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Energy spectra of muonic atoms in quantum electrodynamics

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Contents

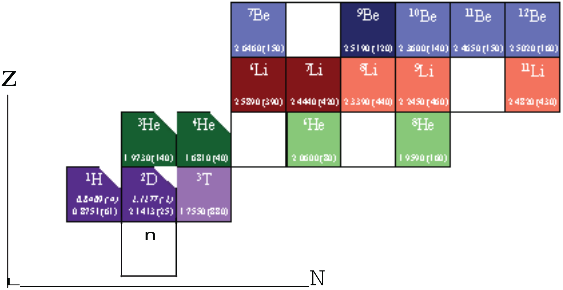
The task of the experiments of the CREMA Collaboration (Charge Radius Experiment with Muonic Atoms) in 2010-2018 is the measurement the fine and hyperfine structure of the spectrum in light muonic atoms (muonic hydrogen, muonic deuterium, muonic helium ions ...); determination of the charge radii of the proton, deuteron, helion, alpha particle with an accuracy of 0.0005 fm. One of the obtained results is connected with the hyperfine splitting (HFS) of 2S-state in muonic hydrogen:

$$\Delta E^{hfs}_{exp}(2S) = 22.8089(51)\ meV. \quad (1)$$

A. Antognini et al., Science **339**, 417 (2013).

One of the future tasks of CREMA is to extend the laser spectroscopy experiments of muonic systems to the elements of lithium and beryllium... and to improve the values of charge radii and the Zemach radii of light nuclei

- S. Schmidt et al., The next generation of laser spectroscopy experiments using light muonic atoms arXiv:1808.07240.
- R. Swainson and G. W. F. Drake, Phys. Rev. A **34**, 620 (1986).



A part of the Breit Hamiltonian, responsible for hyperfine splitting, has a known form in the coordinate representation:

$$\Delta V_B^{hfs}(r)=\frac{4\pi\alpha(1+a_\mu)\mu_N}{3m_1m_ps_2}(\mathbf{s}_1\mathbf{s}_2)\delta(\mathbf{r}), \qquad (2)$$

where the masses of the muon and nuclear will be denoted further m_1, m_2, m_p is the proton mass, μ_N is the nuclear magnetic moment in nuclear magnetons, a_μ is the muon anomalous magnetic moment (AMM), $\mathbf{s}_1$ and $\mathbf{s}_2$ are the spins of a muon and nucleus. The potential ΔV_B^{hfs} gives the main part of hyperfine splitting of order α^4 which is called the Fermi energy:

$$E_F(nS)=\frac{2Z^3\alpha^4\mu^3\mu_N}{3m_1m_pn^3s_2}(2s_2+1), \qquad (3)$$

where n is the principal quantum number, $\mu=m_1m_2/(m_1+m_2)$.

The Fermi energy is obtained after averaging (2) over the Coulomb wave functions. In the case of 1S- and 2S-states they have the form:

$$\psi_{100}(r) = \frac{W^{3/2}}{\sqrt{\pi}} e^{-Wr}, \quad W = \mu Z\alpha, \tag{4}$$

$$\psi_{200}(r) = \frac{W^{3/2}}{2\sqrt{2\pi}} e^{-Wr/2} \left(1 - \frac{Wr}{2}\right). \tag{5}$$

The muon AMM correction to hyperfine splitting is presented separately taking experimental value of muon AMM:

$$\Delta E^{hfs}_{a_\mu}(nS) = a_\mu E_F(nS). \tag{6}$$

Numerical value of relativistic correction of order α^6 to HFS can be obtained by means of known analytical expression:

$$\Delta E^{hfs}_{rel}(nS) = \begin{cases} \frac{3}{2}(Z\alpha)^2 E_F(1S) \\ \frac{17}{8}(Z\alpha)^2 E_F(2S) \end{cases}. \tag{7}$$

Next, we investigate a number of basic corrections to the hyperfine structure of S-states in order to obtain acceptable total result. Numerical values of different corrections are presented for definiteness with the accuracy 10^{-2} meV.

Nuclear structure corrections play important role in the calculation of hyperfine structure. They are determined by the electromagnetic form factors of the nuclei. Among the nuclei that we are considering, several nuclei have a spin $s_2 = 3/2$. The amplitude of the one-photon interaction:

$$iM_{1\gamma} = -\frac{Ze^2}{k^2}[\bar{u}(q_1)\gamma_\mu u(p_1)][\bar{v}_\alpha(p_2)\mathcal{O}_{\alpha\mu\beta}v_\beta(q_2)] = \tag{8}$$

$$-\frac{Ze^2}{k^2}[\bar{u}(q_1)\gamma_\mu u(p_1)]\bar{v}_\alpha(p_2)\{g_{\alpha\beta}\frac{(p_2+q_2)_\mu}{2m_2}F_1(k^2) - g_{\alpha\beta}\sigma_{\mu\nu}\frac{k^\nu}{2m_2}F_2(k^2)+$$

$$\frac{k_\alpha k_\beta}{4m_2^2}\frac{(p_2+q_2)_\mu}{2m_2}F_3(k^2) - \frac{k_\alpha k_\beta}{4m_2^2}\sigma_{\mu\nu}\frac{k^\nu}{2m_2}F_4(k^2)\}v_\beta(q_2).$$

$\mathcal{O}_{\alpha\mu\beta}$ is the vertex function of the spin 3/2 nucleus. Nuclei with a spin 3/2 are described by the spin-vector $v_\alpha(p)$. Four form factors $F_i(k^2)$ are related to the charge G_{E0}, electroquadrupole G_{E2}, magnetic dipole G_{M1} and magnetic octupole G_{M3} form factors:

$$G_{E0} = \left(1+\frac{2}{3}\tau\right)[F_1+\tau(F_1-F_2)] - \frac{\tau}{3}(1+\tau)[F_3+\tau(F_3-F_4)], \tag{9}$$

$$G_{E2} = F_1+\tau(F_1-F_2) - \frac{1+\tau}{2}[F_3+\tau(F_3-F_4)],$$

$$G_{M1} = (1+\frac{4}{3}\tau)F_2 - \frac{2}{3}\tau(1+\tau)F_4,\ G_{M3} = F_2 - \frac{1}{2}(1+\tau)F_4, \quad \tau = -\frac{k^2}{4m_2^2}.$$

