Neutrino propagator in media: spin properties and spectral representation

Alexander Kaloshin

Irkutsk State University

20 сентября 2018 г.

in collaboration with D.Voronin

XXIV Baldin seminar, September 17-22, 2018, Dubna

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Details:

D.Voronin, A.Kaloshin, JETP Lett. **106**, 209 (2017)

A.Kaloshin, D.Voronin, arXiv:1808.05514

Introduction

Neutrino physics, is related with a wide spectrum of physical problems, including the astrophysical ones. The most prominant effect in neutrinos passing through matter is related with resonance amplification of oscillations (MSW effect), which solves the solar neutrino problem.

Neutrino can interact with electromagnetic field due to anomalious magnetic moment and present-day interest for this subject is related first of all with search of new physics.

Most justified way to describe mixing and oscillations phenomena in neutrinos system is the Quantum Field Theory (QFT) approach, and the necessary element of QFT description is the neutrino propagator.

Here we biuld a spectral representation of neutrino propagator in matter moving with constant velocity or in constant homogenious magnetic field. A spectral representation was discussed earlier for dressed fermion propagator in theory with parity violation (Kaloshin, Lomov EPJ (2012)) and for matrix propagator with mixing of few fermionic fields (Kaloshin, Lomov IJMP (2016)).

Propagator in moving matter and spin projectors

In media there exist two 4-vectors: momentum of particle p and matter velocity u. Most general expression for inverse propagator

$$S(p,u) = G^{-1} = s_1 I + s_2 \hat{p} + s_3 \hat{u} + s_4 \sigma^{\mu\nu} p_\mu u_\nu + \\ + s_5 i \varepsilon^{\mu\nu\lambda\rho} \sigma^{\mu\nu} u_\lambda p_\rho + s_6 \gamma^5 + s_7 \hat{p} \gamma^5 + s_8 \hat{u} \gamma^5, \quad (1)$$

where s_i are scalar function.
We will solve the eigenvalue problem for inverse propagator

$$S\Psi_i = \lambda_i \Psi_i. \quad (2)$$

As a starting point it is convenient to introduce γ-matrix basis with simple multiplicative properties.
Let us introduce the 4-vector z^μ, which is linear combination of two vectors p, u and has properties of fermion polarization vector:

$$z^\mu p_\mu = 0, \quad z^2 = -1. \quad (3)$$

Propagator in moving matter and spin projectors

Orthogonal to momentum combination is

$$z^\mu = b\,\left(p^\mu (up) - u^\mu p^2\right), \tag{4}$$

where b is the normalization factor, $b = [p^2((up)^2 - p^2)]^{-1/2}$.
Then one can construct the **generalized** off-shell spin projectors:

$$\Sigma^\pm = \frac{1}{2}(1 \pm \gamma^5 \hat{z}\hat{n}), \quad \Sigma^\pm \Sigma^\pm = \Sigma^\pm, \quad \Sigma^\pm \Sigma^\mp = 0, \tag{5}$$

where $n^\mu = p^\mu / W, \quad W = \sqrt{p^2}$.
One can see that $\Sigma^\pm$ commute with all γ-matrices in inverse propagator (1)

$$[\Sigma^\pm(z), S] = 0. \tag{6}$$

Multiplying the inverse propagator $S(p, u)$ (1) by unit matrix

$$S = (\Sigma^+(z) + \Sigma^-(z))S \equiv S^+ + S^-, \tag{7}$$

one obtains two orthogonal terms S^+, S^-.

Propagator in moving matter and spin projectors

One more useful property of $\Sigma^{\pm}$:

"under their observation" (i.e. in S^{+}, S^{-} terms) γ-matrix structures may be simplified. Namely: γ-matrices, which contain the matter velocity u^{μ} may be transformed to the set of four matrices without velocity: $I, \hat{p}, \gamma^5, \hat{p}\gamma^5$.
For example, one can rewrite the term $\hat{u}$ as a linear combination $\hat{p}$ and $\hat{z}$ and to use the projector property ($\Sigma^{+} \cdot \gamma^5 \hat{z}\hat{n} = \Sigma^{+}$):

$$\Sigma^{+}\hat{u} = \Sigma^{+}(a_1\hat{p} + a_2\hat{z}) = \Sigma^{+}(z)(a_1\hat{p} - \frac{a_2}{W}\hat{p}\gamma^5). \tag{8}$$

