

$$= \frac{1}{4(\sigma_+ - \sigma_-)} \left[\sqrt{\mathcal{Q}\sigma_+} \coth\left(s\sqrt{\mathcal{Q}\sigma_-}\right) - \sqrt{\mathcal{Q}\sigma_-} \coth\left(s\sqrt{\mathcal{Q}\sigma_+}\right) \right] - \frac{1}{4s}$$

$$\sum_{n=1}^{\infty} \frac{1}{2} \frac{s^3 \mathcal{Q}^2}{\pi^4 n^4 + 2 s^2 \pi^2 n^2 \mathcal{R} + \mathcal{Q}^2 s^4} =$$

$$-\frac{1}{4s} + \frac{1}{4(\sigma_+ - \sigma_-)} \left[\sqrt{\mathcal{Q}\sigma_+} \coth\left(s\sqrt{\mathcal{Q}\sigma_-}\right) - \sqrt{\mathcal{Q}\sigma_-} \coth\left(s\sqrt{\mathcal{Q}\sigma_+}\right) \right]$$

$$\exp\{sD^2\}\delta(x-y) = \frac{1}{4\pi^2 s^2}\frac{s^2\mathcal{Q}}{\sinh(s\sqrt{\mathcal{Q}\sigma_-})\sinh(s\sqrt{\mathcal{Q}\sigma_+})}$$

$$\times\exp\left\{-\frac{(x-y)^2}{4s} - \frac{i}{2}x_\mu G_{\mu\nu}y_\nu\right\}\exp\left\{\frac{(x-y)^2}{4s}\right\}$$

$$\times\exp\left\{-\frac{\sqrt{\mathcal{Q}}(x-y)^2}{4(\sigma_+-\sigma_-)}\left[\sqrt{\sigma_+}\coth\left(s\sqrt{\mathcal{Q}\sigma_-}\right)-\sqrt{\sigma_-}\coth\left(s\sqrt{\mathcal{Q}\sigma_+}\right)\right]\right.$$

$$\left.-\frac{G_{\mu\nu}G_{\mu\rho}(x-y)_\nu(x-y)_\rho}{4\sqrt{\mathcal{Q}}(\sigma_+-\sigma_-)}\left[\sqrt{\sigma_+}\coth\left(s\sqrt{\mathcal{Q}\sigma_+}\right)-\sqrt{\sigma_-}\coth\left(s\sqrt{\mathcal{Q}\sigma_-}\right)\right]\right\}$$

$$\exp\{sD^2\}\delta(x-y) = \frac{1}{4\pi^2}\frac{\mathcal{Q}}{\sinh(s\sqrt{\mathcal{Q}\sigma_-})\sinh(s\sqrt{\mathcal{Q}\sigma_+})}\exp\left\{-\frac{i}{2}x_\mu G_{\mu\nu}y_\nu\right\}$$

$$\times\exp\left\{-\frac{\sqrt{\mathcal{Q}}(x-y)^2}{4(\sigma_+-\sigma_-)}\left[\sqrt{\sigma_+}\coth\left(s\sqrt{\mathcal{Q}\sigma_-}\right)-\sqrt{\sigma_-}\coth\left(s\sqrt{\mathcal{Q}\sigma_+}\right)\right]\right.$$

$$\left.-\frac{G_{\mu\nu}G_{\mu\rho}(x-y)_\nu(x-y)_\rho}{4\sqrt{\mathcal{Q}}(\sigma_+-\sigma_-)}\left[\sqrt{\sigma_+}\coth\left(s\sqrt{\mathcal{Q}\sigma_+}\right)-\sqrt{\sigma_-}\coth\left(s\sqrt{\mathcal{Q}\sigma_-}\right)\right]\right\}$$

Finally quark propagator in the presence of an arbitrary homogeneous Abelian gauge field takes the form:

$$S(x,y) = (m + i\not{D}_x)H(x,y)$$

$$H(x,y) = e^{-\frac{i}{2}x_\mu G_{\mu\nu}y_\nu} \frac{\mathcal{Q}}{4\pi^2} \int\limits_0^\infty ds \frac{e^{-m^2 s}}{\sinh(s\sqrt{\mathcal{Q}\sigma_-})\sinh(s\sqrt{\mathcal{Q}\sigma_+})}$$

$$\times \Big[P_+ \cosh(s|\vec{\mathcal{E}} - \vec{\mathcal{H}}|) + P_- \cosh(s|\vec{\mathcal{E}} + \vec{\mathcal{H}}|)$$

$$-\frac{1}{2}\sigma_{\mu\nu}[G_{\mu\nu} - \tilde{G}_{\mu\nu}]\frac{\sinh(s|\vec{\mathcal{E}} - \vec{\mathcal{H}}|)}{|\vec{\mathcal{E}} - \vec{\mathcal{H}}|} - \frac{1}{2}\sigma_{\mu\nu}[G_{\mu\nu} + \tilde{G}_{\mu\nu}]\frac{\sinh(s|\vec{\mathcal{E}} + \vec{\mathcal{H}}|)}{|\vec{\mathcal{E}} + \vec{\mathcal{H}}|}\Big]$$

$$\times \exp\Big\{-\frac{\sqrt{\mathcal{Q}}(x-y)^2}{4(\sigma_+ - \sigma_-)}\left[\sqrt{\sigma_+}\coth\left(s\sqrt{\mathcal{Q}\sigma_-}\right) - \sqrt{\sigma_-}\coth\left(s\sqrt{\mathcal{Q}\sigma_+}\right)\right]$$

$$-\frac{G_{\mu\nu}G_{\mu\rho}(x-y)_\nu(x-y)_\rho}{4\sqrt{\mathcal{Q}}(\sigma_+ - \sigma_-)}\left[\sqrt{\sigma_+}\coth\left(s\sqrt{\mathcal{Q}\sigma_+}\right) - \sqrt{\sigma_-}\coth\left(s\sqrt{\mathcal{Q}\sigma_-}\right)\right]\Big\}$$

$$\mathcal{Q} = (\vec{\mathcal{E}}\vec{\mathcal{H}}), \quad \mathcal{R} = \frac{1}{2}(\vec{\mathcal{E}}^2 + \vec{\mathcal{H}}^2), \quad \sigma_\pm = \frac{\mathcal{R}}{\mathcal{Q}}\left(1 \pm \sqrt{1 - \frac{\mathcal{Q}^2}{\mathcal{R}^2}}\right)$$

$$\vec{\mathcal{H}} = n\vec{B} + \vec{H}, \qquad \vec{\mathcal{E}} = \pm n\vec{B} + \vec{E}.$$

One-loop quark contribution to the effective potential in the presence of arbitrary homogenous Abelian fields

$$U_{\text{eff}}(G) = -\frac{1}{V}\ln\frac{\det(i\not{D}-m)}{\det(i\not{\partial}-m)} = \frac{1}{V}\int\limits_V d^4x\texttt{Tr}\int\limits_m^\infty dm' \left[S(x,x|m') - S_0(x,x|m')\right]|$$

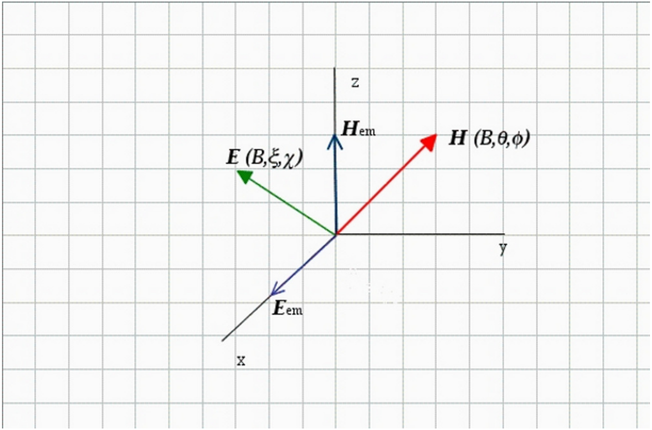
$$U_{\text{eff}}^{\text{ren}}(G) = \frac{B^2}{8\pi^2}\int\limits_0^\infty \frac{ds}{s^3}\texttt{Tr}_n\left[s\varkappa_+\coth(s\varkappa_+)s\varkappa_-\coth(s\varkappa_-) - \mathbf{1} - \frac{s^2}{3}(\varkappa_+^2+\varkappa_-^2)\right]e^{-\frac{m^2}{B}s},$$

$$\varkappa_\pm = \frac{1}{2B}\sqrt{\mathcal{Q}\sigma_\pm} = \frac{1}{2B}\left(\sqrt{2(\mathcal{R}+\mathcal{Q})}\pm\sqrt{2(\mathcal{R}-\mathcal{Q})}\right),$$

$$\mathcal{R} = (H^2-E^2)/2 + \hat{n}^2B^2 + \hat{n}B(H\cos(\theta) + iE\cos(\chi)\sin(\xi))$$

$$\mathcal{Q} = \hat{n}BH\cos(\xi) + i\hat{n}BE\sin(\theta)\cos(\phi) + \hat{n}^2B^2(\sin(\theta)\sin(\xi)cos(\phi-\chi) + cos(\theta)cos(\xi))$$

Y. M. Cho and D. G. Pak, Phys.Rev. Lett., 6 (2001) 1047

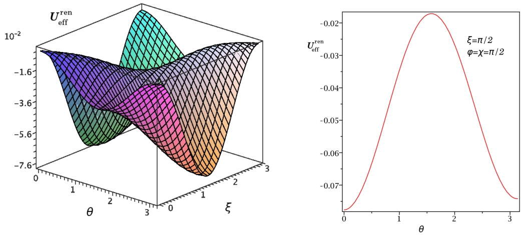


$$H_i = H\delta_{i3}, \quad E_j = E\delta_{j1}, \quad H^{\mathrm{c}} = \{B, \theta, \phi\}, \quad E^{\mathrm{c}} = \{B, \xi, \chi\}$$

$H \neq 0$, $E \neq 0$ and arbitrary gluon field

$$\Im(U_{\rm eff}) = 0 \Longrightarrow \cos(\chi)\sin(\xi) = 0,\ \sin(\theta)\cos(\phi) = 0$$

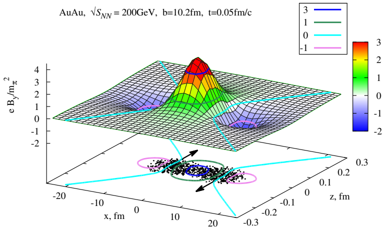
Effective potential (in units of $B^2/8\pi^2$) for the electric $E = .5B$ and the magnetic $H = .9B$ fields as functions of angles θ and ξ ($\phi = \chi = \pi/2$)



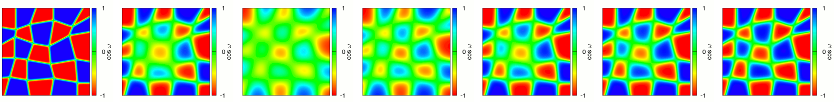
Minimum is at $\theta = \pi$ and $\xi = \pi/2$:
orthogonal to each other chromomagnetic and chromoelectric fields: $\mathcal{Q} = 0$.
Strong electro-magnetic field plays catalyzing role for deconfinement and anisotropies?!