It is useful to consider how the magnitude of the hyperfine splitting in the leading order (the Fermi energy) can be obtained from the amplitude $M_{1\gamma}$. When two moments are added, two states appear with the total angular momentum $F=2$ and $F=1$. To distinguish the contribution of the amplitude $M_{1\gamma}$ to the interaction operator of particles with $F=2$ and $F=1$, we use the method of projection operators, which are constructed from the wave functions of free particles in the rest frame. Thus, the projection operator on a state with $F=2$ is equal to

$$\hat{\Pi}_\alpha=[u(0)\bar{v}_\alpha(0)]_{F=2}=\frac{1+\gamma_0}{2\sqrt{2}}\gamma_\beta\varepsilon_{\alpha\beta}, \tag{10}$$

where the tensor $\varepsilon_{\alpha\beta}$ describes a muonic atom with F $=2$. As a result, the projection of $M_{1\gamma}$ to the state with F $=2$ takes the form:

$$iM_{1\gamma}(F=2)=-\frac{Ze^2}{16k^2m_1^2m_2^2}Tr\Bigg\{(\hat{q}_1+m_1)\gamma_\mu(\hat{p}_1+m_1)\frac{1+\hat{v}}{2\sqrt{2}}\gamma_\rho\varepsilon_{\alpha\rho}(\hat{p}_2-m_2)\times \tag{11}$$

$$\Big[g_{\alpha\beta}\frac{(p_2+q_2)_\mu}{2m_2}F_1(k^2)-g_{\alpha\beta}\sigma_{\mu\nu}\frac{k^\nu}{2m_2}F_2(k^2)+\frac{k_\alpha k_\beta}{4m_2^2}\frac{(p_2+q_2)_\mu}{2m_2}F_3(k^2)-$$

$$\frac{k_\alpha k_\beta}{4m_2^2}\sigma_{\mu\nu}\frac{k^\nu}{2m_2}F_4(k^2)\Big](\hat{q}_2-m_2)\gamma_\lambda\frac{1+\hat{v}}{2\sqrt{2}}\varepsilon^*_{\beta\lambda}\Bigg\},$$

where auxiliary four-vector $v=(1,0,0,0)$.

For further construction of the particle interaction potential from the amplitude, we use the averaging over the projections of the total angular momentum F which is connected with the calculation of the following sum:

$$\sum_{pol} \varepsilon^*_{\beta\lambda}\varepsilon_{\alpha\rho} = \hat{\Pi}_{\beta\lambda\alpha\rho} = \frac{1}{2}X_{\beta\alpha}X_{\lambda\rho}+\frac{1}{2}X_{\beta\rho}X_{\lambda\alpha}-\frac{1}{3}X_{\beta\lambda}X_{\alpha\rho}, \quad X_{\beta\alpha} = (g_{\alpha\beta}-v_\beta v_\alpha). \tag{12}$$

To introduce the projection operators for another state of hyperfine structure with $F=1$ we use the following expansion:

$$\Psi_{s_2=3/2,F=1,F_z} = \sqrt{\frac{2}{3}}\Psi_{S=0,F=1,F_z} + \frac{1}{\sqrt{3}}\Psi_{S=1,F=1,F_z}, \tag{13}$$

the Rarita-Schwinger spinor $v_\alpha(p)$ for the state with $s_2 = 3/2$ is presented as a result of adding spin 1/2 and angular momentum 1. With this method of adding moments, the total spin S can take two values $S=1$ and $S=0$. When calculating the matrix elements for the states Ψ_{01F_z} and Ψ_{11F_z}, we successively perform the projection on the state with spin $S=0$, $S=1$, and then on the state with the total angular momentum $F=1$. The corresponding projection operators have the form:

$$\hat{\Pi}_\alpha(S=0,F=1) = \frac{1+\hat{v}}{2\sqrt{2}}\gamma_5\varepsilon_\alpha, \tag{14}$$

$$\hat{\Pi}_\alpha(S=1,F=1) = \frac{1+\hat{v}}{4}\gamma_\sigma\varepsilon_{\alpha\sigma\rho\omega}v^\rho\varepsilon^\omega, \tag{15}$$

ε^ω is the polarization vector of the state with F=1.

Then the matrix elements of $M_{1\gamma}$ according to the states of Ψ_{01F_z} and Ψ_{11F_z} are reduced to the form:

$$< \Psi_{01F_z}|iM_{1\gamma}(F=1)|\Psi_{01F_z} >= \frac{\pi Z\alpha}{96k^2m_1^2m_2^2}Tr\Bigl\{(\hat{q}_1+m_1)\gamma_\mu(\hat{p}_1+m_1)(1+\hat{v})\gamma_5 \tag{16}$$

$$(\hat{p}_2-m_2)\Bigl[g_{\alpha\beta}\frac{(p_2+q_2)_\mu}{2m_2}F_1(k^2)-g_{\alpha\beta}\sigma_{\mu\nu}\frac{k^\nu}{2m_2}F_2(k^2)+\frac{k_\alpha k_\beta}{4m_2^2}\frac{(p_2+q_2)_\mu}{2m_2}F_3(k^2)-$$

$$\frac{k_\alpha k_\beta}{4m_2^2}\sigma_{\mu\nu}\frac{k^\nu}{2m_2}F_4(k^2)\Bigr](\hat{q}_2-m_2)\gamma_5(1+\hat{v})\Bigr\}(-g_{\alpha\beta}+v_\alpha v_\beta),$$

$$< \Psi_{11F_z}|iM_{1\gamma}(F=1)|\Psi_{11F_z} >= \frac{\pi Z\alpha}{192k^2m_1^2m_2^2}Tr\Bigl\{(\hat{q}_1+m_1)\gamma_\mu(\hat{p}_1+m_1)(1+\hat{v})\gamma_\sigma \tag{17}$$

$$(\hat{p}_2-m_2)\Bigl[g_{\alpha\beta}\frac{(p_2+q_2)_\mu}{2m_2}F_1(k^2)-g_{\alpha\beta}\sigma_{\mu\nu}\frac{k^\nu}{2m_2}F_2(k^2)+\frac{k_\alpha k_\beta}{4m_2^2}\frac{(p_2+q_2)_\mu}{2m_2}F_3(k^2)-$$

$$\frac{k_\alpha k_\beta}{4m_2^2}\sigma_{\mu\nu}\frac{k^\nu}{2m_2}F_4(k^2)\Bigr](\hat{q}_2-m_2)\gamma_\epsilon(1+\hat{v})\Bigr\}\varepsilon_{\alpha\sigma\rho\omega}\varepsilon\beta\epsilon\tau\lambda(-g_{\lambda\omega}+v_\lambda v_\omega).$$