Then it's convenient to introduce the off-shell momentum orthogonal projectors:

$$\Lambda^{\pm} = \frac{1}{2}(1 \pm \hat{n}), \quad n^{\mu} = \frac{p^{\mu}}{W}, \quad W = \sqrt{p^2} \tag{9}$$

Propagator in moving matter: basis

Having the momentum $\Lambda^{\pm}$ and spin projectors $\Sigma^{\pm}$, one can build the basis (R-basis), which will be used below in the eigenvalue problem

$$\begin{aligned} R_1 &= \Sigma^-\Lambda^+, & R_5 &= \Sigma^+\Lambda^+, \\ R_2 &= \Sigma^-\Lambda^-, & R_6 &= \Sigma^+\Lambda^-, \\ R_3 &= \Sigma^-\Lambda^+\gamma^5, & R_7 &= \Sigma^+\Lambda^+\gamma^5, \\ R_4 &= \Sigma^-\Lambda^-\gamma^5, & R_8 &= \Sigma^+\Lambda^-\gamma^5. \end{aligned} \tag{10}$$

The inverse propagator (1) may be written in this basis as

$$S(p,u) = \sum_{i=1}^{4} R_i S_i(p^2, pu) + \sum_{i=5}^{8} R_i S_i(p^2, pu), \tag{11}$$

where these two sums are orthogonal to each other.

Propagator in moving matter: basis

Multiplicative properties of the R-basis (10) are presented in Table 1, where column elements multiply from left the row elements.

Таблица: Multiplicative properties of the matrix basis (10)

	R_1	R_2	R_3	R_4	R_5	R_6	R_7	R_8
R_1	R_1	0	R_3	0	0	0	0	0
R_2	0	R_2	0	R_4	0	0	0	0
R_3	0	R_3	0	R_1	0	0	0	0
R_4	R_4	0	R_2	0	0	0	0	0
R_5	0	0	0	0	R_5	0	R_7	0
R_6	0	0	0	0	0	R_6	0	R_8
R_7	0	0	0	0	0	R_7	0	R_5
R_8	0	0	0	0	R_8	0	R_6	0

So, the eigenvalue problem for inverse propagator (11) is separeted into two different problems: one for $R_1..R_4$ and another for $R_5..R_8$. Every problem has two different eigenvalues.

Spectral representation of propagator

Let us recall that the term spectral representation of linear hermitian operator $\hat{A}$ means the following (see textbook of Messia)

$$\hat{A} = \sum \lambda_i |i\rangle\langle i| = \sum \lambda_i \Pi_i, \tag{12}$$

which contains the eigenvalues λ_i and eigenprojectors $\Pi_i = |i\rangle\langle i|$.

$$\hat{A}|i\rangle = \lambda_i |i\rangle. \tag{13}$$

Orthonornality of the vectors leads to orthogonality of projectors

$$\Pi_i \Pi_k = \delta_{ik} \Pi_k. \tag{14}$$

If an operator is not hermitian, to build a spectral representation one needs to solve two eigenvalue problems: left and right ones.
We want to construct a spectral representation for inverse propagator of general form (1), (11), so we should solve the eigenvalue problem

$$S\Pi_i = \lambda_i \Pi_i. \tag{15}$$

Spectral representation of propagator

After solving we get the spectral representation of inverse propagator in a matter:

$$S(p,u) = \sum_{i=1}^{4} \lambda_i \Pi_i. \tag{16}$$

If the eigenprojectors set is the complete orthogonal system, then propagator is easily obtained by reversing of (16)

$$G(p,u) = \sum_{i=1}^{4} \frac{1}{\lambda_i} \Pi_i, \tag{17}$$

and looks as a sum of single poles, accompanied by corresponding orthogonal projectors.

