Investigation of the $O(n)$-symmetric $\varphi^4 + \varphi^6$ theory using renormalization group method to six loops

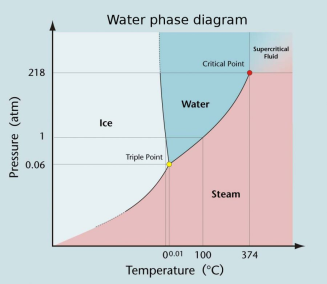
A.V. Trenogin
(with L.Ts. Adzhemyan and M.V. Kompaniets)

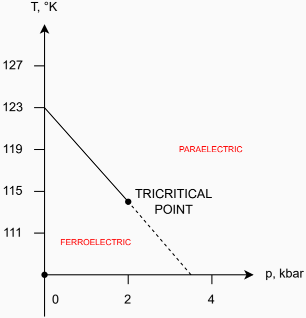
The 30th of October, 2024

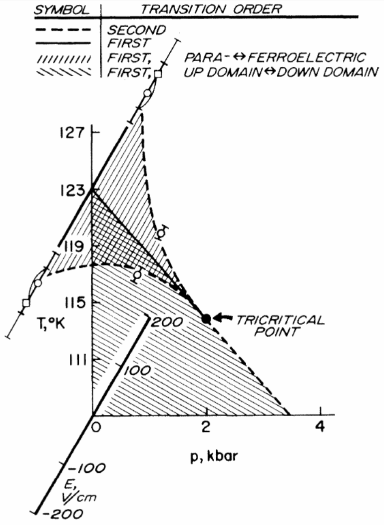
Saint Petersburg State University

Table of contents

Tricritical point in KH_2PO_4 [Schmidt, Western, and Baker 1976]







Some researches

Experiments:

- Zhang et al, Metamagnetic tricritical behavior of the magnetic topological insulator $\mathrm{MnBi_4Te_7}$, **2024**;
- Shang and Solomon, Tricritical scaling and logarithmic corrections for the metamagnet $\mathrm{FeCl_2}$, **1980**;

Theoretic approach:

1. Modeling:
    - Moueddene et al, Logarithmic corrections and criticality in the $d = 3$ Blume-Capel model: Results from small-scale Monte Carlo simulations, **2024**;
    - Moueddene et al, Critical and tricritical singularities from small-scale Monte Carlo simulations: The Blume-Capel model in two dimensions, **2024**;
2. Conformal bootstrap:
    - Gowdigere et al, Conformal Bootstrap Signatures of the Tricritical Ising Universality Class, **2021**;
3. Mean-field theory:
    - Hager et al, Scaling of demixing curves and crossover from critical to tricritical behavior in polymer solutions, **2002**.

Model (mean-field theory)

$O(n)$ $\varphi^4 + \varphi^6$ model (d – Euclidean $(3 - 2\varepsilon)$-space):

$$S_0(\varphi) = \frac{1}{2}\partial_i\varphi_a\partial_i\varphi_a + \frac{\tau_0}{2}\varphi_a\varphi_a + \frac{\lambda_0}{4!}(\varphi_a\varphi_a)^2 + \frac{g_0}{6!}(\varphi_a\varphi_a)^3,$$

where $\varphi = \{\varphi_a, a = 1, \ldots, n\}$ – n-component order parameter; τ_0, $\lambda_0 = \bar{\lambda}_0\tau_0^{\phi}$ and g_0 – parameters.

Renormalized action:

$$S_R(\varphi) = \frac{(Z_1\Delta + Z_2\tau + Z_5\lambda^2)}{2}\varphi^2 + \frac{Z_4\lambda\mu^{2\varepsilon}}{4!}\varphi^4 + \frac{Z_3 g\mu^{4\varepsilon}}{6!}\varphi^6,$$

where

$$\hat{\varphi} = Z_\varphi\hat{\varphi}_R; \qquad Z_1 = Z_\varphi^2; \qquad Z_4 = Z_\lambda Z_\varphi^4;$$

$$\tau_0 = Z'_\tau\tau = Z_\tau\tau + \bar{Z}\lambda^2; \qquad Z_2 = Z_\tau Z_\varphi^2; \qquad Z_5 = \bar{Z}Z_\varphi^2.$$

$$g_0 = Z_g g\mu^{4\varepsilon}; \qquad Z_3 = Z_g Z_\varphi^6;$$

$$\lambda_0 = Z_\lambda\lambda\mu^{2\varepsilon}.$$

A.N. Vasil’ev notations [Vasil’ev 2004]

Tricritical exponents

φ^6:

α – the exponent of the specific heat;

β and $1/\delta$ – different order parameter exponents;

γ – the susceptibility exponent;

ν – the exponent of the correlation length;

η – the Fisher exponent (the critical-point correlation exponent).

$\varphi^4 + \varphi^6$ (additional to φ^6):

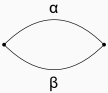
ϕ_t – crossover exponent (the limiting value of the ϕ when both interactions (φ^4 and φ^6) are significant).