B.V. Galilo and S.N., Phys. Rev. D84 (2011) 094017.
M. D'Elia, M. Mariti and F. Negro, Phys. Rev. Lett. **110**, 082002 (2013)
G. S. Bali, F. Bruckmann, G. Endrodi, F. Gruber and A. Schaefer, JHEP **1304**, 130 (2013)

V. Voronyuk, V. D. Toneev, W. Cassing, E. L. Bratkovskaya, V. P. Konchakovski and S. A. Voloshin, Phys. Rev C 84 (2011)



Magnetic field $eB \gtrsim m_\pi^2$ in the region $5fm \times 5fm \times .2fm \times .2fm/c$



Green region (**"Spaghetti vacuum"**) and the color charged quasi-particles

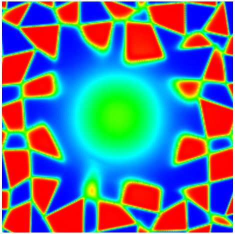
Two-stage phase transformation:

I.Colorless hadrons: $\langle|F\tilde{F}|\rangle = \langle F^2\rangle^2 \neq 0$

$\longrightarrow$ II. Charged quasiparticles: $\langle|F\tilde{F}|\rangle = 0$, $\langle F^2\rangle \neq 0$

$\longrightarrow$ III Weakly interacting QGP: $\langle|F\tilde{F}|\rangle = 0$, $\langle F^2\rangle = 0$

Free energy density $F(B, R; H_{\rm em}, T, \mu)$, EOS, ...: finite size effects, anisotropy, instabilities, ...



Photons as a signal of deconfinement in hadronic matter under extreme conditions

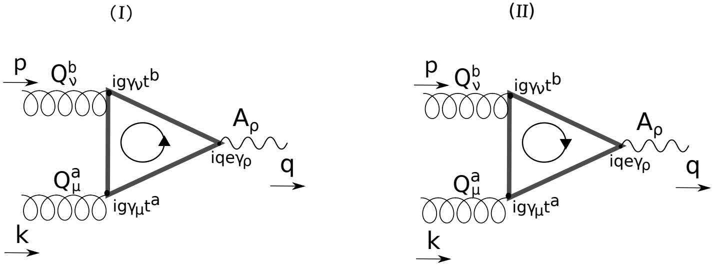


Figure: The diagrams for the process $gg \to \gamma$. Here p, k - are momenta of the gluons, q is the photon momentum. The arrows inside loop indicate the direction of loop momentum.

- S. Nedelko, A. Nikolskii. Eur.Phys. J. A **59** (2023) 4, 70; arXiv: 2208.00842 [hep-ph]

Conversion of the gluons in the photon $gg \to \gamma$ (via quark loop) can contribute to *direct photon flow puzzle*.

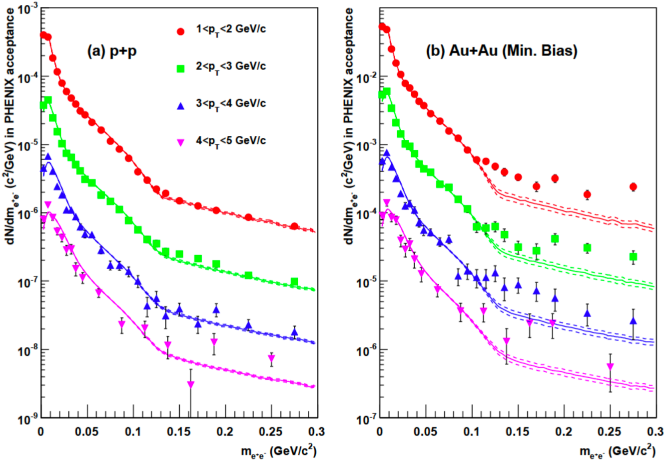


Figure: PHENIX Collaboration, Phys. Rev. Lett. 104, 132301, 2010. Direct photon flow puzzle.

The confinement phase. $gg \to \gamma$ (via quark loop) in the presence of a random ensemble of almost everywhere homogeneous Abelian (anti-)self-dual gluon field:

$$\hat{B}_\mu = \frac{1}{2}\hat{B}_{\mu\nu}x_\nu,\ \hat{B}_{\mu\nu} = \hat{n}B_{\mu\nu},\ \hat{n} = t^8 = \frac{\lambda^8}{2},$$
$$\widetilde{B}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}B_{\alpha\beta} = \pm B_{\mu\nu},\ \hat{B}_{\rho\mu}\hat{B}_{\rho\nu} = 4v^2B^2\delta_{\mu\nu},$$
$$\hat{f}_{\alpha\beta} = \frac{\hat{n}}{2vB}B_{\alpha\beta},\ v = \mathrm{diag}\left(\frac{1}{6},\frac{1}{6},\frac{1}{3}\right),\ \hat{f}^{ik}_{\mu\alpha}\hat{f}^{kj}_{\nu\alpha} = \delta^{ij}\delta_{\mu\nu},$$

where λ^8 is the Gell-Mann matrix. Field strength B sets the scale related to the value of the scalar gluon condensate.
The propagator of the quark field with mass m_f in the presence of the (anti-)self-dual field determined in Eq. (53) has the form

$$S_f(x,y) = \exp\left(\frac{i}{2}x_\mu\hat{B}_{\mu\nu}y_\nu\right)H_f(x-y), \quad (1)$$

$$H_f(z) = \frac{vB}{8\pi^2}\int_0^1\frac{ds}{s^2}\exp\left(-\frac{vB}{2s}z^2\right)\left(\frac{1-s}{1+s}\right)^{\frac{m_f^2}{4vB}}$$
$$\times \left[-i\frac{vB}{s}z_\mu\left(\gamma_\mu \pm is\hat{f}_{\mu\nu}\gamma_\nu\gamma_5\right) + m_f\left(P_\pm + \frac{1+s^2}{1-s^2}P_\mp + \frac{i}{2}\gamma_\mu\hat{f}_{\mu\nu}\gamma_\nu\frac{s}{1-s^2}\right)\right],$$

- B. Galilo, S. Nedelko. Phys. Rev. D 84, 094017 (2011); arXiv:1107.4737 [hep-ph].

The confinement phase. The terms with **odd** powers $f_{\mu\nu}$ **violate** the conditions of the Furry theorem

$$M^{(I)} = ieg^2 q_f (2\pi)^4 \delta^{(4)}(p+k-q) \left(\frac{vB}{8\pi^2}\right)^3 \int_0^1 \int_0^1 \int_0^1 \frac{ds_1}{s_1^2}\frac{ds_2}{s_2^2}\frac{ds_3}{s_3^2}\frac{(-ivB)^3}{s_1 s_2 s_3}$$

$$\left(\frac{1-s_1}{1+s_1}\right)^{\frac{m_f^2}{4vB}} \left(\frac{1-s_2}{1+s_2}\right)^{\frac{m_f^2}{4vB}} \left(\frac{1-s_3}{1+s_3}\right)^{\frac{m_f^2}{4vB}} \int d^4x d^4y \; e^{-i(px+ky)}$$

$$\Big\langle \mathrm{Tr}\Big[e^{ivBx^\mu \hat{f}_{\mu\nu} y^\nu - \frac{v}{2s_1}x^2 - \frac{v}{2s_2}y^2 - \frac{v}{2s_3}(y-x)^2} \hat{f}_{\alpha\omega}\hat{f}_{\beta\chi}\hat{f}_{\lambda\eta} K^{\mu\nu\rho}_{\alpha\omega\beta\chi\lambda\eta} +$$

$$\hat{f}_{\alpha\eta}\hat{f}_{\beta\omega}\Pi^{\mu\nu\rho}_{\alpha\eta\beta\omega} + \hat{f}_{\alpha\omega}\Gamma^{\mu\nu\rho}_{\alpha\omega}\Big]\Big\rangle \; \epsilon^a_\mu(k)\epsilon^b_\nu(p)\epsilon_\rho(q).$$

The amplitudes differ by the **sign** of the phase factor

$$M^{(II)} = ieg^2 q_f (2\pi)^4 \delta^{(4)}(p+k-q) \left(\frac{vB}{8\pi^2}\right)^3 \int_0^1 \int_0^1 \int_0^1 \frac{ds_1}{s_1^2}\frac{ds_2}{s_2^2}\frac{ds_3}{s_3^2}\frac{(-ivB)^3}{s_1 s_2 s_3}$$

$$\left(\frac{1-s_1}{1+s_1}\right)^{\frac{m_f^2}{4vB}} \left(\frac{1-s_2}{1+s_2}\right)^{\frac{m_f^2}{4vB}} \left(\frac{1-s_3}{1+s_3}\right)^{\frac{m_f^2}{4vB}} \int d^4x d^4y \; e^{-i(px+ky)}$$

$$\Big\langle \mathrm{Tr}\Big[e^{-ivBx^\mu \hat{f}_{\mu\nu} y^\nu - \frac{v}{2s_1}(x-y)^2 - \frac{v}{2s_2}y^2 - \frac{v}{2s_3}x^2} \hat{f}_{\alpha\omega}\hat{f}_{\beta\chi}\hat{f}_{\lambda\eta} K^{\mu\nu\rho}_{\alpha\omega\beta\chi\lambda\eta} +$$

$$\hat{f}_{\alpha\eta}\hat{f}_{\beta\omega}\Pi^{\mu\nu\rho}_{\alpha\eta\beta\omega} + \hat{f}_{\alpha\omega}\Gamma^{\mu\nu\rho}_{\alpha\omega}\Big]\Big\rangle \; \epsilon^a_\mu(k)\epsilon^b_\nu(p)\epsilon_\rho(q).$$

