Describing phase transitions in field theory by self-similar approximants

V.I. Yukalov and E.P. Yukalova

Bogolubov Laboratory of Theoretical Physics
and
Laboratory of Information Technologies

Joint Institute for Nuclear Research
Dubna, Russia

Problems in perturbation theory

$$f(x) \simeq f_k(x) \qquad (x \to 0)$$

$$f_k(x) = f_0(x)\left(1 + \sum_{n=1}^{k} a_n x^n\right)$$

Asymptotic series, even for small x

x can be not small
strong divergence!

Padé approximants
Borel summation

→ not always applicable

Ideas of general theory

1. Incorporating control functions

$$\{f_k(x)\} \rightarrow \{F_k(x, u_k)\}$$

2. Defining control functions

$$u_k \rightarrow u_k(x)$$

$\{F_k(x, u_k)\}$ convergent

3. Finding transformation law

$$F_k \rightarrow F_{k+1}$$

$$F_k \rightarrow F_{k+1} \rightarrow F_{k+2} \rightarrow \ldots \rightarrow F^*$$

Incorporating control parameters

1. *Initial conditions*

H Hamiltonian

$$H_\varepsilon = H_0(u) + \varepsilon [\, H - H_0(u) \,]$$

$$F_k(x, u_k) = \langle \hat{A}(x) \rangle_k$$

Perturbation theory with respect to ε

finally

$$\varepsilon \to 1$$

Incorporating control parameters

2. *Reexpansion trick*

(i) parameter substitution

$$m \rightarrow u + \varepsilon(m - u), \qquad x \rightarrow \varepsilon x$$

reexpand $f_k(\varepsilon x, u + \varepsilon(m-u))$ in ε

set $\varepsilon \rightarrow 1$

(ii) change of variable

$x = x(z,u)$, reexpand $f_k(x(z,u))$ in z

Conformal mapping: $x = \dfrac{4\mathrm{u}^2 z}{(1-z)^2}, \qquad z = \dfrac{\sqrt{x+u^2}-u}{\sqrt{x+u^2}+u}$

