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Incomplete fractal showers and restoration of dimension

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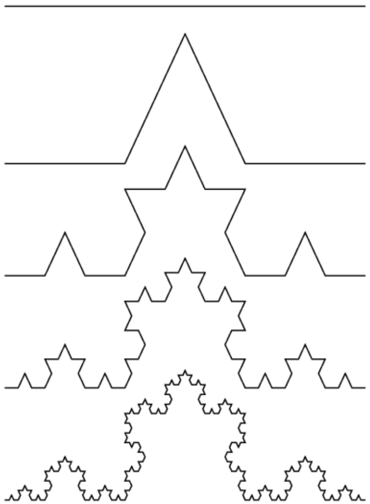
Outline

- Introduction (fractals in physics)
- BC, SePaC methods of fractal analysis
- Results of reconstruction of incomplete fractal and dimension
- Summary

What is a Fractal and Fractal dimension?

Fractal is the self-similar object whose $D_F > D_T$

Fractal curves



N(l) - number of probes with size l covering the object

$$M = \lim_{l \to 0} \sum_{1}^{N} l^{D_T}$$

Euclidean: M = const independent of scale, determines size

Fractal: M $\to \infty$ N(l) increases faster than l reduced

In order to evaluate the speed of N(l) growth
Hausdorff measure is introduced

$$M_H = \lim_{\delta \to 0} \sum_{i=1}^{N} l_i^{d}$$

N - is number of probes with size $l_i < \delta$

$$\begin{aligned} M_H &\to 0, & d &> D_F \\ M_H &\to const, & d &= D_F \\ M_H &\to \infty, & d &< D_F \end{aligned}$$

Fractal dimension is the value D_F which provides the finite M_H

Box dimension

$$D_b = -\lim_{l \to 0} \frac{\ln N(l)}{\ln(l)}$$

Fractals in physics

Fractal solution for the Schrödinger equation

Berry M.V. // J.Phys.A: Math.Gen. 1996. V.29. P. 6617–6629.
Wojcik D. et. all // Phys.Rev.Lett. 2000. V. 85. P. 5022 - 5026.
E. Akkermans et. all. // Phys.Rev.Lett. 105 (2010) 230407

Fractal space-time in quantum field theory as a new method of regularization

K.Svozil //J. Phys. A: Math. Gen.20 (1987) 3861-3875.
A. Kar, S.G. Rajeev //Annals Phys. 327 (2012) 102-117
G. Calcagni // Phys.Rev. D84 (2011) 061501
D.Moore et. all. //Phys.Rev. D90 (2014) no.2, 024075

Topological charge density distribution in lattice gluodynamics as fractals

P.Buividovich et. all. //Phys.Rev. D86 (2012) 074511

Intermittency in the spectra of secondary particles

I.Dremin (Lebedev Inst.). // Surveys High Energ.Phys. 6 (1992) 141-175
M.K. Ghosh et all. // DAE Symp.Nucl.Phys. 54 (2009) 590-591
M.Rasool, S.Ahmad. //Chaos Solitons Fractals 84 (2016) 58-68

Fractal structures in thermodynamical functions of hadrons

A. Deppman // Phys.Rev. D93 (2016) 054001

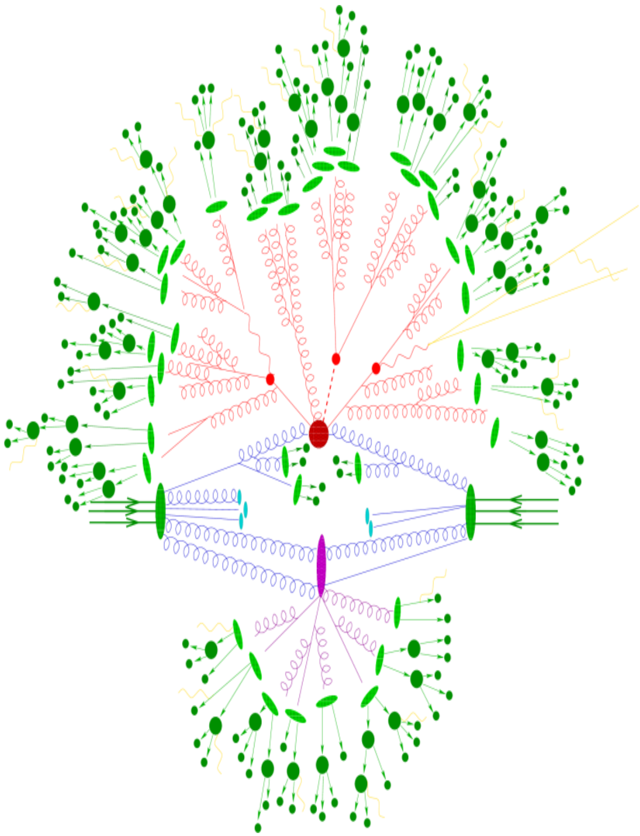
Fractality in momentum space as a signal of criticality in nuclear collisions