In addition, the off-diagonal matrix element $< \Psi_{01F_z}|iM_{1\gamma}(F=1)|\Psi_{11F_z} >$ is also nonzero. The sum of all the matrix elements gives in the nonrelativistic approximation the following value of the hyperfine splitting (the Fermi energy):

$$\Delta E^{hfs}_{1\gamma} = E_F(nS) = \frac{16}{9}\frac{\pi Z\alpha}{m_1 m_2}F_2(0)\frac{W^3}{\pi n^3} = \frac{16\alpha(Z\alpha)^3\mu^3}{9 m_1 m_p n^3}\mu_N. \qquad (18)$$

The expressions for the amplitude $M_{1\gamma}$ are presented in a form that is convenient for the subsequent calculation of the contribution in the Form package. We present in detail the results of calculating the amplitude $M_{1\gamma}$, since this calculation technique is used later in the calculation of two-photon exchange amplitude shown in Fig. 1. In the case of nuclei with spin $s_2 = 1/2$ and $s_2 = 1$ the similar technique of projection operators was used in our previous works.

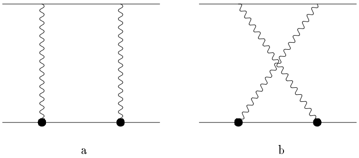


Figure: Nuclear structure effects of order α^5. The bold point denotes the nucleus vertex function.

Basic contribution of the nuclear structure effects of order α^5 to the hyperfine splitting is determined by two-photon exchange diagrams. It is expressed in terms of electric $G_E(k^2)$ and magnetic $G_M(k^2)$ nuclear form factors in the form (the Zemach correction):

$$\Delta E^{hfs}_{str} = E_F \frac{2\mu Z\alpha}{\pi}\int \frac{d\mathbf{k}}{\mathbf{k}^4}\left[\frac{G_E(k^2)G_M(k^2)}{G_M(0)} - 1\right]. \qquad (19)$$

We have analysed numerical values of correction (19) for different parameterizations of nuclear form factors: Gaussian $G_E^G(k^2)$, dipole $G_E^D(k^2)$ and uniformly charged sphere $G_E^U(k^2)$:

$$G_E^G(k^2) = e^{-\frac{1}{6}r_N^2 k^2},\ G_E^D(k^2) = \frac{1}{(1+\frac{k^2}{\Lambda^2})^2},\ G_E^U(k^2) = \frac{3}{(kR)^3}\left[\sin kR - kR\cos kR\right], \qquad (20)$$

where $R = \sqrt{5}r_N/\sqrt{3}$ is the nucleus radius, $\Lambda^2 = 12/r_N^2$.

A comparison of functions $G_E^2(k^2)$ for different parameterizations is presented in Fig. 2 for the nucleus ${}^6_3Li$. In the range $0.1 \leq k \leq 0.4$ GeV there is a difference between functions (19) which leads to different numerical values of the Zemach correction.

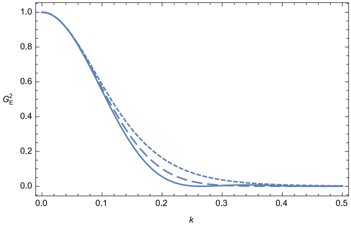


Figure: Gaussian (dashed), dipole (dotted) and uniformly charged sphere (solid) parameterizations of nuclear form factor $G_E^2(k^2)$.

The momentum integration in (19) can be done analytically, so that the Zemach correction with the Gaussian and uniformly charged sphere parameterizations has the form:

$$\Delta E^{hfs}_{str,G} = -E_F \frac{72}{\sqrt{3\pi}} \mu Z\alpha r_N, \quad \Delta E^{hfs}_{str,U} = -E_F \frac{72\sqrt{5}}{35\sqrt{3}} \mu Z\alpha r_N. \qquad (21)$$

Acting as in the case of the one-photon interaction, we can present the contribution of two-photon interactions to HFS at $F=2$:

$$\Delta E^{hfs}_{2\gamma}(F=2)=-\frac{(Z\alpha)^2}{640\pi^2m_1^2m_2^2}|\psi(0)|^2\int\frac{id^4k(k^2-2k_0m_2)}{k^4(k^4-4k_0^2m_1^2)(k^4-4k_0^2m_2^2)}Tr\{(\hat q_1+m_1)\times \tag{22}$$

$$[\gamma_\mu(\hat p_1-\hat k+m_1)\gamma_\nu(k^2+2k_0m_1)+\gamma_\nu(\hat p_1+\hat k+m_1)\gamma_\mu(k^2-2k_0m_1)](\hat p_1+m_1)(1+\hat v)\gamma_\rho(\hat p_2-m_2)\times$$

$$\mathcal{O}_{\alpha\nu\sigma}(k)(-\hat p_2-\hat k+m_2)[g_{\sigma\tau}-\frac{1}{3}\gamma_\sigma\gamma_\tau-\frac{2}{3m_2^2}(p_2+k)_\sigma(p_2+k)_\tau+\frac{1}{3m_2}(\gamma_\sigma(p_2+k)_\tau-\gamma_\tau(p_2+k)_\sigma)]\times$$

$$\mathcal{O}_{\tau\mu\beta}(-k)(\hat q_2-m_2)\gamma_\lambda(1+\hat v)\}\hat\Pi_{\beta\alpha\lambda\rho},$$