Incorporating control parameters

3. Functional transformation

$$\hat{T}[u_k] f_k(x) = F_k(x, u_k)$$

$$f_k(x) = \hat{T}^{-1}[u_k] F_k(x, u_k)$$

fractal transform

$$\hat{T}[u] f_k(x) = x^u f_k(x)$$

Defining control functions

Cauchy criterion

$$| F_{k+p}(x, u_{k+p}) - F_k(x, u_k) | < \varepsilon$$

for

$$k \geq n_\varepsilon, \qquad p \geq 0$$

Cauchy cost functional

$$\boxed{C[u] = \sum_{k=0}^{\infty} | F_{k+p}(x, u_{k+p}) - F_k(x, u_k) |}$$

$$\min_u C[u] \rightarrow u_k(x)$$

Optimized approximant

$$\tilde{f}_k(x) = F_k(x, u_k(x))$$

Transformation law

$$\tilde{f}_k(x) \to \tilde{f}_{k+1}(x) \quad ?$$

Reonomic constraint

$$F_0(x, u_k(x)) = f$$

Expansion function

$$x = x_k(f)$$

Endomorphism

$$y_k(f) = \tilde{f}_k(x_k(f))$$

Self-similar relation

$$\{\ \tilde{f}_k(x)\ \} \ \rightarrow \ f^*(x)$$

Sequence limit ⟷ fixed point

$$\{\ y_k(f)\ \} \ \rightarrow \ y^*(f)$$

$$y_k(y^*(f)) \ = \ y^*(f)$$

In the vicinity of a fixed point

$$y_{k+p}(f) \ = \ y_k(y_p(f))$$

Approximation cascade

Dynamical system in discrete time

$$\{\, y_k : \; k = 0, 1, 2, \ldots \,\}$$

$$\{\, y_k(f) \,\} \longleftrightarrow \{\, \tilde{f}_k(x) \,\}$$

trajectory sequence

$$y^*(f) \longleftrightarrow f^*(x)$$

fixed point sequence limit

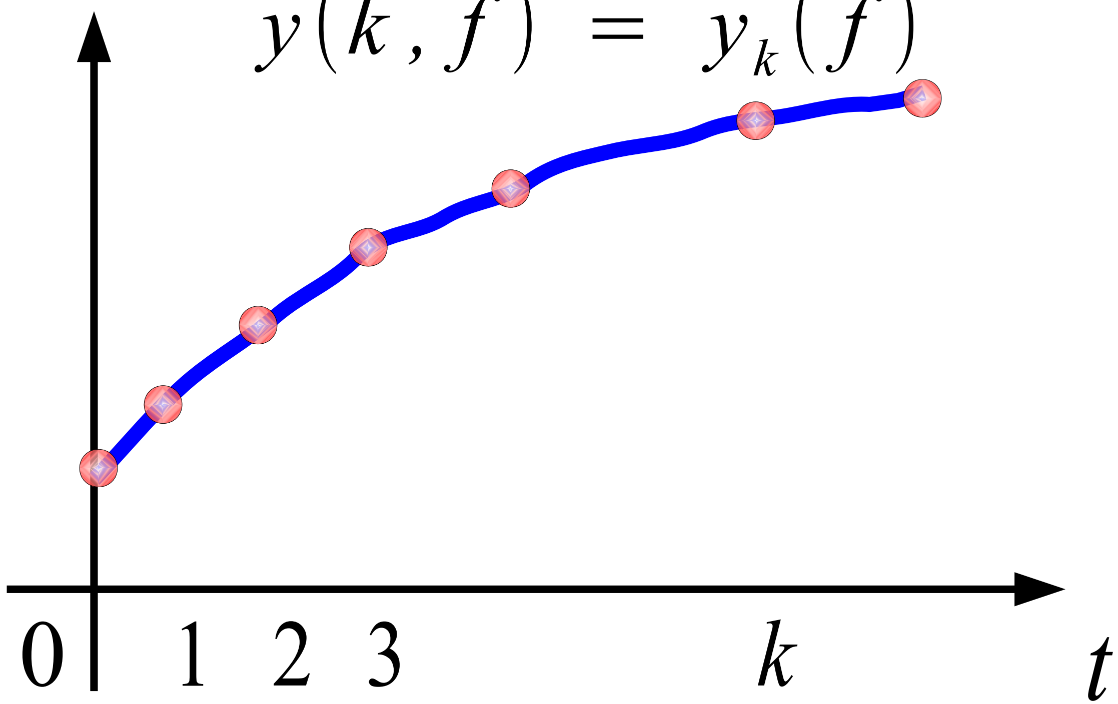
Approximation flow

Embedding cascade into flow

$$\{\, y_k(f):\ k = 0,1,2,\ldots\} \subset \{y(t,f):\ t \geq 0\,\}$$

$$y(t+t', f) = y(t, y(t', f))$$

$$y(k, f) = y_k(f)$$



Evolution equation

Differential Lie equation

$$\frac{\partial}{\partial t} y(t, f) = v(y(t, f))$$

$v(y)$ velocity (Gell-Mann-Low function)

Integral equation

$$\int_{y_k}^{y_k^*} \frac{dy}{v(y)} = t_k$$

Self-similar factor approximants

$$f_k(x) = f_0(x)\left(1 + \sum_{n=1}^{k} a_n x^n\right)$$

$$f_k^*(x) = f_0(x) \prod_{i=1}^{N_k} (1 + A_i x)^{n_i}$$

$$N_k = \begin{cases} k/2 & k = 2, 4, \ldots \\ (k+1)/2 & k = 3, 5, \ldots \end{cases}$$

All $A_i, \quad n_i$ from

$$f_k^*(x) \simeq f_k(x) \qquad (x \to 0)$$

Self-similar exponential approximants

$$f_k^*(x) = f_0(x)\exp\left(C_1 x \exp\left(C_2 x \ldots \exp\left(C_k t_k x\right)\right)\right)$$

$$C_j = \frac{a_j}{a_{j-1}}, \qquad t_k = t_k(x)$$

$$t_k = \exp\left(C_k x t_k \right)$$

Critical temperature

Scattering length a_s

Particle density ρ

Interaction parameter $\gamma = \rho^{1/3} a_s$

$O(N)$ – symmetric φ^4 field theory in $3d$ space

Free field $(\gamma \to 0)$

$$T_0 = \frac{2\pi}{m} \left[\frac{\rho}{\zeta(3/2)} \right]^{2/3}$$

Critical temperature

$T_c(\gamma)$?