Spectral representation of propagator in matter

The use of R-basis (10) simplifies essentially solution of eigenvalue problem.
So, the eigenstate problem for first quartet of basis elements

$$\left(\sum_{k=1}^{4} R_k S_k\right) \cdot \left(\sum_{i=1}^{4} R_i A_i\right) = \lambda \left(\sum_{i=1}^{4} R_i A_i\right) \tag{18}$$

coinsides with the eigenstate problem for dressed vacuum propagator with parity violation (Kaloshin, Lomov EPJ (2012)). The presence of matter leads only to appearence of spin projector in (10) and modification of scalar coefficients.

Spectral representation of propagator in matter

Repeating the algebraic operations from (Kaloshin, Lomov EPJ (2012)) , we have answer for eigenvalues and eigenprojectors:

$$\lambda_{1,2} = \frac{S_1 + S_2}{2} \pm \sqrt{\left(\frac{S_1 - S_2}{2}\right)^2 + S_3 S_4} \,,$$
$$\Pi_1 = \frac{1}{\lambda_2 - \lambda_1}\Big((S_2 - \lambda_1)R_1 + (S_1 - \lambda_1)R_2 - S_3 R_3 - S_4 R_4\Big), \tag{19}$$
$$\Pi_2 = \frac{1}{\lambda_1 - \lambda_2}\Big((S_2 - \lambda_2)R_1 + (S_1 - \lambda_2)R_2 - S_3 R_3 - S_4 R_4\Big),$$

$$\lambda_{3,4} = \frac{S_5 + S_6}{2} \pm \sqrt{\left(\frac{S_5 - S_6}{2}\right)^2 + S_7 S_8} \,,$$
$$\Pi_3 = \frac{1}{\lambda_4 - \lambda_3}\Big((S_6 - \lambda_3)R_5 + (S_5 - \lambda_3)R_6 - S_7 R_7 - S_8 R_8\Big), \tag{20}$$
$$\Pi_4 = \frac{1}{\lambda_3 - \lambda_4}\Big((S_6 - \lambda_4)R_5 + (S_5 - \lambda_4)R_6 - S_7 R_7 - S_8 R_8\Big).$$

Recall that the indexes $1, 2$ refer to S^- (i.e to first quartet in (11), and $3, 4$ to contribution S^+.

Spectral representation of propagator in matter

The introduced by us four-vector z^μ (4) plays role of the complete polarization axis and all eigenvalues are classified by the projection of spin onto this axis. In contrast to vacuum, this axis is not arbitrary. As it will be seen from discussion of SM case, the projection on this axis is not conserved in general case.

Neutrino propagator in matter (SM)

In the case of SM a fermion propagator in matter looks like:

$$S(p,u) = \hat{p} - m - \alpha \hat{u}(1-\gamma^5), \tag{21}$$

where α is some constant. For example, in case of electron neutrino

$$\alpha^{(\nu_e)} = \frac{G_F}{\sqrt{2}}(n_e(1+4\sin^2\theta_W) + n_p(1-4\sin^2\theta_W) - n_n),$$

where n_e, n_p, n_n are densities of matter particles.

Neutrino propagator in matter (SM)

The solutions of the eigenvalue problem (15) in this case have the form:

$$\lambda_{1,2} = -m \pm W\sqrt{1+2K^+},$$

$$\lambda_{3,4} = -m \pm W\sqrt{1+2K^-}, \tag{22}$$

$$\Pi_{1,2} = \Sigma^- \cdot \frac{1}{2}\left[1 \pm \hat{n}\,\frac{1+K^+-\gamma^5 K^+}{\sqrt{1+2K^+}}\right],$$

$$\Pi_{3,4} = \Sigma^+ \cdot \frac{1}{2}\left[1 \pm \hat{n}\,\frac{1+K^--\gamma^5 K^-}{\sqrt{1+2K^-}}\right]. \tag{23}$$

Notation: $K^\pm = -\alpha\Big((pu) \pm \sqrt{(up)^2 - p^2}\Big)/p^2$.