N. G. Antoniou // Phys.Rev. C93 (2016) no.1, 014908

Fractal properties of nuclei and events

Wei-Hu Ma et. all.// Chin.Phys. C39 (2015) no.10, 104101
I.Bunzarov, N.Chankova-Bunzarova, O.Rogachevsky// Phys.Part.Nucl.Lett. 11 (2014) 404-409

Multiple production & Fractals



Set of hadrons produced in inelastic interaction are set of points of the phase-space (p_T,η,φ)

The distribution of points in phase-space:
is determined by the interaction dynamics
is non-uniformly and is considered as a fractal

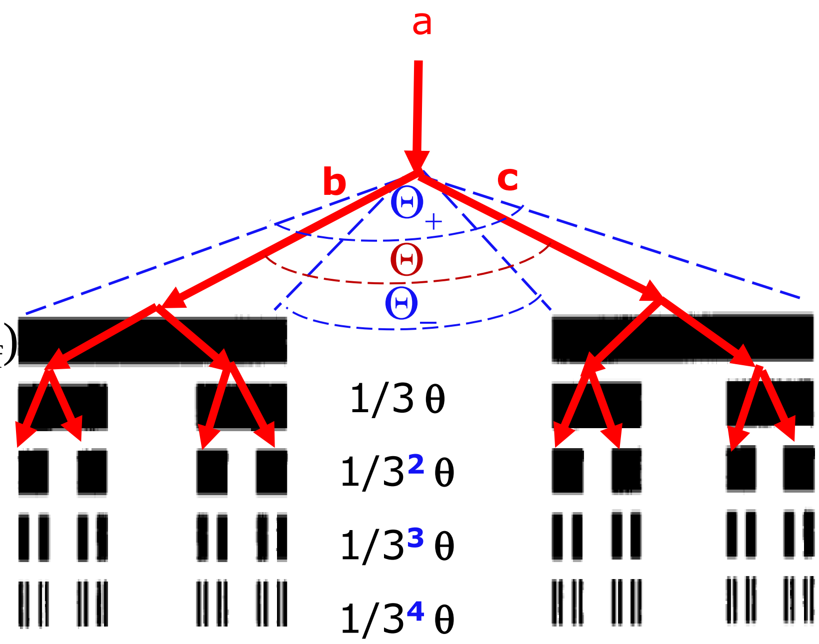
Perturbative evolution includes parton showers. The mechanism of their formation is realized in many Monte Carlo generators (PYTHIA, SHERPA and HERWIG).

The final-state parton shower (PYTHIA) & Fractal

- At each step of shower
  parton branch into two daughter partons **a→ bc**
- The kinematic of process is described by
  - the energy fraction **z** carried away by **b, c**
  - the opening angle **Θ**
- There are permissible ranges for $z_+(m_{ef}) > \mathbf{z} > z_-(m_{ef})$
  and $\Theta_+(z_+) > \Theta > \Theta_-(z_-)$ **(Black rectangles)**

- Branching process is repeated
  The opening angle of a daughter parton
  can't be more parent $\Theta_b, \Theta_c < \Theta_a$

The evolutionary variable is related
to the parton mass ($\mathbf{Q^2 = m_a^2}$)
If $\mathbf{m_a^2 \geq Q_0^2}$ the parton branch



Fractal with independent partition
Triad Cantor set (base of formation P=3)