We also give for completeness analogous expressions for two states in (13) with $F=1, S=0$ and $F=1, S=1$:

$$\Delta E^{hfs}_{2\gamma}(F=1,S=0)=-\frac{(Z\alpha)^2}{384\pi^2m_1^2m_2^2}|\psi(0)|^2\int\frac{id^4k(k^2-2k_0m_2)}{k^4(k^4-4k_0^2m_1^2)(k^4-4k_0^2m_2^2)}Tr\{(\hat q_1+m_1)\times \tag{23}$$

$$[\gamma_\mu(\hat p_1-\hat k+m_1)\gamma_\nu(k^2+2k_0m_1)+\gamma_\nu(\hat p_1+\hat k+m_1)\gamma_\mu(k^2-2k_0m_1)](\hat p_1+m_1)(1+\hat v)\gamma_5(\hat p_2-m_2)\times$$

$$\mathcal{O}_{\alpha\nu\sigma}(k)(-\hat p_2-\hat k+m_2)[g_{\sigma\tau}-\frac{1}{3}\gamma_\sigma\gamma_\tau-\frac{2}{3m_2^2}(p_2+k)_\sigma(p_2+k)_\tau+\frac{1}{3m_2}(\gamma_\sigma(p_2+k)_\tau-\gamma_\tau(p_2+k)_\sigma)]\times$$

$$\mathcal{O}_{\tau\mu\beta}(-k)(\hat q_2-m_2)\gamma_5(1+\hat v)\}(-g_{\alpha\beta}+v_\alpha v_\beta),$$

$$\Delta E^{hfs}_{2\gamma}(F=1,S=1)=-\frac{(Z\alpha)^2}{768\pi^2m_1^2m_2^2}|\psi(0)|^2\int\frac{id^4k(k^2-2k_0m_2)}{k^4(k^4-4k_0^2m_1^2)(k^4-4k_0^2m_2^2)}Tr\{(\hat q_1+m_1)\times \tag{24}$$

$$[\gamma_\mu(\hat p_1-\hat k+m_1)\gamma_\nu(k^2+2k_0m_1)+\gamma_\nu(\hat p_1+\hat k+m_1)\gamma_\mu(k^2-2k_0m_1)](\hat p_1+m_1)(1+\hat v)\gamma_{\sigma_1}\varepsilon_{\alpha\sigma_1\rho_1\omega_1}v_{\rho_1}\times$$

$$(\hat p_2-m_2)\mathcal{O}_{\alpha\nu\sigma}(k)(-\hat p_2-\hat k+m_2)[g_{\sigma\tau}-\frac{1}{3}\gamma_\sigma\gamma_\tau-\frac{2}{3m_2^2}(p_2+k)_\sigma(p_2+k)_\tau+\frac{1}{3m_2}(\gamma_\sigma(p_2+k)_\tau-$$

$$\gamma_\tau(p_2+k)_\sigma)]\mathcal{O}_{\tau\mu\beta}(-k)(\hat q_2-m_2)\gamma_{\epsilon_1}(1+\hat v)\varepsilon_{\beta\epsilon_1\tau_1\lambda_1}v_{\tau_1}\}(-g_{\lambda_1\omega_1}+v_{\lambda_1}v_{\omega_1}).$$

As a result the value of the hyperfine splitting is determined in Euclidean space by the following formula:

$$\Delta E^{hfs}(nS)=|\psi_{nS}(0)|^2\int d^4kV_{2\gamma}(k)=\frac{64}{9}\frac{(Z\alpha)^2}{\pi^2}|\psi_{nS}(0)|^2\int\frac{d^4k}{k^4(k^4+4m_1^2k_0^2)(k^4+4m_2^2k_0^2)}\times \tag{25}$$

$$\Big[F_1F_2\Big(k^6-k^4k_0^2+\frac{4}{15}\frac{k^4k_0^4}{m_2^2}-\frac{7}{10}\frac{k^6k_0^2}{m_2^2}+\frac{13}{30}\frac{k^8}{m_2^2}\Big)+F_2F_4\Big(-\frac{1}{30}\frac{k^2k_0^6}{m_2^2}+\frac{1}{15}\frac{k^4k_0^4}{m_2^2}-\frac{1}{30}\frac{k^6k_0^2}{m_2^2}\Big)+$$

$$F_2F_3\Big(-\frac{1}{15}\frac{k^2k_0^6}{m_2^2}+\frac{11}{60}\frac{k^4k_0^4}{m_2^2}-\frac{7}{60}\frac{k^8}{m_2^2}\Big)+F_1F_4\Big(-\frac{1}{5}\frac{k^2k_0^6}{m_2^2}+\frac{3}{10}\frac{k^4k_0^4}{m_2^2}-\frac{1}{10}\frac{k^8}{m_2^2}\Big)+$$

$$F_2^2\Big(\frac{1}{15}\frac{k^2k_0^6}{m_2^2}-\frac{1}{6}k^2k_0^4-\frac{2}{15}\frac{k^4k_0^4}{m_2^2}+\frac{1}{6}k^4k_0^2+\frac{23}{120}\frac{k^6k_0^2}{m_2^2}-\frac{1}{4}\frac{k^8}{m_2^2}\Big)\Big].$$

When investigating this expression, it is useful to distinguish the Zemach correction, which is determined by the integral

$$J=\int_0^\infty\int_0^\pi\frac{k\sin^2\phi dkd\phi F_1(k^2)F_2(k^2)}{(k^2+4m_1^2\cos^2\phi)(k^2+4m_2^2\cos^2\phi)}=\frac{\pi}{2(m_1+m_2)}\int_0^\infty\frac{dk}{k^2}F_1(k^2)F_2(k^2)+ \tag{26}$$

$$\frac{\pi}{4(m_1^2-m_2^2)}\int_0^\infty\frac{dk}{k^2}[\sqrt{k^2+4m_1^2}-2m_1-\sqrt{k^2+4m_2^2}+2m_2]F_1(k^2)F_2(k^2).$$

The divergence in the first term on the right-hand side of (26) is compensated by the subtraction term

$$\Delta E^{hfs}_{iter}=\frac{64}{9}\frac{(Z\alpha)^2}{\pi^2}|\psi(0)|^2\int_0^\infty\frac{2\pi^2F_2(0)}{(m_1+m_2)k^2}dk. \tag{27}$$

Thus, we have in (25) the main contribution (the Zemach correction) and the recoil correction m_1/m_2.