Neutrino propagator in matter (SM)

In case of SM it is easy to verify that the spin projection on the axis of complete polarization is not conserved. The Hamiltonian is defined by Dirac operator (21)

$$H = p^0 - \gamma^0 S. \tag{24}$$

We can use a known zeroth commutator

$$[R, S] = 0, \qquad R = \gamma^5 \hat{z}\hat{n}, \tag{25}$$

for simple calculation of commutator R with Hamiltonian

$$[R, H] = \gamma^0 [S, R] + [\gamma^0, R] S = [\gamma^0, R] S, \tag{26}$$

which may be reduced to $[\gamma^0, R]$. With use of the standard representation of γ-matrices we have

$$R = \begin{pmatrix} \boldsymbol{\sigma}\mathbf{v} & -i\boldsymbol{\sigma}\boldsymbol{\xi} \\ -i\boldsymbol{\sigma}\boldsymbol{\xi} & \boldsymbol{\sigma}\mathbf{v} \end{pmatrix}, \quad \mathbf{v} = n^0\mathbf{z} - z^0\mathbf{n}, \quad \boldsymbol{\xi} = [\mathbf{z} \times \mathbf{n}]. \tag{27}$$

Neutrino propagator in matter (SM)

If to require $[\gamma^0, R] = 0$, we come to condition $\boldsymbol{\xi} = 0$, i.e.

$$\boldsymbol{\xi} \equiv [\mathbf{z} \times \mathbf{n}] = bW[\mathbf{p} \times \mathbf{u}] = 0. \tag{28}$$

In this case the found polarization vector z^μ (4) takes the form

$$z^\mu = \frac{1}{W}\left(|\mathbf{p}|,\ p^0 \frac{\mathbf{p}}{|\mathbf{p}|}\right), \tag{29}$$

which corresponds to helicity state of fermion, but the off-shell one since $W \neq m$.
In general case we have

$$[\Sigma^\pm, S] = 0, \qquad \textbf{but} \qquad [\Sigma^\pm, H] \neq 0$$

The rest matter

In this case, according to Eq. (28), spin projection is concerved and polarization vector z^μ corresponds to helicity state.
Straight calculation gives

$$\Sigma^{\pm} = \frac{1}{2}\Big(1 \pm \boldsymbol{\Sigma}\frac{\mathbf{p}}{|\mathbf{p}|}\Big), \;\; \boldsymbol{\Sigma} = \gamma^0 \boldsymbol{\gamma}\gamma^5. \tag{30}$$

Eigenvalues:

$$\lambda_{1,2} = -m \pm W\sqrt{1 - 2\alpha(E + |\mathbf{p}|)/W^2}, \tag{31}$$

$$\lambda_{3,4} = -m \pm W\sqrt{1 - 2\alpha(E - |\mathbf{p}|)/W^2}. \tag{32}$$

Thus, for the rest matter the well-known fact (D. Mannheim (1988), J.Pantaleone (1992)) is reproduced that neutrino with definite helicity has a definite law of dispersion in matter.

If some eigenvalue is vanished, we obtain a dispersion relation – energy and momentum connection. We have for $\lambda_{1,2} = 0$

$$E^2 - 2\alpha E - m^2 - \mathbf{p}^2 - 2\alpha|\mathbf{p}| = 0, \tag{33}$$

$$E_{1,2} = \alpha \pm \sqrt{(|\mathbf{p}| + \alpha)^2 + m^2}, \tag{34}$$

and for $\lambda_{3,4} = 0$:

$$E^2 - 2\alpha E - m^2 - \mathbf{p}^2 + 2\alpha|\mathbf{p}| = 0, \tag{35}$$

$$E_{3,4} = \alpha \pm \sqrt{(|\mathbf{p}| - \alpha)^2 + m^2}, \tag{36}$$

Well-known expressions

Neutrino propagator in magnetic field

We found that in moving matter there exists an axis of complete polarization z^μ, and corresponding spin projectors commute with the propagator.

A similar situation arises for neutrino in a magnetic field.

An inverse propagator of a neutral fermion with an anomalous magnetic moment μ in a constant external electromagnetic field:

$$S = \hat{p} - m - \frac{i}{2}\mu\sigma^{\alpha\beta}F_{\alpha\beta}, \qquad \sigma^{\alpha\beta} = \frac{1}{2}[\gamma^\alpha, \gamma^\beta]. \tag{37}$$

In the case of a magnetic field, it takes more customary form:

$$S = \hat{p} - m + \mu\mathbf{\Sigma}\mathbf{B}, \qquad \mathbf{\Sigma} = \gamma^0\boldsymbol{\gamma}\gamma^5. \tag{38}$$