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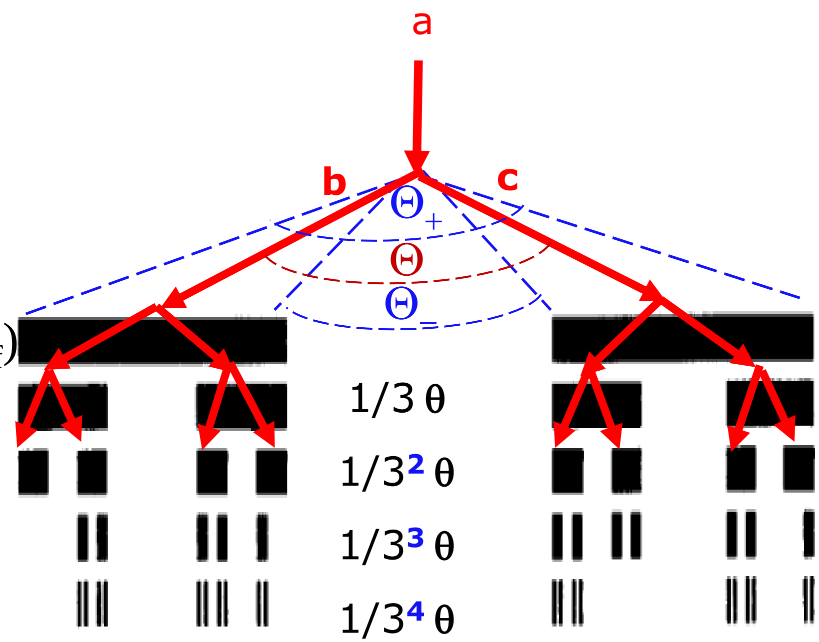

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If $\mathbf{m_a^2 < Q_0^2}$: incomplete fractal

Types of fractals

Independent: dividing ranges
consisting of one part
are divided independently

Dependent: dividing ranges
may consist of several parts
are divided dependently

Combined: dividing ranges
may consist of several parts
some parts are dependent
other - independent

$D_F = D_b$ (independent partition)
$D_F \neq D_b$ (dependent and combined partition)

Fractal analysis

Tasks: Identification and classification of the fractals by dimension

Separation of the events by fractal types

Discovery of new types of events

Steps:

Search for the phase space

Choice of the method

original data set

Verification of the availability of the fractal

Estimation of the number of fractals

Separation of the fractals and background

selected data set

Estimation of the contamination

Determination of the characteristics of fractals

Construction of the characteristic distribution

Estimation of the errors (method, contamination)

Identification of the various fractal processes

Separation of the various fractal processes

Box Counting (BC) method

BC
B. Mandelbrot
"The Fractal Geometry of Nature". Freeman, San Francisco, 1982.

1. Read out data – {X = η , p_T, …} of particles in events
2. Construction of set of distributions of variable X.

   The number of bins M_i in distributions are changed

   $M_i = (P)^i$ <u>*or*</u> $M_{i+1}=M_i+\text{step}$
3. Counting the number of non-zero bins $N(M_i)$ for each distribution
4. Finding slope parameter D_F and χ^2 dependence of ln N vs. ln M
5. Accuracy condition $\chi^2 < \chi^2_{\text{lim}}$: the set of particles is a fractal $\chi^2 = \sum \frac{(x_i-\mu)^2}{\delta^2}, \delta = 1$

BC-method determines D_b
has parameter χ^2_{lim}

$$D_b = -\lim_{l\to 0} \frac{\ln N(l)}{\ln(l)}$$

SePaC: System of the Equation of P-adic Coverage method

1. Read out data – {X = η , p_T, …} of particles in event
2. Construction of P-adic Coverages: $P_i = 3,..P_{Max}$

   Each coverage is a set of distributions of variable X. The number of bins M_i in distributions are changed as a degree of basis P: $M_i = (P)^i$
3. Count a number of non-zero bins N(lev,P):
4. Analysis of system of equations for verification of hypothesis of independent/dependent partition:

- Construction of a system of the equations for all levels N_{lev} and d_{lev} are number and length of permissible ranges $\sum_{i=1}^{N_{lev}} (d_{lev})^{D_F^{lev}} = 1$
- Finding solution D_F^{lev} by using a dichotomy method for each level
- Defining average value $< D_F^{lev} >$ and deviation ΔD_F^{lev}
- Accuracy condition $\frac{\Delta D_F^{lev}}{< D_F^{lev} >} < Dev$: set of particles is a fractal