The form factors $F_i(k^2)$ are expressed in terms of G_{E0}, G_{E2}, G_{M1}, G_{M3} for which the Gaussian parametrization is used in numerical calculations of integrals with respect to k. The values of the form factors at zero have the form:

$$G_{E0}(0) = 1, \; G_{M1}(0) = \frac{m_2 \mu_N}{m_p Z}, \; G_{E2}(0) = m_2^2 Q, \; G_{M3}(0) = \frac{m_2}{m_p Z} m_2^2 \Omega. \quad (28)$$

The nucleus parameters of lithium, beryllium and boron.

Nucleus	Spin	Mass , GeV	Magnetic dipole moment, nm	Charge radius, fm	Electroquadrupole moment, fm^2	Magnetic octupole, moment, nm·fm^2
^6_3Li	1	5.60152	0.8220473(6)	2.5890 $\pm$ 0.0390	-0.083(8)	0
^7_3Li	3/2	6.53383	3.256427(2)	2.4440 $\pm$ 0.0420	-4.06(8)	7.5
^9_4Be	3/2	8.39479	-1.177432(3)	2.5190 $\pm$ 0.0120	5.29(4)	4.1
$^{10}_5B$	3	9.32699	0.8220473(6)	2.4277 $\pm$ 0.0499	8.47(6)	0
$^{11}_5B$	3/2	10.25510	0.8220473(6)	2.4060 $\pm$ 0.0294	4.07(3)	7.8

Different parameters of light nucleus (Li, Be, B) were investigated in electron scattering experiments (H. Uberall, G. H. Fuller). Some of them are unknown with good accuracy, but, nevertheless, one can obtain approximate estimates of the corresponding contributions.

H. Uberall, Electron scattering from complex nuclei, Academic press, NY, London, 1971.

G. H. Fuller, Jour. Phys. and Chem. Ref. Data **5**, 835 (1976).

Another correction for the structure of the nucleus of order α^6, which must be discussed, is obtained as a result of the expansion of nucleus magnetic form factor. The contribution to the interaction potential and HFS in this case has the form:

$$\Delta V^{hfs}_{1\gamma,str}(r)=\frac{4\pi\alpha\mu_N}{9m_1m_p}r_M^2(\mathbf{s}_1\mathbf{s}_2)\nabla^2\delta(\mathbf{r}), \tag{29}$$

$$\Delta E^{hfs}_{1\gamma,str}=\frac{2}{3}\mu^2Z^2\alpha^2r_M^2\frac{3n^2+1}{n^2}E_F(nS). \tag{30}$$

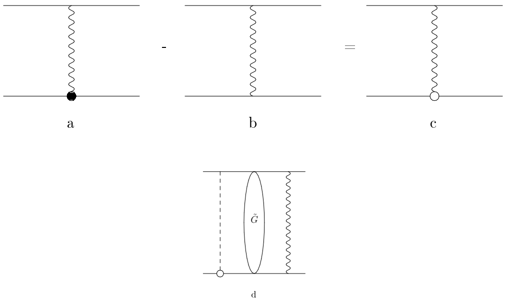


Figure: Nuclear structure effects in one-photon interaction (c) and in second order perturbation theory (d). $\tilde G$ is the reduced Coulomb Green's function.

In second order PT we should take into account a term in which the potential

$$\Delta V^C_{str,1\gamma}(k) = -\frac{Z\alpha}{\mathbf{k}^2}\left[\frac{3}{(kR)^3}\left(\sin kR - kR\cos kR\right) - 1\right] \qquad (31)$$

is considered as a perturbation. After the Fourier transform we obtain:

$$\Delta V^C_{str,1\gamma}(r) = -\frac{Z\alpha}{4R^3 r}(r-R)(r+2R)(R-r+|r-R|). \qquad (32)$$

Using the Green's function we perform the analytical integration in second order PT. It gives the following result:

$$\Delta E^{hfs}_{str,sopt}(1S) = -E_F(1S)\frac{R^2W^2}{4}\left[-\frac{4}{75}(-53+15C+15\ln RW + \frac{RW}{12}(-15+4C+4\ln RW)\right], \qquad (33)$$

$$\Delta E^{hfs}_{str,sopt}(2S) = E_F(2S)\frac{R^2W^2}{4}\left[\frac{4}{75}(-107+60C+60\ln RW) + \frac{RW}{3}(17-8C-8\ln RW)\right], \qquad (34)$$

where we present an expansions in (RW) up to terms of first order in square brackets ($RW(^6_3Li) = 0.038$, $RW(^7_3Li) = 0.036$, $RW(^9_4Be) = 0.050$, $RW(^{10}_5B) = 0.060$, $RW(^{11}_5B) = 0.060$).

One-loop vacuum polarization potential in coordinate representation is defined by the following integral expression:

$$\Delta V^{hfs}_{1\gamma,vp}(r) = \frac{4\alpha g_N(1+a_\mu)}{3m_1 m_p}(\mathbf{s}_1\mathbf{s}_2)\frac{\alpha}{3\pi}\int_1^\infty \rho(\xi)d\xi\left(\pi\delta(\mathbf{r}) - \frac{m_e^2\xi^2}{r}e^{-2m_e\xi r}\right), \quad (35)$$

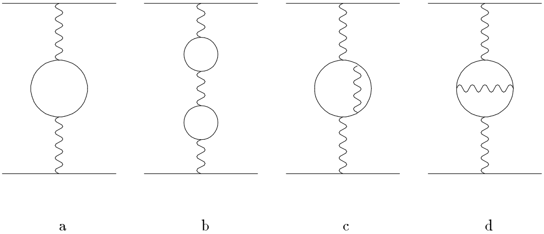


Figure: Effects of one- and two-loop vacuum polarization in one-photon interaction.