Having electromagnetic field tensor and 4-momentum, we can construct a polarization vector z^μ ($z^2 = -1$ and $z_\mu p^\mu = 0$):

$$z^\mu = b\epsilon^{\mu\nu\lambda\rho}F_{\nu\lambda}p_\rho, \qquad b = (p_0^2\mathbf{B}^2 - (\mathbf{pB})^2)^{-1/2}. \tag{39}$$

Neutrino in magnetic field

Using this vector we can construct a spin projector:

$$\Sigma^{\pm} = \frac{1}{2}(1 \pm \gamma^5 \hat{z}). \tag{40}$$

It is easy to see that the spin projectors commute with the inverse propagator (38). In the case of magnetic field:

$$z^{\mu} = b((\mathbf{Bp}), p^0 \mathbf{B}), \qquad b = (p_0^2 \mathbf{B}^2 - (\mathbf{Bp})^2)^{-1/2} \tag{41}$$

and matrix $\gamma^5 \hat{z}$ looks as

$$R \equiv \gamma^5 \hat{z} = b(\gamma^5 \gamma^0 (\mathbf{Bp}) + p^0 \gamma^0 (\mathbf{\Sigma B})), \qquad R^2 = 1. \tag{42}$$

After this, it is easy to see that $[S, \Sigma^{\pm}] = 0$.
Further we can apply the same trick that was used for matter: "under observation" of the spin projector, the gamma-matrix structures are simplified.

Neutrino in magnetic field

Again:

$$S = (\Sigma^+(z) + \Sigma^-(z))S \equiv S^+ + S^-. \tag{43}$$

Since $[S, R] = 0$, two matrices have a common eigenvector:

$$S\Psi = \lambda\Psi, \quad \gamma^5 \hat{z}\Psi = \sigma\Psi, \quad \sigma = \pm 1. \tag{44}$$

The eigenvector of the operator R is obvious: $\Psi^\pm = \Sigma^\pm \Psi_0$, therefore the system looks like this:

$$S^\pm \Psi^\pm = \lambda\Psi^\pm, \qquad \gamma^5 \hat{z}\Psi^\pm = \pm\Psi^\pm. \tag{45}$$

Since the eigenvalues of the matrix R are equal to ± 1, from (42) we can find the useful relation

$$(\mathbf{\Sigma}\mathbf{B})\Psi^\pm = \frac{1}{p^0}(\gamma^5(\mathbf{pB}) \pm \gamma^0 \frac{1}{b})\Psi^\pm. \tag{46}$$

Then, in analogy with the case of matter, in the $S^\pm$ contributions the γ-matrix structure can be transformed. Instead of (38) we get

$$S^\pm = \Sigma^\pm(z)\Big[\hat{p} - m + \frac{\mu}{p^0}(\gamma^5(\mathbf{pB}) \pm \gamma^0 \frac{1}{b})\Big]. \tag{47}$$

Let us recall that for covariant matrix of the form

$$S = aI + b\hat{p} + c\gamma^5 + d\hat{p}\gamma^5 \tag{48}$$

solutions of the matrix eigenvalue problem are known.
The inverse propagator in the external field (38), (47) is non-covariant (in particular, it contains γ^0), but for algebraic problem this is not so important. Therefore, if we redefine the vector p^μ in $S^\pm$, we can get rid of γ^0 and use the ready answer for eigenvalues and eigenprojectors.
So, we can introduce "4-vector"

$$p^\mu_\pm = (p^0 \pm \frac{\mu}{bp_0},\ \mathbf{p}) \tag{49}$$

and after this, the inverse propagator takes the form:

$$S^\pm = \hat{p}_\pm - m + \mu\gamma^5 \frac{(\mathbf{Bp})}{p_0}, \tag{50}$$

in which there are only I, $\hat{p}_\pm$ and γ^5 matrix.