Critical temperature shift

$$\frac{\Delta T_c}{T_0} \equiv \frac{T_c(\gamma) - T_0}{T_0}$$

$$\frac{\Delta T_c}{T_0} \simeq C_1 \gamma \qquad (\gamma \to 0)$$

Loop expansion in $x = (N+2)\frac{\lambda}{\sqrt{\mu}}$

Effective coupling $\lambda \sim \gamma$

Effective chemical potential μ

$$C_1(x) \simeq \sum_{n=1}^{5} a_n x^n \qquad (x \to 0)$$

but $x \to \infty \qquad (\mu \to 0)$

Calculating critical temperature shift

We have: $C_1(x)$ for $x \rightarrow 0$

We need:

$$\boxed{C_1 = \lim_{x \rightarrow \infty} C_1(x)}$$

$$C_1(x) \rightarrow f_k^*(x) = a_1 x \prod_{i=1}^{N_k} (1 + A_i x)^{n_i}$$

$$f_k^*(x) \simeq B_k x^{\beta_k} \qquad (x \rightarrow \infty)$$

$$B_k = a_1 \prod_{i=1}^{N_k} A_i^{n_i}, \qquad \beta_k = 1 + \sum_{i=1}^{N_k} n_i \rightarrow 0$$

$$f_k^*(\infty) = B_k \rightarrow C_1(\infty)$$

The coefficient C_1 of the critical temperature shift, for a different number of the field components N, compared with the available Monte Carlo simulations

N	C_1	*Monte Carlo*
0	0.77 ± 0.03	———
1	1.06 ± 0.05	1.09 ± 0.09
2	1.29 ± 0.07	1.29 ± 0.05 1.32 ± 0.02
3	1.46 ± 0.08	———
4	1.60 ± 0.09	1.60 ± 0.10

Critical exponents

$O(N)$ – symmetric φ^4 field theory in $3d$ space

Wilson ε – expansion $\quad (\varepsilon = 4 - d)$

$$f_k(\varepsilon) = \sum_{n=0}^{k} c_n \varepsilon^n$$

Factor approximants

$$f_k^*(\varepsilon) = f_0(\varepsilon) \prod_{i=1}^{N_k} (1 + A_i \varepsilon)^{n_i}$$

<u>Result</u>

$$\boxed{f_k^* = \frac{1}{2}\left[f_k^*(1) + f_{k-1}^*(1) \right]}$$

Error bar $\quad \pm \frac{1}{2}\left[f_k^*(1) - f_{k-1}^*(1) \right]$

$O(1)$ universality class

$$\eta \simeq 0.0185185\,\varepsilon^2 + 0.01869\,\varepsilon^3 - 0.00832877\,\varepsilon^4 + 0.0256565\,\varepsilon^5$$

$$\nu^{-1} \simeq 2 - 0.333333\,\varepsilon - 0.117284\,\varepsilon^2 + 0.124527\,\varepsilon^3 - 0.30685\,\varepsilon^4 - 0.95124\,\varepsilon^5$$

$$\omega \simeq \varepsilon - 0.62963\,\varepsilon^2 + 1.61822\,\varepsilon^3 - 5.23514\,\varepsilon^4 + 20.7498\,\varepsilon^5$$

Scaling relations $\alpha = 2 - 3\nu, \quad \beta = \frac{\nu}{2}(1+\eta)$

$$\gamma = \nu(2-\eta), \quad \delta = \frac{5-\eta}{1+\eta}$$

Critical exponents for $O(1)$ – symmetric φ^4 field theory in $3d$, calculated using self-similar factor approximants (FA), conformal bootstrap conjecture (CB), and Monte Carlo simulations (MC).