SePaC determines $D_F = < D_F^{lev} >$, P, N_{lev}, has two parameters: P_{Max}, Dev

SePaC is developed
DT, M.Tokarev
Phys.Part.Nucl.Lett. 9(2011) №6 6552

Fractal with independent partition
DT, M.Tokarev
Phys.Part.Nucl.Lett. 10(2013) №6 481

Fractal with dependent partition
DT, M.Tokarev
Phys.Part.Nucl.Lett. 10(2013) №6 491

Fractal with combined partition
DT, M.Tokarev
Phys.Part.Nucl.Lett. 2(2016) 169

Two-step SePaC
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Reconstruction of the incomplete fractal

Data set (2000 events) contains 50% incomple fractals and 50% background

Incomplete fractals are independent fractals with P=3-8 and $N_{point} = 64$
(unambiguous separation of complete fractals and background)

Recursive generating procedure

$$\boxed{x_1 = x_0} \qquad \boxed{x_i = f_j(x_{i-1})}$$

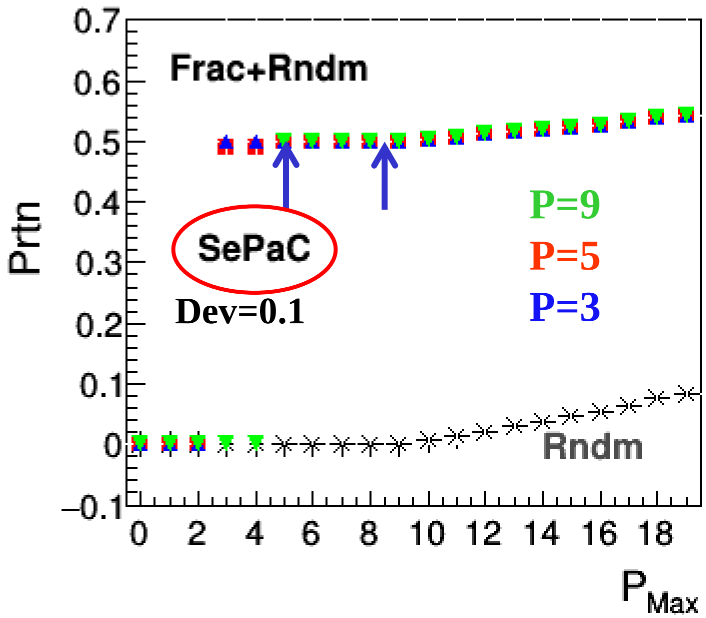
$$\boxed{f_1(x) = \frac{1}{P}x} \qquad \boxed{f_2(x) = \frac{P-1}{P} + \frac{1}{P}x}$$

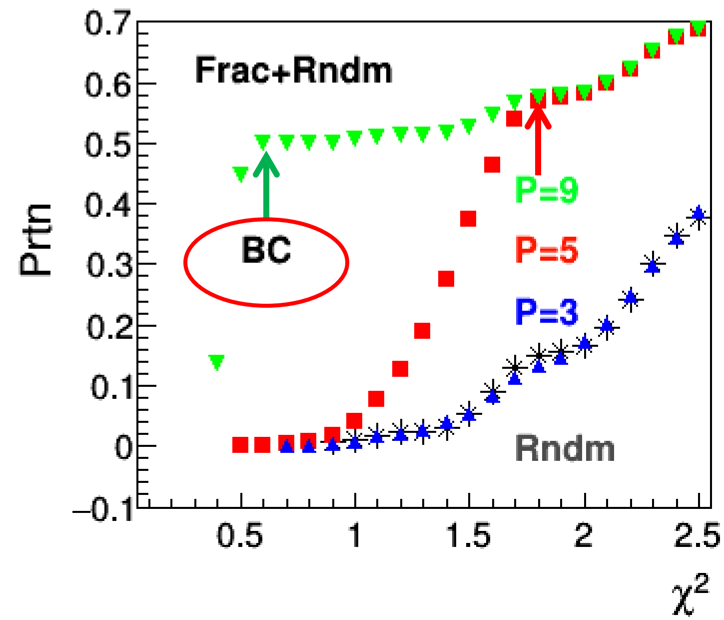
x_1 - initial value
f_j – generating function
j = 0 or 1 randomly
P - basis of fractal formation