After this simplification, we can use general formulas (19) :

$$\lambda_1^{\pm} = -m + \sqrt{W_{\pm}^2 + \frac{\mu^2}{p_0^2}(\mathbf{Bp})^2}, \quad \lambda_2^{\pm} = -m - \sqrt{W_{\pm}^2 + \frac{\mu^2}{p_0^2}(\mathbf{Bp})^2}, \quad (51)$$

$$\Pi_1^{\pm} = \frac{\Sigma^{\pm}}{2}\Big(1 - \frac{1}{A^{\pm}}(\hat{p}_{\pm} + \frac{\mu(\mathbf{Bp})}{p_0}\gamma^5)\Big), \quad (52)$$

$$\Pi_2^{\pm} = \frac{\Sigma^{\pm}}{2}\Big(1 + \frac{1}{A^{\pm}}(\hat{p}_{\pm} + \frac{\mu(\mathbf{Bp})}{p_0}\gamma^5)\Big). \quad (53)$$

Notations: $W_{\pm} = \sqrt{p_{\pm}^2}, \quad A^{\pm} = \sqrt{W_{\pm}^2 + \mu^2(\mathbf{Bp})^2/p_0^2}$.
If the eigenvalue is vanishing, we can obtain the well-known dispersion law for movement of anomalous magnetic moment in magnetic field (I.M. Ternov, V.G. Bagrov, A.M. Khapaev. JETP (1965))

$$E^2 = m^2 + \mathbf{p}^2 + \mu^2\mathbf{B}^2 \pm 2\mu\sqrt{m^2\mathbf{B}^2 + \mathbf{B}_{\perp}^2}. \quad (54)$$

Here $\pm$ corresponds to different signs in (50), i.e. to terms $S^{\pm}$ in propagator, which are accompanied by spin projectors $\Sigma^{\pm}$.

Neutrino in magnetic field

The spectral representation of the inverse propagator with found eigenvalues and eigenprojectors can be written as:

$$S = \sum_{i=1}^{2} \lambda_i^+ \Pi_i^+ + \sum_{i=1}^{2} \lambda_i^- \Pi_i^- . \qquad (55)$$

So, in constant magnetic field all eigenvalues are classified by the spin projection on the fixed axis z (41). The inverse propagator (38) can be connected with the Dirac Hamiltonian

$$S = \gamma^0 (p^0 - H_D), \qquad H_D = \alpha \mathbf{p} + \beta m + \mu \gamma^0 (\mathbf{\Sigma B}). \qquad (56)$$

Using the zero commutator of the matrix $R = \gamma^5 \hat{z}$ with the inverse propagator

$$0 = [R, S] = \gamma^0 [H_D, R] + [R, \gamma^0](p^0 - H_D), \qquad (57)$$

we can reduce the case to the commutator $[R, \gamma^0]$. Calculating it in the standard representation of gamma-matrices, we have

$$[\gamma^5 \hat{z}, \gamma^0] = \begin{pmatrix} 0 & 2z^0 \\ 2z^0 & 0 \end{pmatrix}, \qquad z^0 = b\,(\mathbf{Bp}). \tag{58}$$

So we see that the projection of the spin on the axis of complete polarization (39) is conserved only in case of a transverse magnetic field.

Summary

So, we have constructed a spectral representation for neutrino propagator

- in matter, moving with constant 4-velocity u^μ;
- in external magnetic field $\boldsymbol{B} = const$

Starting point: eigenvalue problem for inverse propagator S

$$S\ \Psi = \lambda\ \Psi.$$

As a result, we have representation of propagator $G = S^{-1}$

$$G = \sum_i \frac{1}{\lambda_i} \Pi_i$$

In both cases there exists the fixed 4-axis of polarization z^μ

$$[\Sigma^{\pm}(z),\ S] = 0.$$

In both cases projection is not conserved (only in special cases)

$$[\Sigma^{\pm}(z),\ \hat{H}_D] \neq 0.$$

Summary

Presence of such axis simplifies essentially the problem.
As for eigenvalues of inverse propagator S, they are classified according to projection on this fixed axis

$$\lambda_i^{\pm}$$

Therefore, all laws of dispersy in media also separated into two classes

$$E_i^{\pm}(p)$$

Summary

After all:

- We have very simple and convenient approach for propagation of fermion in media;

  Spectral representation $\Leftrightarrow$ Diagonalization
- The approach can be applied to problem of mixing of few fermion fields;
- There are some interesting questions, conserning to role of the found axis of complete polarization in spin dynamics.

Thank you for attention!