	FA	CB	MC
α	0.10645	0.11008	0.11026
β	0.32619	0.32642	0.32630
γ	1.24117	1.23708	1.23708
δ	4.80502	4.78984	4.79091
η	0.03359	0.03630	0.03611
ν	0.63118	0.62997	0.62991
ω	0.78755	0.82966	0.830

$O(N)$ universality class $\left(-2 \le N < 10\,000 \right)$

Exact critical exponents for $O(N)$ – symmetric φ^4 field theory in $3d$ for $N = -2$ and $N \to \infty$

	$N = -2$	$N \to \infty$
α	0.5	-1
β	0.25	0.5
γ	1	2
δ	5	5
η	0	0
ν	0.5	1
ω	0.8	1

Quark-gluon plasma

QCD, three colours $N_c = 3$

High temperature, weak coupling

Massless fermions n_f
zero chemical potential

Dimensional regularization,
minimal subtraction scheme $\overline{\text{MS}}$

Is it possible to get confinement – deconfinement phase transition, considering expansions in asymptotically small coupling ?

$$\alpha_s = \frac{g^2}{4\pi}$$

Asymptotic pressure $\left(g \rightarrow 0 \right)$

$$P(g) \simeq \frac{8\pi^2}{45} T^4 \left(\sum_{n=0}^{5} c_n g^n + c_4' g^4 \ln g \right)$$

$$c_1 = 0$$

Stefan – Boltzmann limit

$$P_0 = P(0) = \frac{8\pi^2}{45} T^4 \left(1 + \frac{21}{32} n_f \right)$$

Reduced pressure

$$p_k \equiv \frac{P_k}{P_0} = p_k(g, \mu, T)$$

Renormalization scale in $\overline{\text{MS}}$ - μ

$$p_k = 1 + \sum_{n=2}^{5} \bar{c}_n g^n + \bar{c}_4' g^4 \ln g, \quad \bar{c}_n \equiv \frac{c_n}{c_0}$$

Renormalization scale - μ

Minimal difference condition

$$p_4(g, \mu, T) - p_3(g, \mu, T) = 0$$

$$\mu = \mu(g, T)$$

Reduced pressure

$$p_5 = 1 + \bar{c}_2 g^2 + \bar{c}_3 g^3 + \bar{c}_5 g^5$$

Reduced pressure

$$p_2^* = \exp\left(C_2 g^2\right) \qquad \left(C_2 = \frac{c_2}{c_0} \right)$$

$$p_3^* = \exp\left(C_2 g^2 t_3\right)$$

$$t_3 = \exp\left(C_3 g\, t_3\right) \qquad \left(C_3 = \frac{c_3}{c_2} \right)$$

$$p_5^* = \exp\left(C_2 g^2 \exp\left(C_3 g\, t_5\right)\right)$$

$$t_5 = \exp\left(C_5 g^2 t_5\right) \qquad \left(C_5 = \frac{c_5}{c_3} \right)$$

Renormalization group equation

$$\mu \frac{\partial g}{\partial \mu} = \beta(g)$$

$$\beta(g) \simeq \beta_k(g) \quad (g \to 0)$$

$$\beta_k(g) = -\sum_{n=0}^{k} b_n g^{2n+3}$$

$$\beta_k(g) = -b_0 g^3 \left(1 + \sum_{n=1}^{k} \frac{b_n}{b_0} g^{2n} \right)$$

Initial condition: for Z^0 boson mass

$$g(m_Z) = 1.22285 \quad (m_Z = 91187\,\mathrm{MeV})$$

Gell – Mann – Low function

$$\beta_k^*(g) = -b_0 g^3 \exp\left(B_1 g^2 \exp\left(B_2 g^2 \ldots \exp\left(B_k t_k g^2\right)\right)\right)$$