Background is random data sets

A procedure for analyzing incomplete fractals by SePaC method is proposed
(accuracy condition in the method is verified only for 3 level)

Reconstruction of the incomplete fractals



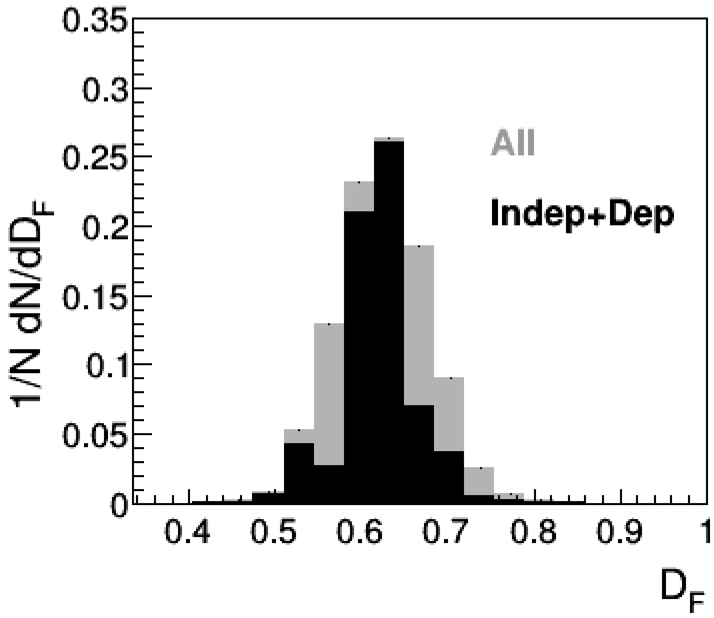


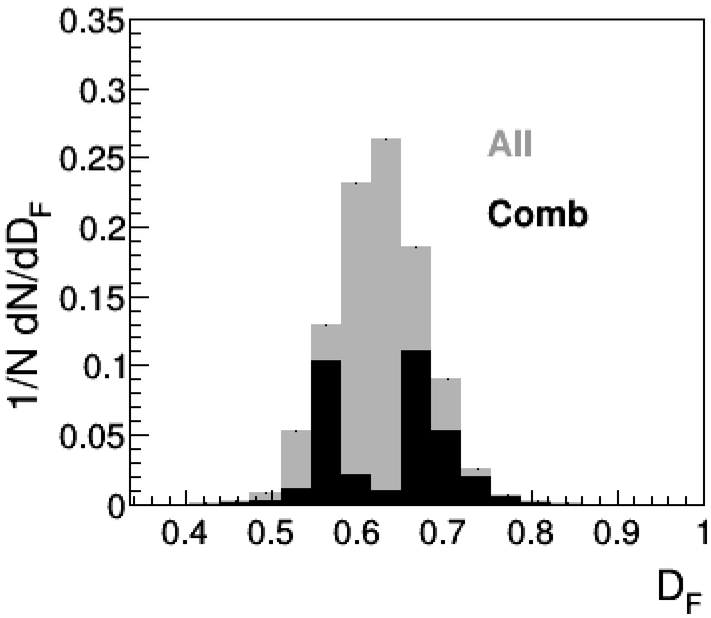
- The portion for mixed event and background is significantly different
- Starting from P_{Max}=5 the portion for all data set is the same and has a plateau
- Choice P_{Max} as the minimal value on the plateau allows to full reconstruction of fractals and unambiguously suppress the background ($Prtn_{Frac}$=1, $Prtn_{Rnd}$=0)
- Separation of fractals with P=3 and background is impossible (the function Prtn(P_{Max}) is the same)
- Choice χ^2 as the minimal value on the plateau allows to best split fractals with P=5,9 and background
- Data extracted from mixed event containing fractal with P = 5 includes background

Reconstruction of the event distribution by D_F

Data set (2001 fractal) includes an equal number of incomplete independent, complete dependent and combined fractals