The contribution of order α^5 to hyperfine structure of $1S-$ and $2S$-states ($a_1 = m_e/W$, $W = \mu Z\alpha$):

$$\Delta E^{hfs}_{1\gamma,vp}(1S) = \frac{4\alpha^2(Z\alpha)^3\mu^3 g_N(1+a_\mu)}{9m_1 m_p \pi} < (\mathbf{s}_1\mathbf{s}_2) > \int_1^\infty \rho(\xi)d\xi \left[1 - \frac{m_e^2\xi^2}{W^2}\int_0^\infty xdxe^{-x\left(1+\frac{m_e\xi}{W}\right)}\right] = \tag{36}$$

$$E_F(1S)\frac{\alpha(1+a_\mu)}{9\pi\sqrt{1-a_1^2}}\left[\sqrt{1-a_1^2}(1+6a_1^2-3\pi a_1^3)+(6-3a_1^2+5a_1^4)\ln\frac{1+\sqrt{1-a_1^2}}{a_1}\right],$$

$$\Delta E^{hfs}_{1\gamma,vp}(2S) = \frac{\alpha^2(Z\alpha)^3\mu^3 g_N(1+a_\mu)}{18m_1 m_p \pi} < (\mathbf{s}_1\mathbf{s}_2) > \int_1^\infty \rho(\xi)d\xi\times \tag{37}$$

$$\left[1 - \frac{4m_e^2\xi^2}{W^2}\int_0^\infty x\left(1-\frac{x}{2}\right)^2 dxe^{-x\left(1+\frac{2m_e\xi}{W}\right)}\right] =$$

$$E_F(2S)\frac{\alpha(1+a_\mu)}{18\pi(4a_1^2-1)^{5/2}}\left\{\sqrt{4a_1^2-1}[11+2a_1^2(-29+8a_1(-22a_1+48a_13-3\pi(4a_1^2-1)^2))]+\right.$$

$$\left.12(1-10a_1^2+66a_1^4-160a_1^6+256a_1^8)\arctan\sqrt{4a_1^2-1}\right\}.$$

We present in detail these results to demonstrate the general structure of the obtained analytical expressions. After integrating over particle coordinates, the results have a fairly simple form, but the following integration over the spectral parameters gives, as a rule, rather cumbersome expressions.

Two-loop vacuum polarization potentials have the form of a double and a single spectral integral in coordinate space:

$$\Delta V^{hfs}_{1\gamma,vp-vp}(r)=\frac{4\pi\alpha g_N(1+a_\mu)}{3m_1m_p}(\mathbf{s}_1\mathbf{s}_2)\left(\frac{\alpha}{3\pi}\right)^2\int_1^\infty\rho(\xi)d\xi\int_1^\infty\rho(\eta)d\eta\times \quad (38)$$

$$\times\left[\delta(\mathbf{r})-\frac{m_e^2}{\pi r(\eta^2-\xi^2)}\left(\eta^4e^{-2m_e\eta r}-\xi^4e^{-2m_e\xi r}\right)\right],$$

$$\Delta V^{hfs}_{1\gamma,2-loop\ \ vp}(r)=\frac{8\alpha^3 g_N(1+a_\mu)}{9\pi^2m_1m_p}(\mathbf{s}_1\mathbf{s}_2)\int_0^1\frac{f(v)dv}{1-v^2}\left[\pi\delta(\mathbf{r})-\frac{m_e^2}{r(1-v^2)}e^{-\frac{2m_er}{\sqrt{1-v^2}}}\right], \quad (39)$$

$$f(v)=v\Big\{(3-v^2)(1+v^2)\left[Li_2\left(-\frac{1-v}{1+v}\right)+2Li_2\left(\frac{1-v}{1+v}\right)+\frac{3}{2}\ln\frac{1+v}{1-v}\ln\frac{1+v}{2}-\ln\frac{1+v}{1-v}\ln v\right] \quad (40)$$

$$+\left[\frac{11}{16}(3-v^2)(1+v^2)+\frac{v^4}{4}\right]\ln\frac{1+v}{1-v}+\left[\frac{3}{2}v(3-v^2)\ln\frac{1-v^2}{4}-2v(3-v^2)\ln v\right]+\frac{3}{8}v(5-3v^2)\Big\},$$

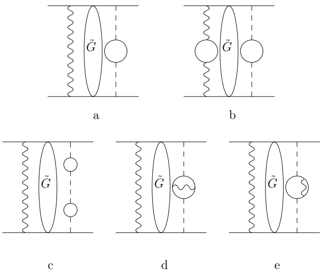


Figure: One- and two-loop vacuum polarization in second order PT.

$$\tilde{G}_{1S}(\mathbf{r},0)=\frac{Z\alpha\mu^2}{4\pi}\frac{e^{-x}}{x}g_{1S}(x),\;\; g_{1S}(x)=\Big[4x(\ln 2x+C)+4x^2-10x-2\Big], \tag{41}$$

$$\tilde{G}_{2S}(\mathbf{r},0)=-\frac{Z\alpha\mu^2}{4\pi}\frac{e^{-x/2}}{2x}g_{2S}(x),\;\; g_{2S}(x)=\Big[4x(x-2)(\ln x+C)+x^3-13x^2+6x+4\Big], \tag{42}$$

General structure of two-loop contribution (b) takes the form:

$$\Delta E^{hfs}_{sopt~vp~2} = 2 < \psi|\Delta V^{hfs}_{1\gamma,vp} \cdot \tilde{G} \cdot \Delta V^{C}_{vp}|\psi > . \quad (43)$$

$$\Delta E^{hfs}_{sopt~vp~21}(1S) = -2E_F(1S)\frac{\alpha^2}{9\pi^2}(1+a_\mu)\int_1^\infty \rho(\xi)d\xi \int_1^\infty \rho(\eta)d\eta \int_0^\infty dx e^{-2x\left(1+\frac{m_e\eta}{W}\right)} g_{1S}(x), \quad (44)$$

$$\Delta E^{hfs}_{sopt~vp~22}(1S) = 2E_F(1S)\frac{\alpha^2}{9\pi^2}(1+a_\mu)\frac{16m_e^2}{W^2}\int_1^\infty \rho(\xi)\xi^2 d\xi \times \quad (45)$$