The confinement phase. The sign of the phase factor is reflected in the result of averaging over the spacial orientation of the background field (S. Nedelko, V. Voronin. Phys. Rev. D 95, 074038 (2017); arXiv: 1612.02621 [hep-ph])

$$\left\langle \prod_{j=1}^{n} f_{\alpha_j\beta_j} e^{\pm i f_{\mu\nu} J_{\mu\nu}} \right\rangle = \frac{(\pm 1)^n}{(2i)^n} \prod_{j=1}^{n} \frac{\partial}{\partial J_{\alpha_j\beta_j}} \frac{\sin\sqrt{2\left(J_{\mu\nu}J_{\mu\nu} \pm J_{\mu\nu}\widetilde{J}_{\mu\nu}\right)}}{\sqrt{2\left(J_{\mu\nu}J_{\mu\nu} \pm J_{\mu\nu}\widetilde{J}_{\mu\nu}\right)}}, \qquad (2)$$

and

$$\left\langle \prod_{j=1}^{n} f_{\alpha_j\beta_j} e^{-i f_{\mu\nu} J_{\mu\nu}} \right\rangle = (-1)^n \left\langle \prod_{j=1}^{n} f_{\alpha_j\beta_j} e^{i f_{\mu\nu} J_{\mu\nu}} \right\rangle. \qquad (3)$$

Thus, the terms in $M = M^{(I)} + M^{(II)}$ with the product of an even number of the tensor $\hat{f}_{\mu\nu}$ cancel each other out identically just as in the case of the "usual" Furry theorem in QED, and the terms with the product of an odd number of the field strength tensor cancel each other upon averaging.

The amplitude $M = M^{(I)} + M^{(II)}$ vanishes in the confinement phase where averaging over random ensemble of almost everywhere homogeneous (anti-)self-dual vacuum gluon fields must be applied. **The conversion $gg \to \gamma$ does not occur in the presence of the random ensemble of confining vacuum fields.**

The deconfinement phase. This phase is characterized by the presence of the chromomagnetic field with the singled direction (B. Galilo, S. Nedelko. Phys. Rev. D 84, 094017 (2011); arXiv:1107.4737 [hep-ph].)

$$\hat{B}_{\mu\nu} = \hat{n} B_{\mu\nu} = \hat{n} B f_{\mu\nu}, \; f_{12} = -f_{21} = 1,$$

all other components of $f_{\mu\nu}$ are equal to zero.
The coordinates and momenta (in Euclidean space-time)

$$x_\perp = \big(x_1, x_2, 0, 0\big), \; x_{||} = (0, 0, x_3, x_4)$$
$$p_\perp = (p_1, p_2, 0, 0)\,, \; p_{||} = (0, 0, p_3, p_4)\,.$$

The complete propagator of the quark field with mass m_f in the presence of an external chromomagnetic field , accounting for contribution of all Landau levels, has the form

$$S(x, y) = \exp\left\{-\frac{i}{2} x_\perp^\mu \hat{B}_{\mu\nu} y_\perp^\nu\right\} H_f(x - y), \tag{4}$$

$$H_f(z) = \frac{\mathcal{B}}{16\pi^2} \int_0^\infty \frac{ds}{s} \left[\coth(\mathcal{B}s) - \sigma_{\rho\lambda} f_{\rho\lambda}\right] \exp\left\{-m_f^2 s - \frac{1}{4s} z_{||}^2 - \frac{1}{8s}\left[\mathcal{B}s \coth(\mathcal{B}s) + 1\right] z_\perp^2\right\}$$
$$\left\{m_f - \frac{i}{2s}\gamma_\mu z_{||}^\mu - \frac{1}{2}\gamma_\mu \hat{B}_{\mu\nu} z_\perp^\nu - \frac{i}{4s}\left[\mathcal{B}s\coth(\mathcal{B}s) + 1\right]\gamma_\mu z_\perp^\mu\right\},$$
$$\mathcal{B} = B|\hat{n}|, \sigma_{\rho\lambda} = \frac{i}{2}\left[\gamma_\rho, \gamma_\lambda\right].$$

The deconfinement phase. The probability of photon production is given by the squared amplitude averaged over the initial gluon polarization states and summed over the final polarizations of photon

$$\overline{T}(p,k,q) = \Delta v \Delta \tau (2\pi)^4 \delta^4(p+k-q)\ T(p,k)\ ,$$

here $\Delta v \Delta \tau$ - is a space-time volume,

$$\begin{aligned} T(p,k) \quad &= \quad \frac{2\alpha\alpha_s^2}{\pi}\int_0^\infty ds_1 ds_2 ds_3 dr_1 dr_2 dr_3\ F(s_1,s_2,s_3,r_1,r_2,r_3|p,k) \\ &\times \quad \exp\left\{p_\perp^2\Phi_1(s_1,s_2,s_3,r_1,r_2,r_3) + p_\perp k_\perp \Phi_2(s_1,s_2,s_3,r_1,r_2,r_3) + \right. \\ &\left. \quad k_\perp^2\Phi_3(s_1,s_2,s_3,r_1,r_2,r_3) - m_f^2(s_1+s_2+s_3+r_1+r_2+r_3)\right\}, \end{aligned} \tag{5}$$

where α and α_s – electromagnetic and strong coupling constants, and Φ_1, Φ_2, Φ_3, - some functions of quark proper time
SN, A. Nikolskii. Eur.Phys. J. A **59** (2023) 4, 70; arXiv: 2208.00842 [hep-ph].

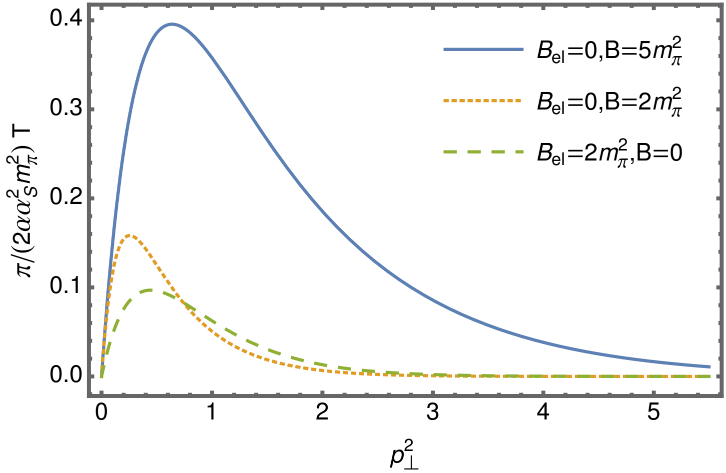


Figure: Dependence of $T(p,k)$ in regime $k_\perp^2 = p_\perp^2$. The dashed line corresponds to the purely magnetic field $B_{\rm el}$, dotted and solid lines represent the case of pure chromomagnetic field B with different values of strength.

- the gluon conversion in pure magnetic field $B_{\rm el}$: A. Ayala et al. Eur. Phys. J. A 56, 53 (2020); arXiv: 1904.02938 [hep-ph].

Collective excitations, composite fields

Nonlocal Wick-Cutkosky model with confinement

Starting point is the Euclidean functional integral

$$\mathcal{Z} = \mathcal{N} \int \mathcal{D}\phi \mathcal{D}\Phi \mathcal{D}\Phi^\dagger \exp\left\{ \int d^4x \, [\mathcal{L}(x)] \right\},$$

$$\mathcal{L} = -\Phi^\dagger S^{-1}(-\partial^2)\Phi - \frac{1}{2}\phi D^{-1}(-\partial^2)\phi - g\Phi^\dagger\Phi\phi.$$

taken over the fields with finite classical action.
The Lagrangian of the model has the same structure as that of the well-known Wick-Cutkosky model [G.C. Wick, Phys. Rev. 96 (1954) 1124; R.E. Cutkosky, *ibid* 1135.;]. Originally, The Wick-Cutkosky model describes interaction of real massless scalar field ϕ and massive charged scalar field Φ – a prototype theory for the relativistic bound state problem in quantum electrodynamics. In the nonlocal version with confinement one replaces massive and massless scalar fields by the fields with Euclidean momentum space propagators the form

$$S(p^2) = D(p^2) = \frac{1 - e^{-p^2/\Lambda^2}}{p^2} \quad \text{or} \quad S(p^2) = D(p^2) = \frac{1}{\Lambda^2} e^{-p^2/\Lambda^2} \tag{6}$$

Propagators $S(p^2)$ and $D(p^2)$ are entire functions such that no free particles can be associated with the fields Φ and ϕ. Eqs. of motion

$$S^{-1}(-\partial^2)\Phi(x) = 0, \quad D^{-1}(-\partial^2)\phi(x) = 0$$

have no solutions which can describe free particles. These fields represent fluctuations localized in space and time - so called **virtons**. A typical space-time size of fluctuations is set by scale Λ.

The spectrum of collective excitations and effective action for their interaction with each other can be derived analytically due to the simple Gaussian form of propagators.