Details for rest matter

$$\Pi_1 = \frac{1}{4}(1 - \mathbf{\Sigma}\frac{\mathbf{p}}{|\mathbf{p}|})\Big(1 + \frac{\hat{n}}{B^+}[1 - \frac{\alpha(E + |\mathbf{p}|)}{W^2}(1 - \gamma^5)]\Big), \tag{59}$$

$$\Pi_2 = \frac{1}{4}(1 - \mathbf{\Sigma}\frac{\mathbf{p}}{|\mathbf{p}|})\Big(1 - \frac{\hat{n}}{B^+}[1 - \frac{\alpha(E + |\mathbf{p}|)}{W^2}(1 - \gamma^5)]\Big), \tag{60}$$

$$\Pi_3 = \frac{1}{4}(1 + \mathbf{\Sigma}\frac{\mathbf{p}}{|\mathbf{p}|})\Big(1 + \frac{\hat{n}}{B^-}[1 - \frac{\alpha(E - |\mathbf{p}|)}{W^2}(1 - \gamma^5)]\Big), \tag{61}$$

$$\Pi_4 = \frac{1}{4}(1 + \mathbf{\Sigma}\frac{\mathbf{p}}{|\mathbf{p}|})\Big(1 - \frac{\hat{n}}{B^-}[1 - \frac{\alpha(E - |\mathbf{p}|)}{W^2}(1 - \gamma^5)]\Big), \tag{62}$$

where $B^\pm = \sqrt{1 - 2\alpha(E \pm |\mathbf{p}|)/W^2}$.

Spectral representation of matrix of general form

In order to build a spectral representation of the matrix S of general form, one needs to solve two eigenvalue problems.
Left eigenvalue problem:

$$S\psi = \lambda\psi \tag{63}$$

and right one:

$$\phi^T S = \phi^T \lambda. \tag{64}$$

Here S is matrix of dimension n and ψ, ϕ are the columns of this dimension.
Let us indicate the main properties of these problems.

- The spectra of the left and right problems coinsides. Indeed, the eigenvalues of the left problem are defined by equation $det(S - \lambda E) = 0$, as for spectrum of the right – it is defined by transpose matrix $det(S - \lambda E)^T = 0$.

Spectral representation of matrix of general form

- Orthogonality of eigenvectors. Let us write down two equations

$$S\psi_i = \lambda_i \psi_i. \tag{65}$$

$$\phi_k^T S = \phi_k^T \lambda_k. \tag{66}$$

Let us multiply (65) by ϕ_k^T from the left, (66) by ψ_i from the right and subtruct one equation from another. We have

$$0 = (\lambda_i - \lambda_k)\phi_k^T \psi_i, \tag{67}$$

i.e. eigenvectors of left and right problems ϕ_k, ψ_i are orthogonal at $i \neq k$.

$$\phi_k^T \psi_i = \psi_i^T \phi_k \equiv (\psi_i, \phi_k) = 0 \quad \text{at} \quad i \neq k \tag{68}$$

One can require the orthonormality of these two sets of vectors

$$(\psi_i, \phi_k) = \delta_{ik}. \tag{69}$$

Spectral representation of matrix of general form

- Having solutions of both left and right problems with the property (69), one can build matrices of the form

$$\Pi_i = \psi_i \phi_i^T, \qquad i = 1 \dots n, \tag{70}$$

which are the set of orthogonal projectors.

$$\Pi_i \Pi_k = \delta_{ik} \Pi_k \tag{71}$$

Note that the projectors Π_i (eigenprojectors) are the matrix solution of both left and right eigenvalue problems.

- In particular case of hermitian matrix S, solutions of left and right problems are related as follows

$$\phi_i = \psi_i^* \tag{72}$$

and eigenprojectors look like:

$$\Pi_i = \psi_i \psi_i^\dagger, \qquad i = 1 \dots n. \tag{73}$$

Spectral representation of matrix of general form

Having solutions of left and right problems, one can represent matrix in a form

$$S = \sum_{i=1}^{n} \lambda_i \Pi_i = \sum_{i=1}^{n} \lambda_i \psi_i \phi_i^T. \tag{74}$$

This is a spectral representation of a general form matrix, which includes solutions of both left ψ_i and right ϕ_i eigenvalue problem.