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Covariance and noncovariance of relativistic spin equations

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OUTLINE

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- **Three the most important definitions of fundamental operators**

Evidence of noncovariance of relativistic spin equations

Physics – Uspekhi **43** (10) 1055 – 1066 (2000)

Spinning relativistic particles in external fields

A A Pomeranskiĭ, R A Sen'kov, I B Khriplovich

Problems with covariant equations of motion

$$H = -\frac{eg}{2m}\,\mathbf{sB} + \frac{e(g-1)}{2m^2}\,\mathbf{s}[\mathbf{p}\times\mathbf{E}]\,.$$

Coordinate x instead of r, v=dr/dt

$$\mathbf{x} = \mathbf{r} + \frac{\gamma}{m(\gamma+1)}\,\mathbf{s}\times\mathbf{v}\,, \quad \gamma = \frac{1}{\sqrt{1-v^2}}$$

center of mass

center of charge

Center of mass corresponds to the lab-frame spin.

Center of charge corresponds to the rest-frame spin.

Spin definition in quantum mechanics

In quantum mechanics, the spin definition in the lab frame in the equation j=l+s is useless because it confuses the following analysis. In the Dirac and Foldy-Wouthuysen representations, the relativistic spin operator is a matrix $\hbar\boldsymbol{\Sigma}/2$. Quantum mechanics uses only the rest-frame spin coupled with the cenfer-of-charge coordinates.

Position definition in quantum mechanics

In quantum mechanics, an electromagnetic force and even a gravitational one are determined relative of center of charge of a particle. Therefore, quantum mechanics uses cenfer-of-charge coordinates.

Position and spin in the framework of Poincaré group

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Position and spin in relativistic quantum mechanics

There are ten independent *fundamental quantities* $p_\mu = (H, \mathbf{p})$, $j_{\mu\nu}$ ($\mu, \nu = 0, 1, 2, 3$) describing the momentum and total angular momentum and characteristic for the dynamical system. The antisymmetric tensor $j_{\mu\nu}$ is defined by the two vectors, $\boldsymbol{j}$ and $\boldsymbol{K}$. As a result, there are the ten infinitesimal generators of the Poincaré group, namely, the generators of the infinitesimal space translations $\boldsymbol{p} = (p_i)$, the generator of the infinitesimal time translation H, the generators of infinitesimal rotations $\boldsymbol{j} = (j_i)$, and the generators of infinitesimal Lorentz transformations (boosts) $\boldsymbol{K} = (K_i)$ ($i = 1, 2, 3$).

Poincaré group and fundamental operators in relativistic quantum mechanics

The operators being counterparts of fundamental classical variables should satisfy the relations

$$[p_i,p_j]=0,\quad [p_i,\mathcal{H}]=0,\quad [j_i,\mathcal{H}]=0,$$

$$[j_i,j_j]=ie_{ijk}j_k,\quad [j_i,p_j]=ie_{ijk}p_k,\quad [j_i,K_j]=ie_{ijk}K_k,$$

$$[K_i,\mathcal{H}]=ip_i,\quad [K_i,K_j]=-ie_{ijk}j_k,\quad [K_i,p_j]=i\delta_{ij}\mathcal{H},$$

$$[q_i,p_j]=i\delta_{ij},\quad [q_i,j_j]=ie_{ijk}q_k,\quad [q_i,s_j]=0,\quad [j_i,p_j]=ie_{ijk}p_k,$$

$$[s_i,p_j]=0,\quad [l_i,l_j]=ie_{ijk}l_k,\quad [s_i,s_j]=ie_{ijk}s_k,\quad [l_i,s_j]=0,$$

$$[q_i,q_j]=0,$$

key definitions

$$[q_i,K_j]=\frac{1}{2}\left(q_j\left[q_i,\mathcal{H}\right]+\left[q_i,\mathcal{H}\right]q_j\right)-it\delta_{ij}.$$

There is a difference for the Dirac and FW representations!

Commutation relations should agree with Poisson brackets:

$$\{Q_i, P_j\} = \delta_{ij}, \quad \{Q_i, J_j\} = e_{ijk}Q_k,$$

$$\{Q_i, K_j\} = \frac{1}{2}(Q_j\{Q_i, H\} + \{Q_i, H\}Q_j) - t\delta_{ij}.$$

It follows from the above equations that

$$\{L_i, P_j\} = e_{ijk}P_k, \qquad \{S_i, P_j\} = 0.$$

The Poisson brackets for the conventional particle position defining *the center of charge* are equal to zero:

$$\{Q_i, Q_j\} = 0.$$

As a result, for a free particle

$$\{Q_i, L_j\} = e_{ijk}Q_k, \quad \{Q_i, S_j\} = 0, \quad \{P_i, S_j\} = 0,$$

$$\{L_i, L_j\} = e_{ijk}L_k, \quad \{S_i, S_j\} = e_{ijk}S_k.$$

$$\{L_i, S_j\} = 0.$$

No spin-orbit interaction for a free particle

The quantity $\boldsymbol{\zeta}$ defines the three-component laboratory-frame spin and can be written in the form

$$\zeta = s - \frac{p \times (p \times s)}{m(\epsilon + m)}.$$

$$j = l + s = \mathcal{L} + \zeta, \quad \mathcal{L} = \mathcal{X} \times p.$$

$$\mathcal{X}_{\mathrm{FW}} = x + \frac{s \times p}{m(\epsilon + m)}, \quad \mathcal{L}_{\mathrm{FW}} = \mathcal{X}_{\mathrm{FW}} \times p,$$

where $\boldsymbol{x}$ is the FW center-of-charge position operator.

Three the most important definitions of fundamental operators

Any set of fundamental operators p_μ and $j_{\mu\nu}$ is correct if all commutation relations for H, $\mathbf{p}$, $\mathbf{j}$ and $\mathbf{K}$ are satisfied. The most important sets of operators characterize Dirac operators in the Dirac representation and operators in the Foldy-Wouthuysen representation based on the center-of-charge and center-of-mass position operators. For the Dirac operators, some commutators which are not fundamental (like $[H,l_i]=0$, $[H,s_i]=0$) are not satisfied. As a result, the Dirac representation distorts a physical picture as compared with a classical picture in the Minkowski spacetime but this representation uses a commutative geometry. The main preference of the Dirac representation is a covariant form of initial equations. The relativistic Foldy-Wouthuysen transformation allows one to derive correct (but noncovariant) equations of motion. The Foldy-Wouthuysen representation also uses

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a commutative geometry. Applying the fundamental operators based on the center-of-mass position operator leads to covariant equations of motion but needs the use of a noncommutative geometry.


Thank you for your attention