Integrating out the field ϕ one arrives at a quartic interaction of the scalar fields Φ:

$$\mathcal{Z} = \mathcal{N} \int \mathcal{D}\Phi\mathcal{D}\Phi^\dagger \exp\left\{ \int d^4x \left[-\Phi^\dagger S^{-1}(-\partial^2)\Phi \right] + L_2[\Phi] \right\},$$

$$L_2[\Phi] = \frac{g^2}{2} \int d^4x_1 d^4x_2 \Phi^\dagger(x_1)\Phi(x_1)D(x_1-x_2)\Phi^\dagger(x_2)\Phi(x_2)$$

$$x_1 = x + y/2, \quad x_2 = x - y/2, \quad \overleftrightarrow{\partial} = \overleftarrow{\partial} - \overrightarrow{\partial}$$

$$= \frac{g^2}{4} \int d^4x \int d^4y D(y) \left[\Phi^\dagger(x+y/2)\Phi(x-y/2) \right] \left[\Phi^\dagger(x-y/2)\Phi(x+y/2) \right]$$

$$= \frac{g^2}{4} \int d^4x \int d^4y D(y) \left[\Phi^\dagger(x) e^{\frac{y}{2}\overleftrightarrow{\partial}} \Phi(x) \right] \left[\Phi^\dagger(x) e^{-\frac{y}{2}\overleftrightarrow{\partial}} \Phi(x) \right]$$

Introducing a complete orthonormal set of functions $U_\mathcal{Q}$, corresponding to the radial quantum number the angular momentum $\mathcal{Q} = \{n, l\}$, the four point interaction can be rewritten as an infinite sum of products of a non-local currents

$$L_2 = \frac{g^2}{2} \sum_\mathcal{Q} \int d^4x J_\mathcal{Q}(x) J_\mathcal{Q}(x)$$

$$J_\mathcal{Q}(x) = \Phi^\dagger(x) V_\mathcal{Q}(\overleftrightarrow{\partial})\Phi(x), \; V_\mathcal{Q}(\overleftrightarrow{\partial}) = \int d^4y \sqrt{D(y)} U_\mathcal{Q}(y) e^{\frac{y}{2}\overleftrightarrow{\partial}}$$

Collective excitation mode fields $\Psi_{\mathcal{Q}}(x)$ are introduced

$$e^{\frac{g^2}{2}\sum_{\mathcal{Q}}\int d^4x J_{\mathcal{Q}}(x)J_{\mathcal{Q}}(x)} = \prod_{\mathcal{Q}}\int \mathcal{D}\Psi_{\mathcal{Q}}\exp\left\{-\frac{\Lambda^2}{2}\int d^4x\Psi^2_{\mathcal{Q}}(x) + g\Lambda J_{\mathcal{Q}}(x)\Psi_{\mathcal{Q}}(x)\right\}$$

with a subsequent Gaussian integration over the field Φ

$$Z = \prod_{\mathcal{Q}}\int \mathcal{D}\Psi_{\mathcal{Q}}\exp\left\{-\frac{\Lambda^2}{2}\int d^4p\Psi_{\mathcal{Q}}(-p)(\delta_{\mathcal{Q}\mathcal{Q}'} - \alpha\Pi_{\mathcal{Q}\mathcal{Q}'}(p))\Psi_{\mathcal{Q}'}(p) + W_I[g\Psi]\right\}$$

where the dimensionless coupling constant $\alpha = g^2/(4\pi\Lambda)^2$, and

$$W_I[g\Psi] = -\mathrm{Tr}[\ln(1 - g\Psi_{\mathcal{Q}}V_{\mathcal{Q}}S) + \frac{g^2}{2}\Psi_{\mathcal{Q}}V_{\mathcal{Q}}S\Psi_{\mathcal{Q}'}V_{\mathcal{Q}'}S] \quad (7)$$

describes interactions between collective excitations fields from which the quadratic term in the expansion of the logarithm is subtracted and added to the original quadratic part,

$$\alpha\tilde{\Pi}_{\mathcal{Q}\mathcal{Q}'}(x-y) = \frac{g^2}{2\Lambda^2}V_{\mathcal{Q}}(\overleftrightarrow{\partial}_x)S(x-y)V_{\mathcal{Q}'}(\overleftrightarrow{\partial}_y)S(y-x)$$

The nontrivial requirement is that the basis $U_{\mathcal{Q}}$ be chosen such that the self-energy $\Sigma_{\mathcal{Q}\mathcal{Q}'}(p)$ and hence the quadratic part of the effective action is diagonal in the quantum numbers Q,

$$\Pi_{\mathcal{Q}\mathcal{Q}'}(p) = \Pi_{\mathcal{Q}}(-p^2)\delta_{\mathcal{Q}\mathcal{Q}'} \quad (8)$$

Masses of the collective modes with quantum numbers $\mathcal{Q} = \{n, l\}$ are then real solutions to

$$1 - \alpha\Pi_{\mathcal{Q}}(M^2_{\mathcal{Q}}) = 0. \quad (9)$$

Calculation of the self-energy gives

$$\Pi_{nl}(-p^2) \quad = \quad \frac{e^{-\frac{p^2}{2\Lambda^2}}}{(2+\sqrt{3})^{2n+l+2}},$$

the square of the collective excitations masses reads

$$M_{nl}^2 = 2\Lambda^2 \left[\ln\frac{(2+\sqrt{3})^2}{\alpha} + (2n+l)\ln(2+\sqrt{3})\right] \qquad (10)$$

which manifests a linear Regge spectrum. Finally the fields should be rescaled

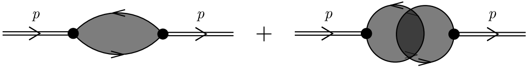
$$\begin{aligned} \Psi_{\mathcal{Q}} &= \Psi_{\mathcal{Q}} g^{-1}\Lambda h_{\mathcal{Q}}, \\ h_{\mathcal{Q}}^{-2} &= \frac{\Lambda^2}{(4\pi)^2}\frac{dPi_{\mathcal{Q}}(-p^2)}{dp^2}\Big|_{p^2=-M_{\mathcal{Q}}^2} \end{aligned} \qquad (11)$$

in order to ensure the correct residue of the propagator at the mass pole. The coupling constant $h_{\mathcal{Q}}$ is defined as a dimensionless quantity. Thus the final form of the functional integral for composite fields is

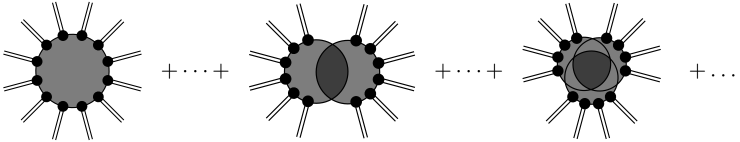
$$Z \quad = \quad \prod_{\mathcal{Q}}\int \mathcal{D}\Psi_{\mathcal{Q}} \exp\left\{-\frac{\Lambda^4 h_{\mathcal{Q}}^2}{2g^2}\int d^4p\Psi_{\mathcal{Q}}(-p)(1-\alpha\Pi_{\mathcal{Q}}(-p^2))\Psi_{\mathcal{Q}}(p) + W_I[h_{\mathcal{Q}}\Psi]\right\} (12)$$

The original coupling constant g enters only the quadratic part of the effective action, while the remaining terms contain the effective coupling constant $h_{\mathcal{Q}}$.

$$\Pi^{(2)}_{\mathcal{Q}_1\mathcal{Q}_2} =$$



$$\Pi^{(n)}_{\mathcal{Q}_1\mathcal{Q}_2\ldots\mathcal{Q}_n} =$$



In the momentum representation propagators and vertices involved in the effective action have the form

$$\begin{aligned} S(p) &= \Lambda^{-2} e^{-p^2/\Lambda^2}, \\ V^{nl}_{\mu_1\dots\mu_l}(K) &= e^{i(q-p)x}\left[e^{ipx} V^{nl}_{\mu_1\dots\mu_l}\left(\overleftrightarrow{\partial}_x\right) e^{-iqx}\right], \\ &= (-1)^{n+l}\Lambda C_{nl} T^l_{\mu_1,\dots,\mu_l}(K) L^{l+1}_n(aK^2) e^{-bK^2} \end{aligned} \quad (13)$$

with $K = (p+q)/2\Lambda$, $a = 2\sqrt{3}$, $b = 2/(1+\sqrt{3}) = 4/(2+a)$ and

$$C_{nl} = \frac{2^{n+2+2l}\sqrt{3}^{l/2+1}}{(1+\sqrt{3})^{2n+l+2}} \sqrt{\frac{n!(l+1)}{(n+l+1)!}}.$$

Here the angular part of the vertex is given by $T^l_{\mu_1\dots\mu_l}$, the irreducible tensors of the Euclidean rotation group $O(4)$,

$$\begin{aligned} &\int_\Omega \frac{d\omega}{2\pi^2} T^l_{\mu_1\dots\mu_l}(n_y) T^k_{\nu_1\dots\nu_k}(n_y) = \frac{1}{2^l(l+1)} \delta^{lk} \delta_{\mu_1\nu_1}\dots\delta_{\mu_l\nu_l} \\ &T^l_{\mu_1\dots\mu\dots\nu\dots\mu_l}(n_y) = T^l_{\mu_1\dots\nu\dots\mu\dots\mu_l}(n_y),\ T^l_{\mu\mu\dots\mu_l}(n_y) = 0, \\ &T^l_{\mu_1\dots\mu_l}(n_y) T^l_{\mu_1\dots\mu_l}(n_x) = \frac{1}{2^l} C^{(1)}_l(n_y n_x), \quad n_x^2 = n_y^2 = 1 \end{aligned} \quad (14)$$

with $C^{(1)}_l$ being Gegenbauer polynomials. The radial part corresponds to the Laguerre polynomials L^{l+1}_n

$$\int_0^\infty du u^{l+1} e^{-u} L^{l+1}_n(u) L^{l+1}_{n'}(u) = \delta_{nn'}$$

QCD bosonization

$$Z = \int_{\mathcal{B}} DB \int_{\Psi} \mathcal{D}\psi \mathcal{D}\bar{\psi} \int_{\mathcal{Q}} \mathcal{D}Q \delta[D(B)Q] \Delta_{\mathrm{FP}}[B,Q] e^{-S^{\mathrm{QCD}}[Q+B,\psi,\bar{\psi}]} =$$

$$\int_{\mathcal{B}} dB \int_{\Psi} \mathcal{D}\psi \mathcal{D}\bar{\psi} \exp\left\{ \int dx \bar{\psi} \left(i \not{\partial} + g \not{B} - m \right) \psi \right\} W[j]$$

$$W[j] = \exp\left\{ \sum_n \frac{g^n}{n!} \int dx_1 \cdots \int dx_n j^{a_1}_{\mu_1}(x_1) \dots j^{a_n}_{\mu_n}(x_n) G^{a_1 \dots a_n}_{\mu_1 \dots \mu_n}(x_1, \dots, x_n | B) \right\}$$

$$j^a_\mu = \bar{\psi} \gamma_\mu t^a \psi,$$

$W[j]$ is truncated up to the term including two-point gluon correlation function. Fierz transform

$$\gamma^{\alpha\beta}_\mu \gamma^{\delta\rho}_\nu = \sum_J C^J_{\mu\nu} \Gamma^J_{\alpha\rho} \Gamma^J_{\delta\beta}, \quad \Gamma^J \in \{I, \gamma_5, \gamma_\mu, i\gamma_5\gamma_\mu, \sigma_{\mu\nu}\}$$