$$B_j = \frac{b_j}{b_{j-1}}, \qquad t_k = \exp\left(B_k g^2 t_k\right)$$

RG equation $\longrightarrow$ $g = g(\mu)$

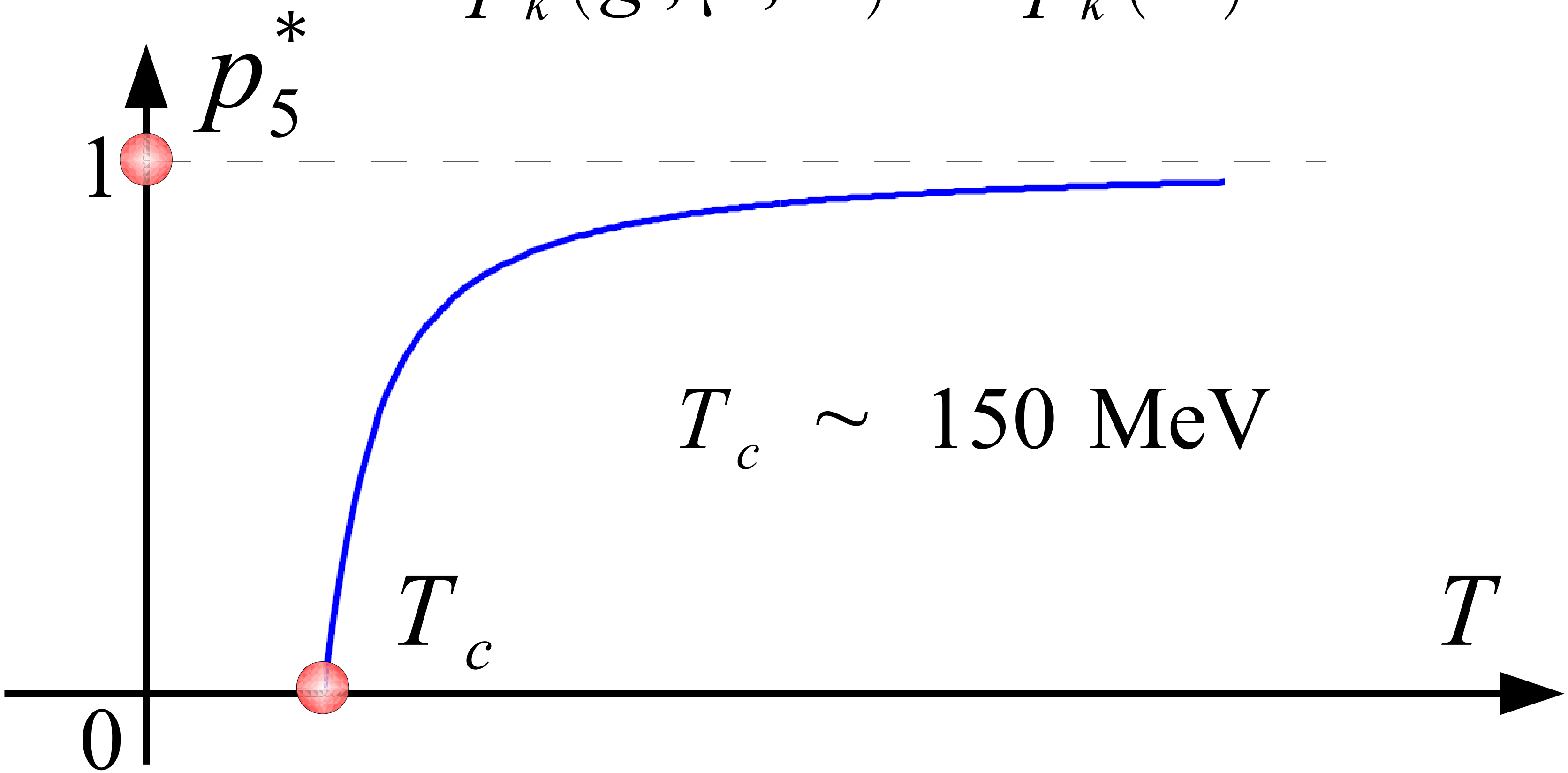
Temperature dependence

Minimal – difference condition → $\mu = \mu(g, T)$

Renormalization – group equation → $g = g(\mu)$

Hence: $\mu = \mu(T), \quad g = g(T)$

Pressure $p_k^*(g, \mu, T) \to p_k^*(T)$



Conclusion

- Self – similar approximation theory is a powerful tool for extrapolating asymptotic series.
- Influence of the coupling – parameter strength on T_c in the $O(N)$ – symmetric φ^4 theory.
- Critical exponents for the phase transition of the $O(N)$ universality class.
- Confinement – deconfinement transition temperature in QCD.
- Good agreement with numerical methods.
- Low cost calculations.