$$\times \int_1^\infty \rho(\eta)d\eta \int_0^\infty x_1 dx_1 e^{-2x_1\left(1+\frac{m_e\eta}{W}\right)} \int_0^\infty x_2 dx_2 e^{-2x_2\left(1+\frac{m_e\xi}{W}\right)} g_{1S}(x_1,x_2),$$

$$\Delta E^{hfs}_{sopt~vp~21}(2S) = E_F(2S)\frac{\alpha^2}{9\pi^2}(1+a_\mu)\int_1^\infty \rho(\xi)d\xi \int_1^\infty \rho(\eta)d\eta \int_0^\infty \left(1-\frac{x}{2}\right) dx e^{-x\left(1+\frac{2m_e\eta}{W}\right)} g_{2S}(x), \quad (46)$$

$$\Delta E^{hfs}_{sopt~vp~22}(2S) = -E_F(2S)\frac{\alpha^2}{9\pi^2}(1+a_\mu)\frac{2m_e^2}{W^2}\int_1^\infty \rho(\xi)\xi^2 d\xi \times \quad (47)$$

$$\times \int_1^\infty \rho(\eta)d\eta \int_0^\infty \left(1-\frac{x_1}{2}\right) dx_1 e^{-x_1\left(1+\frac{2m_e\xi}{W}\right)} \int_0^\infty \left(1-\frac{x_2}{2}\right) dx_2 e^{-x_2\left(1+\frac{2m_e\eta}{W}\right)} g_{2S}(x_1,x_2).$$

$$\Delta E^{hfs}_{vp,vp}(1S) = \begin{cases} {}^6_3Li: 0.05~meV \\ {}^7_3Li: 0.20~meV \\ {}^9_4Be: -0.21~meV; \\ {}^{10}_5B: 0.65~meV \\ {}^{11}_5B: 1.11~meV \end{cases} \quad \Delta E^{hfs}_{vp,vp}(2S) = \begin{cases} {}^6_3Li: 0.01~meV \\ {}^7_3Li: 0.02~meV \\ {}^9_4Be: -0.02~meV. \\ {}^{10}_5B: 0.06~meV \\ {}^{11}_5B: 0.11~meV \end{cases} \quad (48)$$

There is another correction for the polarization of the vacuum, which also includes the effect of the nuclear structure. To calculate it, it is necessary to use the potential $V_{2\gamma}(k)$. As a result, the contribution to the HFS spectrum is determined by the following expression:

$$E^{hfs}_{2\gamma,vp} = -\frac{2\mu^3 Z^3\alpha^4}{9\pi^2 n^3}\int\frac{V_{2\gamma}(k)d^4k}{k^3} \tag{49}$$

$$\left[5k^3 - 12m_e k^2 - 6(k^2-2m_e^2)\sqrt{k^2+4m_e^2}Arcth[\frac{k}{\sqrt{k^2+4m_e^2}}\right].$$

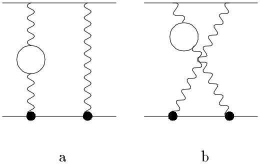


Figure: Two photon exchange amplitudes accounting for effects of vacuum polarization and nuclear structure. The wavy line denotes the photon. The bold point denotes the nucleus vertex function.

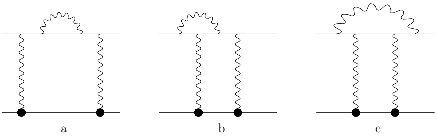
The results already obtained clearly show that the corrections to the structure of the nucleus are dominant. In this connection, it seems useful to consider another correction for the structure of the nucleus of order α^6 shown in Fig. to refine the results. The amplitudes of two-photon exchange with radiative corrections to the muon line can be calculated in the framework of the calculation method formulated in Section II. For a radiative photon, the Fried-Yennie gauge is used, in which each of the amplitudes (muon self-energy, muon vertex correction, amplitude with the spanning photon) can be represented by a finite integral expression. The general structure of the amplitudes is the following:

$$i\mathcal{M}=\frac{(Z\alpha)^2}{\pi^2}\int d^4k\,[\bar u(q_1)L_{\mu\nu}u(p_1)]\,D_{\mu\omega}(k)D_{\nu\lambda}(k)\times \quad (50)$$

$$\Big[\bar v_\rho(p_2)\mathcal{O}_{\rho\omega\beta}\mathcal{D}_{\beta\tau}(p_2+k)\mathcal{O}_{\tau\lambda\alpha}v_\alpha(q_2)\Big].$$

The lepton tensor $L_{\mu\nu}$ is equal to a sum of three terms coming from three amplitudes:

$$L_{\mu\nu}=L^{se}_{\mu\nu}+L^{vertex}_{\mu\nu}+L^{jellyfish}_{\mu\nu}. \quad (51)$$



Three types of contributions of order $E_F\alpha(Z\alpha)$ to HFS of muonic ions:

$$\Delta E^{hfs}_{se} = E_F 6\frac{\alpha(Z\alpha)}{\pi^2}\int_0^1 xdx\int_0^\infty \frac{G_E(k^2)G_M(k^2)dk}{x+(1-x)k^2}, \tag{52}$$

$$\Delta E^{hfs}_{vertex-1} = -E_F 24\frac{\alpha(Z\alpha)}{\pi^2}\int_0^1 dz\int_0^1 xdx\int_0^\infty \frac{G_E(k^2)G_M(k^2)\ln[\frac{x+k^2z(1-xz)}{x}]dk}{k^2}, \tag{53}$$

$$\Delta E^{hfs}_{vertex-2} = E_F 8\frac{\alpha(Z\alpha)}{\pi^2}\int_0^1 dz\int_0^1 dx\int_0^\infty \frac{dk}{k^2}\left\{\frac{G_E(k^2)G_M(k^2)}{[x+k^2z(1-xz)]^2}\left[-2xz^2(1-xz)^2k^4+ zk^2(3x^3z-x^2(9z+1)+x(4z+7)-4)+x^2(5-x)\right]-\frac{1}{2}\right\}, \tag{54}$$

$$\Delta E^{hfs}_{jellyfish} = E_F 4\frac{\alpha(Z\alpha)}{\pi^2}\int_0^1 (1-z)dz\int_0^1 (1-x)dx\int_0^\infty \frac{G_E(k^2)G_M(k^2)dk}{[x+(1-xz)k^2]^3} \tag{55}$$
$$\times\left[6x+6x^2-6x^2z+2x^3-12x^3z-12x^4z+k^2(-6z+18xz+4xz^2+7x^2z-30x^2z^2- 2x^2z^3-36x^3z^2+12x^3z^3+24x^4z^3)+k^4(9xz^2-31x^2z^3+34x^3z^4-12x^4z^5\right]$$

The dependence of basic corrections of order α^5 on the nucleus is shown in Fig.