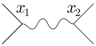
$$t^{ik}_a t^{jl}_a = \frac{1}{2N} \delta_{ki} \delta_{lj} + \frac{1}{2} \delta_{li} \delta_{kj} \quad \delta_{li} \delta_{kj} = \frac{1}{N} \delta_{ki} \delta_{lj} + T^a_{ik} t^a_{jl}$$

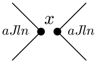
$$\bar{\psi}^{i\alpha}_f(x) \gamma^{\alpha\beta}_\mu t^a_{ik} \psi^{k\beta}_f(x) \times \bar{\psi}^{j\delta}_f(x) \gamma^{\delta\rho}_\mu t^a_{jl} \psi^{l\beta}_f(x)$$

$$\mathcal{Z} = \int dB \int_{\Psi} \mathcal{D}\psi \mathcal{D}\bar{\psi} \exp\left\{ \int dx \bar{\psi} \left(i \not\partial + g \not B - m \right) \psi \right.$$
$$\left. + \frac{g^2}{2} \int dx_1 dx_2 G^{a_1 a_2}_{\mu_1 \mu_2}(x_1, x_2 | B) j^{a_1}_{\mu_1}(x_1) j^{a_2}_{\mu_2}(x_2) \right\}$$

Fierz transform, center of mass coordinates $\longrightarrow \int dz dx G(z|B) J^{aJ}(x,z) J^{aJ}(x,z)$

α_s  $= \alpha_s(0)$  $\left[1 + \Pi^{\mathrm{R}}(p^2) \right]; \ \Pi^{\mathrm{R}}(0) = 0$

0  $z \ \to \frac{e^{-\frac{1}{4}Bz^2}}{4\pi^2 z^2}$ $\quad \int dx_1\, dx_2$ 

$= \int dx \sum_{aJln}$ 

$\to \alpha_s(0) \frac{1 - \exp(-p^2/B)}{p^2}$

$$J^{aJ}(x,z) = \sum_{nl} \left(z^2\right)^{l/2} f^{nl}_{\mu_1 \dots \mu_l}(z) J^{aJln}_{\mu_1 \dots \mu_l}(x), \quad J^{aJln}_{\mu_1 \dots \mu_l}(x) = \bar{q}(x) V^{aJln}_{\mu_1 \dots \mu_l}\left(\frac{\overleftrightarrow{D}(x)}{B} \right) q(x),$$

$$f^{nl}_{\mu_1 \dots \mu_l} = L_{nl}\left(z^2\right) T^{(l)}_{\mu_1 \dots \mu_l}(n_z), \quad n_z = \frac{z}{\sqrt{z}}.$$

$T^{(l)}_{\mu_1 \dots \mu_l}$ are irreducible tensors of four-dimensional rotational group

$$\int_0^\infty du \rho_l(u) L_{nl}(u) L_{n'l}(u) = \delta_{nn'}, \quad \rho_l(u) = u^l e^{-u} \leftrightarrow \frac{e^{-Bz^2}}{z^2} \quad \text{gluon propagator}$$

Effective meson action for composite colorless fields:

$$Z = \mathcal{N} \lim_{V\to\infty} \int D\Phi_{\mathcal{Q}} \exp\left\{ -\frac{B}{2}\frac{h^2_{\mathcal{Q}}}{g^2 C_{\mathcal{Q}}} \int dx \Phi^2_{\mathcal{Q}}(x) - \sum_k \frac{1}{k} W_k[\Phi] \right\}, \quad \mathcal{Q} = (aJln)$$

$$1 = \frac{g^2 C_{\mathcal{Q}}}{B} \tilde{\Gamma}^{(2)}_{\mathcal{Q}\mathcal{Q}}(-M^2_{\mathcal{Q}}|B), \quad h^{-2}_{\mathcal{Q}} = \frac{d}{dp^2} \tilde{\Gamma}^{(2)}_{\mathcal{Q}\mathcal{Q}}(p^2)|_{p^2=-M^2_{\mathcal{Q}}}$$

$$W_k[\Phi] = \sum_{\mathcal{Q}_1\dots\mathcal{Q}_k} h_{\mathcal{Q}_1}\dots h_{\mathcal{Q}_k} \int dx_1 \cdots \int dx_k \Phi_{\mathcal{Q}_1}(x_1)\dots\Phi_{\mathcal{Q}_k}(x_k) \Gamma^{(k)}_{\mathcal{Q}_1\dots\mathcal{Q}_k}(x_1,\dots,x_k|B)$$

$$\Gamma^{(2)}_{\mathcal{Q}_1\mathcal{Q}_2} = \overline{G^{(2)}_{\mathcal{Q}_1\mathcal{Q}_2}(x_1,x_2)} - \Xi_2(x_1-x_2)\overline{G^{(1)}_{\mathcal{Q}_1} G^{(1)}_{\mathcal{Q}_2}},$$

$$\Gamma^{(3)}_{\mathcal{Q}_1\mathcal{Q}_2\mathcal{Q}_3} = \overline{G^{(3)}_{\mathcal{Q}_1\mathcal{Q}_2\mathcal{Q}_3}(x_1,x_2,x_3)} - \frac{3}{2}\Xi_2(x_1-x_3)\overline{G^{(2)}_{\mathcal{Q}_1\mathcal{Q}_2}(x_1,x_2) G^{(1)}_{\mathcal{Q}_3}(x_3)}$$
$$+\frac{1}{2}\Xi_3(x_1,x_2,x_3)\overline{G^{(1)}_{\mathcal{Q}_1}(x_1) G^{(1)}_{\mathcal{Q}_2}(x_2) G^{(1)}_{\mathcal{Q}_3}(x_3)},$$

$$\Gamma^{(4)}_{\mathcal{Q}_1\mathcal{Q}_2\mathcal{Q}_3\mathcal{Q}_4} = \overline{G^{(4)}_{\mathcal{Q}_1\mathcal{Q}_2\mathcal{Q}_3\mathcal{Q}_4}(x_1,x_2,x_3,x_4)} - \frac{4}{3}\Xi_2(x_1-x_2)\overline{G^{(1)}_{\mathcal{Q}_1}(x_1) G^{(3)}_{\mathcal{Q}_2\mathcal{Q}_3\mathcal{Q}_4}(x_2,x_3,x_4)}$$
$$-\frac{1}{2}\Xi_2(x_1-x_3)\overline{G^{(2)}_{\mathcal{Q}_1\mathcal{Q}_2}(x_1,x_2) G^{(2)}_{\mathcal{Q}_3\mathcal{Q}_4}(x_3,x_4)}$$
$$+\Xi_3(x_1,x_2,x_3)\overline{G^{(1)}_{\mathcal{Q}_1}(x_1) G^{(1)}_{\mathcal{Q}_2}(x_2) G^{(2)}_{\mathcal{Q}_3\mathcal{Q}_4}(x_3,x_4)}$$
$$-\frac{1}{6}\Xi_4(x_1,x_2,x_3,x_4)\overline{G^{(1)}_{\mathcal{Q}_1}(x_1) G^{(1)}_{\mathcal{Q}_2}(x_2) G^{(1)}_{\mathcal{Q}_3}(x_3) G^{(1)}_{\mathcal{Q}_4}(x_4)}.$$

$$\overline{G^{(k)}_{\mathcal{Q}_1\ldots\mathcal{Q}_k}(x_1,\ldots,x_k)} = \int dB_j \mathrm{Tr} V_{\mathcal{Q}_1}\left(x_1|B^{(j)}\right) S\left(x_1,x_2|B^{(j)}\right)\ldots$$
$$\ldots V_{\mathcal{Q}_k}\left(x_k|B^{(j)}\right) S\left(x_k,x_1|B^{(j)}\right)$$

$$\overline{G^{(l)}_{\mathcal{Q}_1\ldots\mathcal{Q}_l}(x_1,\ldots,x_l) G^{(k)}_{\mathcal{Q}_{l+1}\ldots\mathcal{Q}_k}(x_{l+1},\ldots,x_k)} =$$
$$\int dB_j \mathrm{Tr}\left\{V_{\mathcal{Q}_1}\left(x_1|B^{(j)}\right) S\left(x_1,x_2|B^{(j)}\right)\ldots V_{\mathcal{Q}_k}\left(x_l|B^{(j)}\right) S\left(x_l,x_1|B^{(j)}\right)\right\}$$
$$\times \mathrm{Tr}\left\{V_{\mathcal{Q}_{l+1}}\left(x_{l+1}|B^{(j)}\right) S\left(x_{l+1},x_{l+2}|B^{(j)}\right)\ldots V_{\mathcal{Q}_k}\left(x_k|B^{(j)}\right) S\left(x_k,x_{l+1}|B^{(j)}\right)\right\},$$

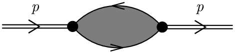
Bar denotes integration over all configurations of the background field with measure dB_j.