Previous results/tricritical exponents

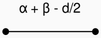
- [1]: $\eta - \varepsilon^3$ (1 six-loop diagram with 2 external edges in φ^6 theory, six-loop contribution into Z_φ) and $\phi_t - \varepsilon^2$;
- [2]: $\nu - \varepsilon^3$ (six-loop contribution into Z_τ) and confirmed ε^3 term in η;
- [3]: calculated full 3 order:
  - ϕ_t – calculated ε^3 term (however incorrect);
  - confirmed ε^2 and ε^3 terms.

[1] Lewis and Adams 1978, "Tricritical behavior in two dimensions. II. Universal quantities from the ϵ expansion".
[2] Hager and Schäfer 1999, "Θ-point behavior of diluted polymer solutions: Can one observe the universal logarithmic corrections predicted by field theory?"
[3] J. S. Hager 2002, "Six-loop renormalization group functions of $O(n)$-symmetric ϕ^6-theory and ϵ-expansions of tricritical exponents up to ϵ^3".

G-functions



$$= \frac{1}{(2\pi)^d} \int \frac{dk}{k^{2\alpha}(k-p)^{2\beta}} = \frac{1}{(4\pi)^{d/2}} \frac{G(\alpha,\beta)}{p^{2(\alpha+\beta-d/2)}} \sim$$

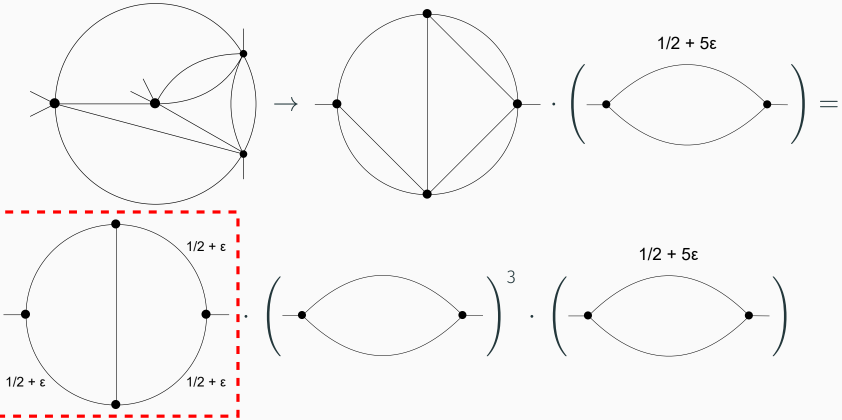


,

$$G(\alpha,\beta) = \frac{\Gamma(d/2-\alpha)\Gamma(d/2-\beta)\Gamma(\alpha+\beta-d/2)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(d-\alpha-\beta)}.$$

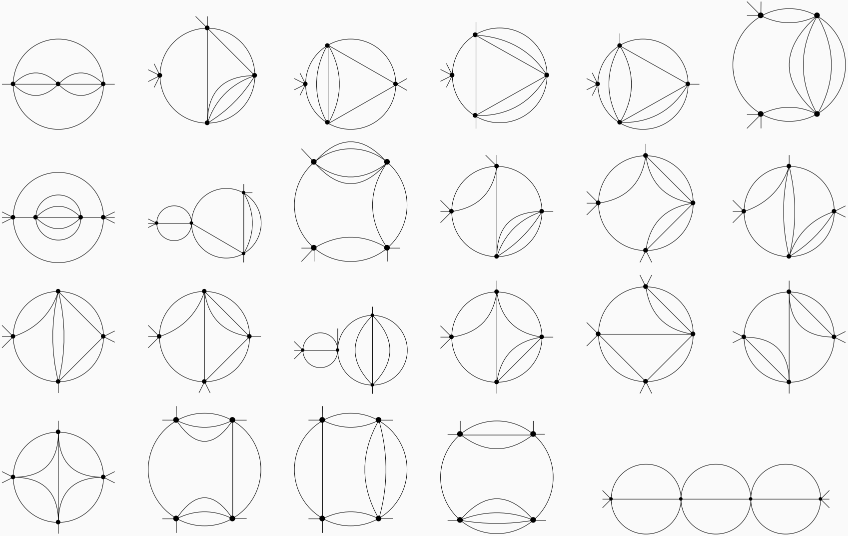
Sector Decomposition

$$d = 3 - 2\varepsilon$$



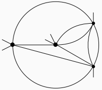
To calculate ↑ (multi-loop irreducible graphs) we use the Sector Decomposition method.

One-loop reducible diagrams, 6 loops



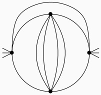
Complex diagrams, 6 loops

Complex diagrams – diagrams that are not one-loop reducible.