The fractal distribution by dimension





SePaC: reconstructed event distribution by D_F

Analysis by two-step SePaC method

step 1: using base SePaC (addinal condition $N_i=(N_1)^i$)
step 2: using SePaC

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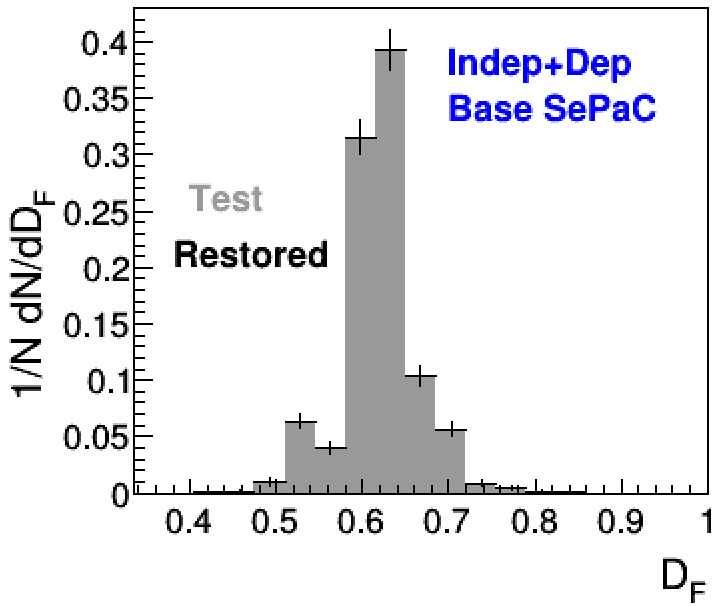
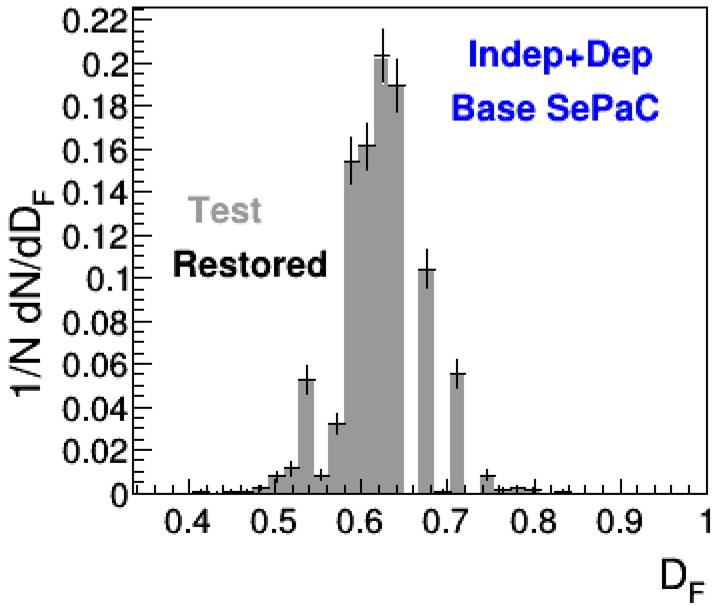
step 1: D_F is exactly reconstructed for independent and dependent fractals