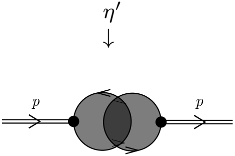
$$\langle \exp(iB_{\mu\nu}J_{\mu\nu})\rangle = \frac{\sin W}{W}$$

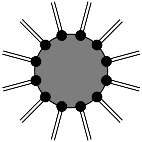
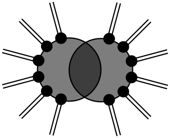
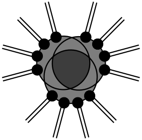
$$W = \sqrt{2B^2\left(J_{\mu\nu}J_{\mu\nu} \pm J_{\mu\nu}\widetilde{J}_{\mu\nu}\right)}$$

$J_{\mu\nu}$ is a tensor, in general composed of the momenta $p_{1\mu_1}\ldots p_{n\mu_n}$ - arguments of the meson interaction vertex

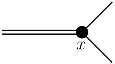
$$\tilde{\Gamma}^{(n)}(p_{1\mu_1}\ldots p_{n\mu_n})$$

$\Gamma^{(2)}_{\mathcal{Q}_1\mathcal{Q}_2} =$ 

$+$ 

$\Gamma^{(n)}_{\mathcal{Q}_1\mathcal{Q}_2\ldots\mathcal{Q}_n} =$  $+\cdots+$  $+\cdots+$  $+\ldots$

Meson-quark vertex operators $\Leftarrow J^{aJln}_{\mu_1\dots\mu_l} = \bar{q}(x) V^{aJln}_{\mu_1\dots\mu_l} q(x)$



$$V^{aJln}_{\mu_1\dots\mu_l}(x) = M^a \Gamma^J \left\{\left\{ F_{nl}\left(\frac{\overleftrightarrow{D}^2(x)}{B^2}\right) T^{(l)}_{\mu_1\dots\mu_l}\left(\frac{1}{i}\frac{\overleftrightarrow{D}(x)}{B}\right)\right\}\right\},$$

$$F_{nl}(s) = s^n \int_0^1 dt t^{n+l} \exp(st) = \int_0^1 dt t^{n+l} \frac{\partial^n}{\partial t^n} \exp(st),$$

$$\overleftrightarrow{D} = \overleftarrow{D}\, \xi_{f'} - \overrightarrow{D}\, \xi_f, \xi_f = \frac{m_f}{m_f + m_{f'}}$$

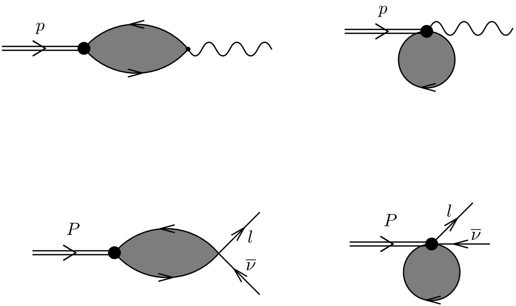
Quark propagator in homogeneous Abelian (anti-)self-dual field

 $=$ $\frac{}{m(0)}\left[1 + \Sigma^{\mathrm{R}}(p^2)\right];\ \Sigma^{\mathrm{R}}(0) = 0 \qquad S(x,y) = \exp\left(-\frac{i}{2} x_\mu B_{\mu\nu} y_\nu\right) H(x-y),$

$$\widetilde{H}_f(p|B) = \frac{1}{\upsilon B^2}\int_0^1 ds e^{(-p^2/\upsilon B^2)s}\left(\frac{1-s}{1+s}\right)^{m_f^2/2\upsilon B^2}\left[p_\alpha\gamma_\alpha \pm is\gamma_5\gamma_\alpha\frac{B_{\alpha\beta}}{\upsilon B^2}p_\beta + \right.$$
$$\left. + m_f\left(P_\pm + P_\mp\frac{1+s^2}{1-s^2} - \frac{i}{2}\gamma_\alpha\frac{B_{\alpha\beta}}{\upsilon B^2}\gamma_\beta\frac{s}{1-s^2}\right)\right]$$

$$\tilde{H}_f(p|B) = \frac{m}{2\upsilon\Lambda^2}\mathcal{H}_S(p^2) \mp \gamma_5\frac{m}{2\upsilon\Lambda^2}\mathcal{H}_P(p^2) + \gamma_\alpha\frac{p_\alpha}{2\upsilon\Lambda^2}\mathcal{H}_V(p^2) \pm i\gamma_5\gamma_\alpha\frac{f_{\alpha\beta}p_\beta}{2\upsilon\Lambda^2}\mathcal{H}_A(p^2) \quad (15)$$
$$+\sigma_{\alpha\beta}\frac{m f_{\alpha\beta}}{4\upsilon\Lambda^2}\mathcal{H}_T(p^2).$$

Weak and electromagnetic interactions



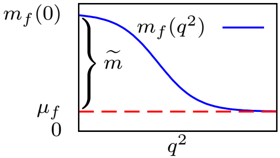
Masses of radially excited mesons

The parameters of the model are

$$\alpha_s\,(0) \quad m_{u/d}(0) \quad m_s(0) \quad m_c(0) \quad m_b(0) \quad B \quad R$$

$$\langle \alpha_s F^2 \rangle = \frac{B^2}{\pi} \quad \chi_{\rm YM} = \frac{B^4 R^4}{128\pi^2}$$

Dynamical chiral symmetry breaking:



$$\widetilde{m} = 136 \text{ MeV}$$

$$\mu_{u/d} = m_{u/d} - \widetilde{m}$$

$$\mu_s = m_s - \widetilde{m}$$

$$\frac{\mu_s}{\mu_{u/d}} = 26.7$$

$$\Lambda^2 \Phi^{(0)}_{\mathcal{Q}_1} = \sum_{k=1}^{\infty} \frac{g^k}{k} \sum_{\mathcal{Q}_1 \dots \mathcal{Q}_k} \Phi^{(0)}_{\mathcal{Q}_2} \dots \Phi^{(0)}_{\mathcal{Q}_k} \Gamma^{(k)}_{\mathcal{Q}_1 \dots \mathcal{Q}_k},$$

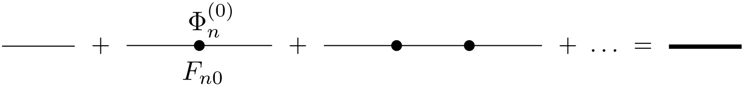


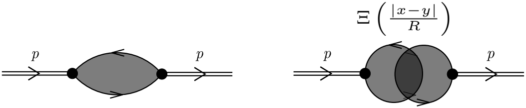
Figure: Mass corrections to the quark propagator due to the constant scalar condensates $\Phi^{(0)}_n$ coupled to nonlocal form factor F_{n0}. Summation over the radial number n is assumed.

Asymptotic Regge spectrum :

$$M_n^2 \sim Bn, \; n \gg 1$$
$$M_l^2 \sim Bl, \; l \gg 1$$

G.V. Efimov and S.N. , Phys. Rev. D 51 (1995)

η and η'!



Polarization operator

Polarization operation for $l=0$:

$$\Pi_J^{nn'}\left(-M^2;m_f,m_{f'};B\right)=$$

$$\frac{B}{4\pi^2}\mathrm{Tr}_\upsilon\int\limits_0^1 dt_1\int\limits_0^1 dt_2\int\limits_0^1 ds_1\int\limits_0^1 ds_2\left(\frac{1-s_1}{1+s_1}\right)^{m_f^2/4\upsilon B}\left(\frac{1-s_2}{1+s_2}\right)^{m_{f'}^2/4\upsilon B}\times$$

$$\times\, t_1^n t_2^{n'}\frac{\partial^n}{\partial t_1^n}\frac{\partial^{n'}}{\partial t_2^{n'}}\frac{1}{\Phi_2^2}\left[\frac{M^2}{B}\frac{F_1^{(J)}}{\Phi_2^2}+\frac{m_f m_{f'}}{B}\frac{F_2^{(J)}}{(1-s_1^2)(1-s_2^2)}+\frac{F_3^{(J)}}{\Phi_2}\right]\exp\left\{\frac{M^2}{2\upsilon B}\frac{\Phi_1}{\Phi_2}\right\}.$$

$$\Phi_1=s_1s_2+2\left(\xi_1^2s_1+\xi_2^2s_2\right)(t_1+t_2)\upsilon,$$

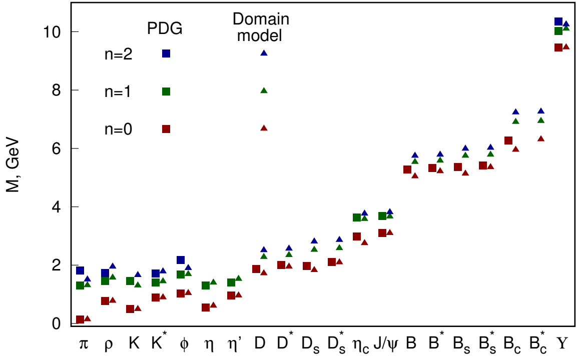
$$\Phi_2=s_1+s_2+2(1+s_1s_2)(t_1+t_2)\upsilon+16(\xi_1^2s_1+\xi_2^2s_2)t_1t_2\upsilon^2,$$

$$F_1^{(P)}=(1+s_1s_2)\left[2(\xi_1s_1+\xi_2s_2)(t_1+t_2)\upsilon+\right.$$
$$\left.4\xi_1\xi_2(1+s_1s_2)(t_1+t_2)^2\upsilon^2+s_1s_2(1-16\xi_1\xi_2t_1t_2\upsilon^2)\right],$$

$$F_1^{(V)}=\left(1-\frac{1}{3}s_1s_2\right)\left[s_1s_2+16\xi_1\xi_2t_1t_2\upsilon^2+2(\xi_1s_1+\xi_2s_2)(t_1+t_2)\upsilon\right]+$$
$$4\xi_1\xi_2(1-s_1^2s_2^2)(t_1-t_2)^2\upsilon^2,$$

$$F_2^{(P)}=(1+s_1s_2)^2,\quad F_2^{(V)}=(1-s_1^2s_2^2),$$

$$F_3^{(P)}=4\upsilon(1+s_1s_2)(1-16\xi_1\xi_2t_1t_2\upsilon^2),\;F_3^{(V)}=2\upsilon(1-s_1s_2)(1-16\xi_1\xi_2t_1t_2\upsilon^2).$$



PDG
Domain model
n=2
n=1
n=0
M, GeV
0
2
4
6
8
10
π ρ K K* ϕ η η' D D* Ds Ds* ηc J/ψ B B* Bs Bs* Bc Bc* Υ

Table: Model parameters fitted to the masses of $\pi, \rho, K, K^*, \eta', J/\psi, \Upsilon$ and used in calculation of all other quantities.

$m_{u/d}$, MeV	m_s, MeV	m_c, MeV	m_b, MeV	Λ, MeV	α_s	R, fm
145	376	1566	4879	416	3.45	1.12

Table: Masses of light mesons. $\tilde{M}$ denotes the value in the chiral limit.