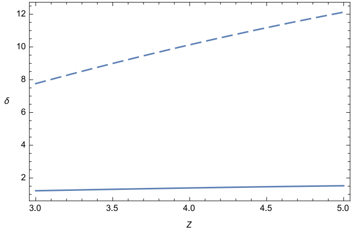


Figure: Relative order contributions δ in percent of vacuum polarization (solid line, order α^5) and nuclear structure (dashed line, order α^5) to hyperfine structure of muonic ions of lithium, beryllium and boron.

The uncertainty, due to the electromagnetic form factors of the nuclei, can be about 1 percent of the correction to the structure of the nucleus of the order α^5. Thus, we estimate approximately the errors in the calculation of the HFS spectrum in the form: $\delta E^{hfs}({}^6_3Li) = \pm 1$ meV, $\delta E^{hfs}({}^7_3Li) = \pm 4$ meV, $\delta E^{hfs}({}^9_4Be) = \pm 4.5$ meV, $\delta E^{hfs}({}^{10}_5B) = \pm 14$ meV, $\delta E^{hfs}({}^{11}_5B) = \pm 24$ meV.

There is another correction for the polarizability of the nucleus, which is not considered in this paper. The correction for the polarizability is of order $O(Z\alpha m_1/m_2)$, so its possible numerical value for different nuclei (0.6 meV (^6_3Li), 1.8 meV (^7_3Li), -1.6 meV (^9_4Be), 4.7 meV ($^{10}_5B$), 7.3 meV ($^{11}_5B$)) is comparable in magnitude to those errors that are connected with errors in measuring nuclear form factors. At the same time, it should be noted that the correction for the polarizability for a deuteron substantially exceeds this estimate. Therefore, its exact calculation becomes a very urgent problem. Our work in this direction is in progress.

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No.	Contribution to the splitting	$(\mu\ {}^6_3Li)^{2+}$, meV		$(\mu\ {}^7_3Li)^{2+}$, meV	
		1S	2S	1S	2S
1	Contribution of order α^4,	1416.07	177.01	5026.00	628.25
2	Muon AMM contribution	1.65	0.21	5.87	0.73
3	Relativistic correction α^6	1.02	0.18	3.62	0.64
4	Nuclear structure correction of order α^5	G; -109.92 U: -112.02	G: -13.74 U: -14.00	G: -369.25 U: -376.31	G: -46.16 U: -47.04
5	Nuclear structure and recoil	G: -0.20	G: -0.03	G: -30.67	G: -3.83
6	Nuclear structure correction of order α^6 in 1γ interaction	3.35	0.34	10.67	1.08
7	Nuclear structure correction in second order perturbation theory	-2.56	-0.90	-8.19	-2.90
8	Vacuum polarization contribution of order α^5 in first order PT	5.22	0.67	18.54	2.38
9	Vacuum polarization contribution of order α^5 in second order PT	12.05	1.11	42.83	3.94
10	Muon vacuum polarization contribution of order α^6 in first order PT	0.08	0.01	0.29	0.04
11	Muon vacuum polarization contribution of order α^6 in second order PT	0.09	0.01	0.31	0.04
12	Vacuum polarization contribution of order α^6 in first order PT	0.07	0.01	0.24	0.03
13	Vacuum polarization contribution of order α^6 in second order PT	0.14	0.02	0.53	0.05
14	Nuclear structure and vacuum polarization correction of order α^6	-1.62	-0.20	-5.85	-0.73
15	Nuclear structure and muon vacuum polarization correction of order α^6	-0.14	-0.02	-0.51	-0.06
16	Hadron vacuum polarization contribution of order α^6	0.06	0.01	0.21	0.03
17	Radiative nuclear finite size correction of order α^6	-0.34	-0.04	-1.24	-0.15
	Summary contribution	1325.02	164.65	4693.40	583.38

No.	Contribution to the splitting	$(\mu\ {}^9_4Be)^{3+}$, meV	
		1S	2S
1	Contribution of order α^4,	-4353.49	-544.19
2	Muon AMM contribution	-5.08	-0.64
3	Relativistic correction α^6	-5.57	-0.99
4	Nuclear structure correction of order α^5	G; 441.09 U: 449.54	G: 55.14 U: 56.19
5	Nuclear structure and recoil	G: -97.71	G: -12.21
6	Nuclear structure correction of order α^6 in 1γ interaction	-17.57	-1.78
7	Nuclear structure correction in second order perturbation theory	12.64	4.36
8	Vacuum polarization contribution of order α^5 in first order PT	-17.97	-2.30
9	Vacuum polarization contribution of order α^5 in second order PT	-42.62	-3.92
10	Muon vacuum polarization contribution of order α^6 in first order PT	-0.34	-0.04
11	Muon vacuum polarization contribution of order α^6 in srcond order PT	-0.36	-0.05
12	Vacuum polarization contribution of order α^6 in first order PT	-0.24	-0.03
13	Vacuum polarization contribution of order α^6 in second order PT	-0.54	-0.05
14	Nuclear structure and vacuum polarization correction of order α^6	5.31	0.66
15	Nuclear structure and muon vacuum polarization correction of order α^6	0.55	0.07
16	Hadron vacuum polarization contribution of order α^6	-0.25	-0.03
17	Radiative nuclear finite size correction of order α^6	1.44	0.18
	Summary contribution	-4080.71	-505.82