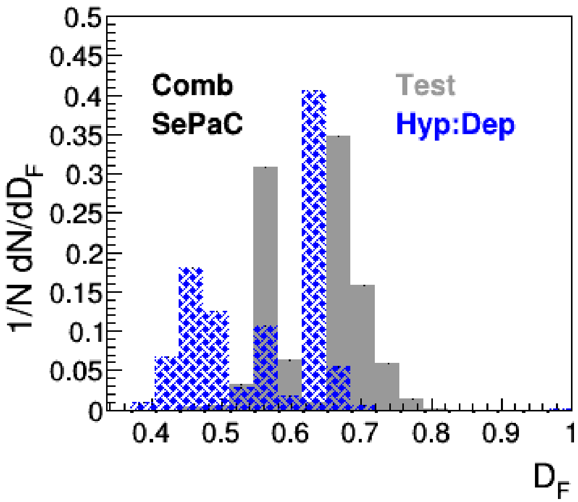


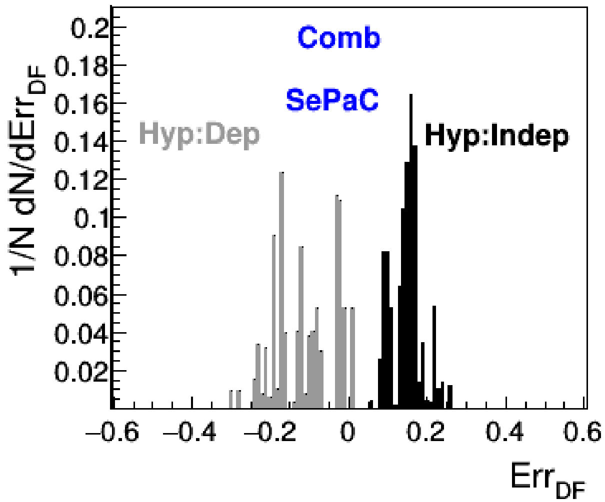


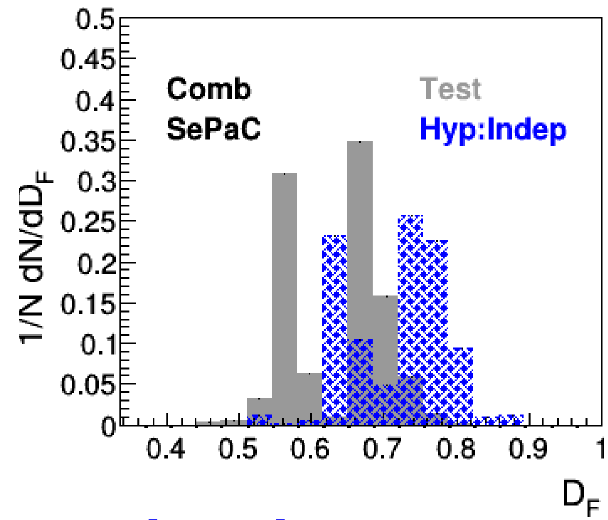
SePaC: reconstructed event distribution by D_F

step 2: analysis of combined fractals by SePaC-method

SePaC method allows to verify hypotheses of independent and dependent partition







Hypothesis Dependent partition

The distribution by D_F

- is shifted to a smaller values $D_F^{restored} < D_F^{test}$
- does not keep a shape
- $Err_{DF} < 0$ and can reach 30%

Hypothesis Independent partition

The distribution by D_F

- is shifted to a larger values $D_F^{restored} > D_F^{test}$
- keep a shape
- $Err_{DF} > 0$ and can reach 20%

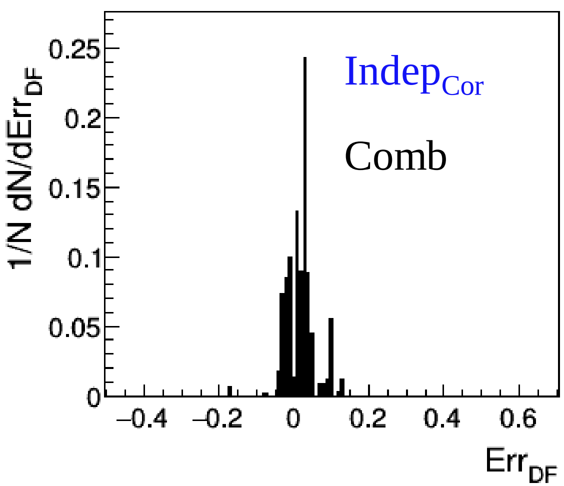
To reduce Err the correction of dimension D_F is required

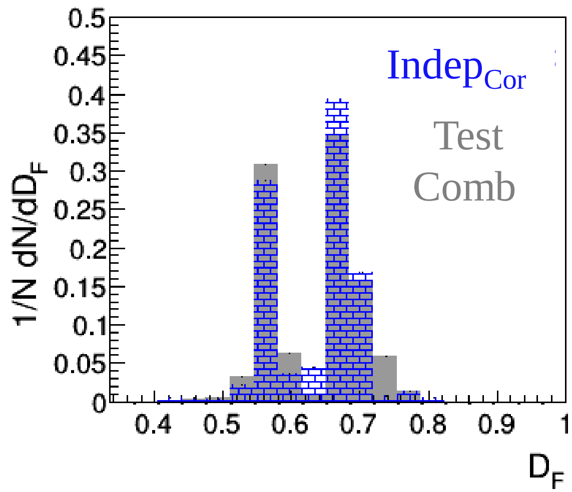
SePaC: reconstructed event distribution by D_F