Meson	n	$M_{\rm exp}$ (MeV)	M (MeV)	$\widetilde{M}$ (MeV)	Meson	n	$M_{\rm exp}$ (MeV)	M (MeV)	$\widetilde{M}$ (MeV)
π	0	140	140	0	ρ	0	775	775	769
$\pi(1300)$	1	1300	1310	1301	$\rho(1450)$	1	1450	1571	1576
$\pi(1800)$	1	1812	1503	1466	ρ	2	1720	1946	2098
K	0	494	494	0	K^*	0	892	892	769
$K(1460)$	1	1460	1302	1301	$K^*(1410)$	1	1410	1443	1576
K	2		1655	1466	$K^*(1717)$	1	1717	1781	2098
η	0	548	621	0	ω	0	775	775	769
η'	0	958	958	872	ϕ	0	1019	1039	769
$\eta(1295)$	1	1294	1138	1361	$\phi(1680)$	1	1680	1686	1576
$\eta(1475)$	1	1476	1297	1516	ϕ	2	2175	1897	2098

Table: Masses of heavy-light mesons and their lowest radial excitations .

Meson	n	$M_{\rm exp}$ (MeV)	M (MeV)	Meson	n	$M_{\rm exp}$ (MeV)	M (MeV)
D	0	1864	1715	D^*	0	2010	1944
D	1		2274	D^*	1		2341
D	2		2508	D^*	2		2564
D_s	0	1968	1827	D_s^*	0	2112	2092
D_s	1		2521	D_s^*	1		2578
D_s	2		2808	D_s^*	2		2859
B	0	5279	5041	B^*	0	5325	5215
B	1		5535	B^*	1		5578
B	2		5746	B^*	2		5781
B_s	0	5366	5135	B_s^*	0	5415	5355
B_s	1		5746	B_s^*	1		5783
B_s	2		5988	B_s^*	2		6021
B_c	0	6277	5952	B_c^*	0		6310
B_c	1		6904	B_c^*	1		6938
B_c	2		7233	B_c^*	2		7260

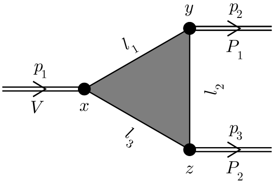
Table: Masses of heavy quarkonia.

Meson	n	$M_{\rm exp}$ (MeV)	M (MeV)
$\eta_c(1S)$	0	2981	2751
$\eta_c(2S)$	1	3639	3620
η_c	2		3882
$J/\psi(1S)$	0	3097	3097
$\psi(2S)$	1	3686	3665
$\psi(3770)$	2	3773	3810
$\Upsilon(1S)$	0	9460	9460
$\Upsilon(2S)$	1	10023	10102
$\Upsilon(3S)$	2	10355	10249

Table: Decay and transition constants of various mesons

Meson	n	$f_P^{\rm exp}$ (MeV)	f_P (MeV)	Meson	n	$g_{V\gamma}^{\rm exp}$	$g_{V\gamma}$
π	0	130	140	ρ	0	0.2	0.2
$\pi(1300)$	1	—	29	ρ	1		0.034
K	0	156	175	ω	0	0.059	0.067
$K(1460)$	1	—	27	ω	1		0.011
D	0	205	212	ϕ	0	0.074	0.069
D	1	—	51	ϕ	1		0.025
D_s	0	258	274	J/ψ	0	0.09	0.057
D_s	1	—	57	J/ψ	1		0.024
B	0	191	187	Υ	0	0.025	0.011
B	1	—	55	Υ	1		0.0039
B_s	0	253	248				
B_s	1	—	68				
B_c	0	489	434				
B_c	1		135				

Stong decays: g_{VPP}



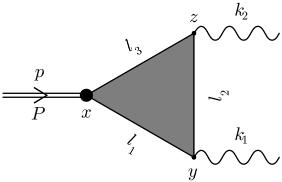
Decay	g_{VPP} [*]	g_{VPP}
$\rho^0 \to \pi^+\pi^-$	5.95	7.58
$\omega \to \pi^+\pi^-$	0.17	0
$K^{*\pm} \to K^\pm\pi^0$	3.23	3.54
$K^{*\pm} \to K^0\pi^\pm$	4.57	5.01
$\varphi \to K^+K^-$	4.47	5.02
$D^{*\pm} \to D^0\pi^\pm$	8.41	7.9
$D^{*\pm} \to D^\pm\pi^0$	5.66	5.59

local color gauge invariance

[*] K.A. Olive et al. (Particle Data Group) Chinese Phys. C 38,090001, 2014

Pion transition form factor

$$T_a^{\mu\nu}(x,y,z) = h_P \sum_n u_n^a \int d\sigma_B \mathrm{Tr}\ t_a e_f^2 V^n(x)\gamma_5 S(x,y|B)\gamma_\mu S(y,z|B)\gamma_\nu S(z,x|B),$$



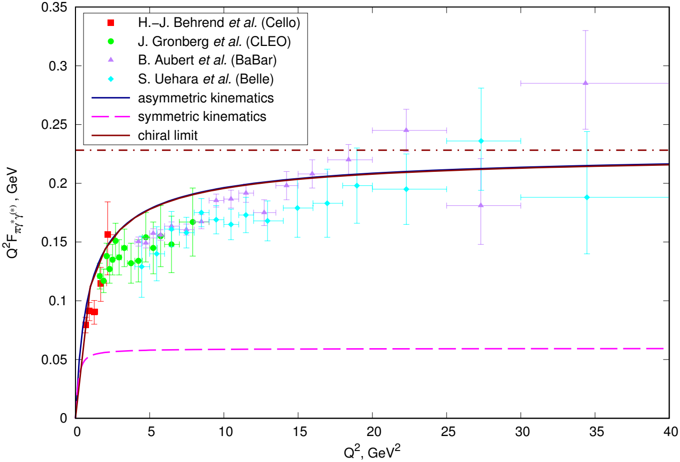
In momentum representation, the diagram has the following structure:

$$T_a^{\mu\nu}(p^2,k_1^2,k_2^2) = ie^2\delta^{(4)}(p-k_1-k_2)\varepsilon_{\mu\nu\alpha\beta}k_{1\alpha}k_{2\beta}T_a(p^2,k_1^2,k_2^2).$$

$$F_{P\gamma}\left(Q^2\right) = T\left(-M_P^2, Q^2, 0\right).$$

$$\Gamma(P\to\gamma\gamma) = \frac{\pi}{4}\alpha^2 M_P^3 g_{P\gamma\gamma}^2$$

$$g_{P\gamma\gamma} = T(-M_P^2,0,0) = F_{P\gamma}(0).$$



$$g_{\pi\gamma\gamma} = 0.272\text{GeV}^{-1} \qquad (g^{\text{exp}}_{\pi\gamma\gamma} = 0.274\text{GeV}^{-1}).$$

$$F_{\pi\gamma^*\gamma^*}\left(Q^2\right) = T\left(-M_P^2, Q^2, Q^2\right).$$

$$F_{\pi\gamma^*\gamma} \sim \varkappa_{\gamma^*\gamma} \frac{\sqrt{2} f_\pi}{Q^2}, \ \varkappa_{\gamma^*\gamma} = 1.23, \tag{16}$$

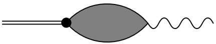
$Q^2 F_{\pi\gamma^*\gamma}$ approaches a constant value at large Q^2 in qualitative agreement with factorization prediction, but the value of constant $\varkappa_{\gamma^*\gamma}$ substantially differs from unity.
At the same time, asymptotic behavior of form factor in symmetric kinematics with two photons with equal virtuality Q^2,

$$F_{\pi\gamma^*\gamma^*} \sim \varkappa_{\gamma^*\gamma^*} \frac{\sqrt{2} f_\pi}{3Q^2}, \ \varkappa_{\gamma^*\gamma^*} = 1. \tag{17}$$

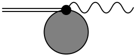
Leptonic decays

The amplitude for leptonic decay of pseudoscalar mesons can be parametrized as

$$A(H(p) \to \ell(k)\bar{\nu}_\ell(k')) = \frac{G_F}{\sqrt{2}} V_{qq'} \bar{\ell}\gamma_\mu(1-\gamma_5)\nu_\ell M_H^\mu = i\frac{G_F}{\sqrt{2}} V_{qq'} \bar{\ell}\gamma_\mu(1-\gamma_5)\nu_\ell f_H p^\mu .$$



(a)



(b)

$$\begin{aligned}(2\pi)^4\delta^{(4)}(p+q)M_\mu^{(\mathrm{a})} &= h_H \int d^4x \int d^4y \exp(ipx+iqy) \\ &\quad \times (-1)\int d\sigma_B \mathrm{Tr} V^{\mathcal{H}}(x) S(x,y) V^{\mathrm{CKM}} \gamma_\mu(1-\gamma_5) S(y,x), \end{aligned} \qquad \text{(a)}$$

$$(2\pi)^4\delta^{(4)}(p+q)M_\mu^{(\mathrm{b})} = h_H \int d^4x \exp(ipx+iqx)\ (-1)\int d\sigma_B \mathrm{Tr} V_\mu^{\mathcal{H}}(x;q) S(x,x). \qquad \text{(b)}$$

$$V^{\mathcal{H}}_{W^+_\mu f_1 f_2}(x;q) = -\frac{g}{\sqrt{2}}\int_0^1 d\tau \frac{1}{\tau}\frac{\partial}{\partial q_\mu}$$
$$\left\{ R \begin{pmatrix} 0 & V^{\mathrm{CKM}} \\ 0 & 0 \end{pmatrix}_{f_1 f} V^{\mathcal{H}}_{ff'}\left(\overset{\leftrightarrow}{\nabla}(x) - iq\tau\xi\right)\delta_{f'f_2} - \delta_{f_1 f} V^{\mathcal{H}}_{ff'}\left(\overset{\leftrightarrow}{\nabla}(x) + iq\tau\xi'\right)\begin{pmatrix} 0 & V^{\mathrm{CKM}} \\ 0 & 0 \end{pmatrix}\right.$$

V.Voronin, in preparation (2024)

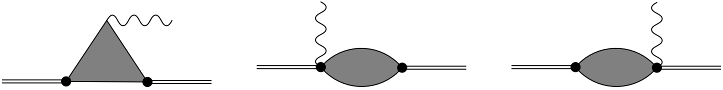
meson	decay constant f_P, MeV	f_P, MeV, this model
π	131.7 [PDG]	140.4
K	157.3 [PDG]	178.6
D	208.5 [PDG]	231.1
D_s	251.8 [PDG]	286.8
B	205.7 [PDG]	203
B_s	230.7 [PDG]	262.8
B_c	$427 \pm 6 \pm 2$ [McNeile, 2012]	450.2

[McNeile, 2012] C. McNeile, C. T. H. Davies, E. Follana, K. Hornbostel and G. P. Lepage, Phys. Rev. D **86** (2012), 074503.