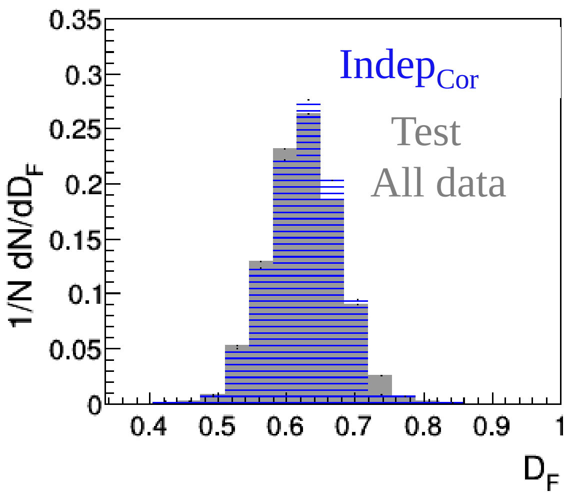
The procedure for correction of D_F for combined fractals: redefinition D_F^{Indep}

$$D_F^{Indep} = D_F^{Indep} - \frac{(< D_F^{Indep} > - < D_F^{Dep} >)}{2}$$

After correction



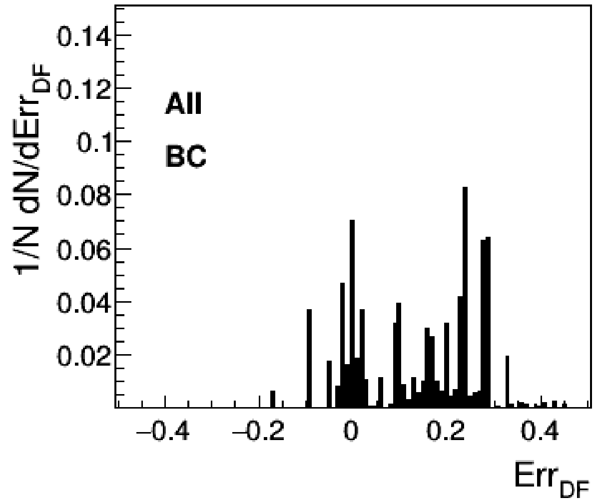


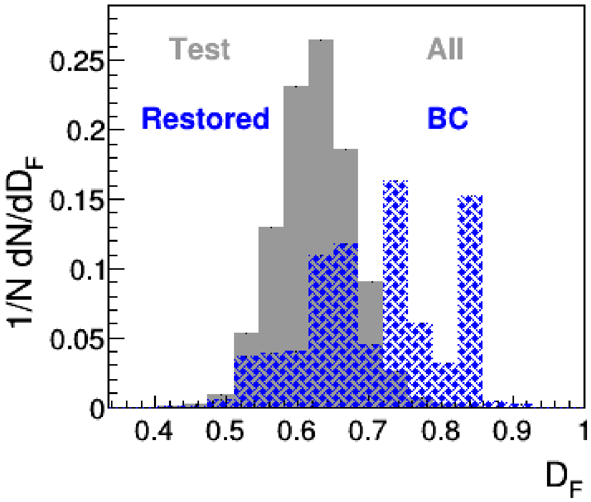


- Decrease of $Err_{DF} \leq 12\%$, symmetric around zero
- The reconstructed distribution by D_F is close to test one for combined fractal and all data set

BC: reconstructed event distribution by D_F

Analysis data set by BC-method





- Err_{DF} can reach 40%
- The reconstructed distribution by D_F is shifted to a larger values and does not keep a shape

Correction to reconstructed distribution by DF is required

Summary

SePaC and BC method were used for:

- Fractal analysis of mixed events containing incomplete fractals with $N=64$
- Reconstruction of the event distribution by D_F

For SePaC-method we proposed the procedures

- for analyzing incomplete fractals
- for correcting the determination D_F of combined fractals

It was found that:

- SePaC full reconstructs of incomplete fractals and unambiguously suppresses the background
- The separation of incomplete fractals and background by BC method depends on the basis P (no separation for P=3, separation with impurities for P=5, pure separation for P=9)
- The distribution of events by D_F is more precisely reconstructed by SePaC method in comparison with BC method.

Thank You

for your attention !