Semileptonic decays

The amplitude for semileptonic decay of pseudoscalar mesons can be parametrized as

$$A(H(p)\to H'(p')\ell\bar{\nu}_\ell)=\frac{G_F}{\sqrt{2}}V_{qq'}\bar{\ell}\gamma_\mu(1-\gamma_5)\nu_\ell M^\mu_{HH'}.$$



(a) (b) (c)

$$(2\pi)^4\delta^{(4)}(p+p'+q)M^{(a)}_\mu = h_H h_{H'}\int d^4x\int d^4y\int d^4z\,\exp(ipx+ip'y+iqz)$$
$$\times(-1)\int d\sigma_B \mathrm{Tr}V^{\mathcal{H}}(x)S(x,y)V^{\mathcal{H}}(y)S(y,z)V^{\mathrm{CKM}}\gamma_\mu(1-\gamma_5)S(z,x) \quad \text{(a)}$$

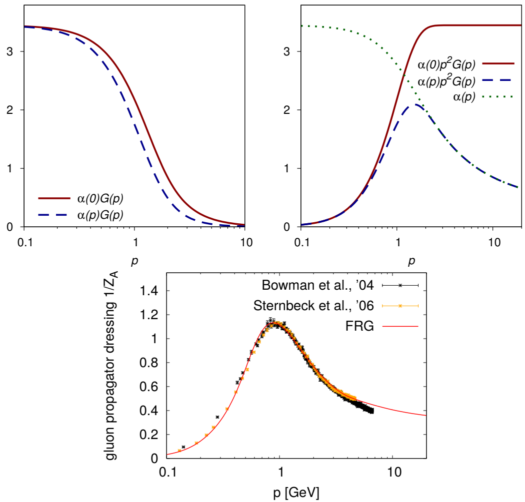
$$(2\pi)^4\delta^{(4)}(p+p'+q)M^{(b)}_\mu = h_H h_{H'}\int d^4x\int d^4y\,\exp(ipx+ip'y+iqx)$$
$$\times(-1)\int d\sigma_B \mathrm{Tr}V^{\mathcal{H}}_\mu(x;q)S(x,y)V^{\mathcal{H}}(y)S(y,x), \quad \text{(b)}$$

$$(2\pi)^4\delta^{(4)}(p+p'+q)M^{(c)}_\mu = h_H h_{H'}\int d^4x\int d^4y\,\exp(ipx+ip'y+iqx)$$
$$\times(-1)\int d\sigma_B \mathrm{Tr}V^{\mathcal{H}}_\mu(x)S(x,y)V^{\mathcal{H}}_\mu(y;q)S(y,x), \quad \text{(c)}$$

V.Voronin, in preparation (2024)

decay	decay width $\Gamma_{\rm exp}$, MeV [PDG]	Γ, MeV, this model
$K^- \to \pi^0 e^- \bar\nu_e$	$(2.70 \pm 0.021) \times 10^{-15}$	2.49×10^{-15}
$K^- \to \pi^0 \mu^- \bar\nu_\mu$	$(1.78 \pm 0.018) \times 10^{-15}$	1.62×10^{-15}
$K_L \to \pi^\pm e^\mp \nu_e$	$(5.22 \pm 0.014) \times 10^{-15}$	4.97×10^{-15}
$K_L \to \pi^\pm \mu^\mp \nu_\mu$	$(3.48 \pm 0.009) \times 10^{-15}$	3.24×10^{-15}
$D^0 \to K^- e^+ \nu_e$	$(5.69 \pm 0.042) \times 10^{-11}$	5.9×10^{-11}
$D^0 \to K^- \mu^+ \nu_\mu$	$(5.47 \pm 0.064) \times 10^{-11}$	5.73×10^{-11}
$D^0 \to \pi^- e^+ \nu_e$	$(4.67 \pm 0.064) \times 10^{-12}$	4.67×10^{-12}
$D^0 \to \pi^- \mu^+ \nu_\mu$	$(4.28 \pm 0.19) \times 10^{-12}$	4.57×10^{-12}
$D^+ \to \bar K^0 e^+ \nu_e$	$(5.56 \pm 0.057) \times 10^{-11}$	5.9×10^{-11}
$D^+ \to \bar K^0 \mu^+ \nu_\mu$	$(5.58 \pm 0.12) \times 10^{-11}$	5.73×10^{-11}
$D^+ \to \pi^0 e^+ \nu_e$	$(2.37 \pm 0.11) \times 10^{-12}$	2.33×10^{-11}
$D^+ \to \pi^0 \mu^+ \nu_\mu$	$(2.23 \pm 0.096) \times 10^{-12}$	2.29×10^{-11}

"Projection" to other methods/models



Functional RG, DSE, Lattice QCD

DSE in combination with Bethe-Salpeter approach

S. Kubrak, C. S. Fischer and R. Williams, arXiv:1412.5395 [hep-ph]
C. S. Fischer, S. Kubrak and R. Williams, Eur. Phys. J. A **51**, no. 1, 10 (2015) [arXiv:1409.5076 [hep-ph]]
C. S. Fischer, S. Kubrak and R. Williams, Eur. Phys. J. A **50**, 126 (2014) [arXiv:1406.4370 [hep-ph]].

S. M. Dorkin, L. P. Kaptari and B. Kampfer, arXiv:1412.3345 [hep-ph]
S. M. Dorkin, L. P. Kaptari, T. Hilger and B. Kampfer, Phys. Rev. C **89**, no. 3, 034005 (2014) [arXiv:1312.2721 [hep-ph]]

$$S^{-1}(p) = Z_2 S_0^{-1}(p) + 4\pi Z_2^2 C_F \int \frac{d^4k}{(2\pi)^4} \gamma^\mu S(k+p)\gamma^\nu (\delta_{\mu\nu} - k_\mu k_\nu/k^2)\, \frac{\alpha_{\rm eff}(k^2)}{k^2},$$

$$\alpha_{\rm eff}(q^2) = \pi\eta^7 x^2 e^{-\eta^2 x} \;+\; \frac{2\pi\gamma_m \left(1-e^{-y}\right)}{\ln\left[e^2 - 1 + (1+z)^2\right]},\; x = q^2/\Lambda^2,\; y = q^2/\Lambda_t^2,\; z = q^2/\Lambda_{\rm QCD}^2$$

Harmonic confinement - 4-dim. oscillator

R. P. Feynman, M, Kislinger, and F. Ravndal, Phys. Rev. D **3** (1971) 2706.

H. Leutwyler and J. Stern, "Harmonic Confinement: A Fully Relativistic Approximation to the Meson Spectrum," Phys. Lett. B **73** (1978) 75;

H. Leutwyler and J. Stern, "Relativistic Dynamics on a Null Plane," Annals Phys. **112** (1978) 94.

Laguerre polynomials

$$\mathcal{S}_2 = -\frac{1}{2}\int d^4x \int d^4z D(z)\Phi^2_{Jc}(x,z)$$

$$-2g^2 \int d^4x d^4x' d^4z d^4z' D(z) D(z') \Phi_{Jc}(x,z) \Pi_{Jc,J'c'}(x,x';z,z') \Phi_{J'c'}(x',z'),$$

$$\Phi^{aJ}(x,z) = \sum_{nl} \left(z^2\right)^{l/2} \varphi^{nl}(z) \Phi^{aJln}(x).$$

Soft-wall AdS/QCD models

A. Karch, E. Katz, D. T. Son and M. A. Stephanov, Phys. Rev. D **74**, 015005 (2006) [hep-ph/0602229]
T. Branz, T. Gutsche, V. E. Lyubovitskij, I. Schmidt and A. Vega, Phys. Rev. D **82**, 074022 (2010) [arXiv:1008.0268 [hep-ph]].

G. F. de Teramond and S. J. Brodsky, Phys. Rev. Lett. **102**, 081601 (2009)

$$S_\Phi = \frac{(-1)^J}{2}\int d^dx\,dz\left(\frac{R}{z}\right)^{d+1} e^{-\kappa^2 z^2}\left(\partial_N\Phi_J\partial^N\Phi_J - \mu_J^2(z)\Phi_J\Phi^J\right)$$

$$M_{nJ} = 4\kappa^2\left(n + \tfrac{l+J}{2}\right)$$

$$\Phi_J(x,z) = \sum_n \phi_{nJ}\Phi_{nJ}(x),$$

$$\phi_{nJ} = R^{J-(d-1)/2}\kappa^{1+l}z^{l-J+2}L_n^l(\kappa^2 z^2)$$

G.V. Efimov and S.N. Nedelko, Phys. Rev. D 51, 176 (1995)

$$J^{aJ}(x,z) = \sum_{nl}\left(z^2\right)^{l/2} f_J^{nl}(z)J_J^{aJln}(x),\quad J_J^{aJln}(x) = \bar q(x)V_J^{aJln}(x)\,q(x)$$

$$M_n^2 \propto Bn$$
$$M_l^2 \propto Bl$$

$$f^{nl}_{\mu_1\dots\mu_l} = L_n^l\left(z^2\right)T^{(l)}_{\mu_1\dots\mu_l}(n_z),\quad n_z = \frac{z}{\sqrt{z}},$$

$$\int_0^\infty du\rho_l(u)L_n^l(u)L_{n'}^l(u) = \delta_{nn'},\quad \rho_l(u) = u^l e^{-u}.$$

THANK YOU!