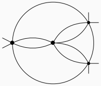
$$\frac{29.13318(6)}{\varepsilon}$$

$$\frac{8(\pi^2\beta(2)+24\beta(4))}{9\varepsilon}\ (IBP)$$



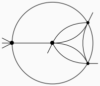
$$\frac{1.3333318(14)}{\varepsilon^2}-\frac{5.33336(3)}{\varepsilon}$$

$$\frac{4}{3\varepsilon^2}-\frac{16}{3\varepsilon}\ (IBP)$$



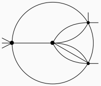
$$\frac{32.46970(7)}{\varepsilon}$$

$$\frac{\pi^4}{3\varepsilon}\ (IBP)$$



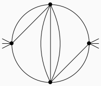
$$-\frac{3.289883(9)}{\varepsilon^2}+\frac{6.72312(16)}{\varepsilon}$$

$$-\frac{\pi^2}{3\varepsilon^2}+\frac{\pi^2(1+\ln 4)-14\zeta(3)}{\varepsilon}\ (IBP)$$



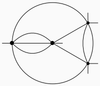
$$\frac{0.16666664(28)}{\varepsilon^3}-\frac{2.000003(6)}{\varepsilon^2}+\frac{1.89371(10)}{\varepsilon}$$

$$\frac{1}{6\varepsilon^3}-\frac{2}{\varepsilon^2}-\frac{8(\pi^2-12)}{9\varepsilon}\ (IBP)$$



$$\frac{0.3333331(3)}{\varepsilon^3}-\frac{2.666661(4)}{\varepsilon^2}+\frac{2.66667(6)}{\varepsilon}$$

$$\frac{1}{3\varepsilon^3}-\frac{8}{3\varepsilon^2}+\frac{8}{3\varepsilon}\ (R^*)$$



$$-\frac{1.644932(2)}{\varepsilon^2}+\frac{4.18125(4)}{\varepsilon}$$

$$-\frac{\pi^2}{6\varepsilon^2}+\frac{\pi^2(5+\ln 4)-42\zeta(3)}{3\varepsilon}\ (IBP)$$

Second line is results of M. Kompaniets and A. Pikelner, Unpublished

Tricritical exponents, $O(n)$

$$\eta = (2.66667 + 2n + 0.333333n^2)\frac{\varepsilon^2}{(22+3n)^2} + (33797.3 + 33534.1n + 10838.6n^2 + 1385.63n^3 + 64.232n^4 + 0.822467n^5)\frac{\varepsilon^3}{(22+3n)^4};$$

$$\nu = 0.5 + (10.6667 + 8n + 1.33333n^2)\frac{\varepsilon^2}{(22+3n)^2} + (86891.3 + 82490.4n + 24328.3n^2 + 2518.52n^3 + 56.9602n^4 - 0.411234n^5)\frac{\varepsilon^3}{(22+3n)^4};$$

$$\phi_t = 0.5 + (6-n)\frac{\varepsilon}{22+3n} + (-47927.4 - 20941.2n - 2312.87n^2 - 8.39119n^3 + 2.4674n^4)\frac{\varepsilon^2}{(22+3n)^3} + (4.726074(15)\cdot 10^8 + 3.191107(10)\cdot 10^8 n + 8.107993(26)\cdot 10^7 n^2 + 9692087(31)n^3 + 538116.4(1.6)n^4 + 11367.367(28)n^5 + 203.17798(17)n^6 + 6.08807n^7)\frac{\varepsilon^3}{(22+3n)^5}.$$

ϕ_t difference

Our result:

$$\phi_t = 0.5 + (6-n)\frac{\varepsilon}{22+3n} + (-47927.4 - 20941.2n - 2312.87n^2 - 8.39119n^3 + 2.4674n^4)\frac{\varepsilon^2}{(22+3n)^3} + (4.726074(15)\cdot 10^8 + 3.191107(10)\cdot 10^8 n + 8.107993(26)\cdot 10^7 n^2 + 9692087(31)n^3 + 538116.4(1.6)n^4 + 11367.367(28)n^5 + 203.17798(17)n^6 + 6.08807n^7)\frac{\varepsilon^3}{(22+3n)^5}.$$

Result of the article [J. S. Hager 2002]:

$$\phi_t = 0.5 + (6-n)\frac{\varepsilon}{22+3n} + (-47927.4 - 20941.2n - 2312.87n^2 - 8.39119n^3 + 2.4674n^4)\frac{\varepsilon^2}{(22+3n)^3} + (5.82218\cdot 10^8 + 4.01209\cdot 10^8 n + 1.04251\cdot 10^8 n^2 + 1.26915\cdot 10^7 n^3 + 702497n^4 + 13218.9n^5 + 158.765n^6 + 6.08807n^7)\frac{\varepsilon^3}{(22+3n)^5}.$$

Conclusion

- We have performed six-loop calculation of the tricritical exponents of the $O(n)$-symmetric $\varphi^4 + \varphi^6$ theory;
- Both η and ν tricritical exponents completely coincided with the results of the work[4];
- ϕ_t tricritical exponent differs from the result presented in the work[9];
- TODO: 8-loop calculations in the $O(n)$-symmetric $\varphi^4 + \varphi^6$ theory.

[4] J. S. Hager 2002, "Six-loop renormalization group functions of $O(n)$-symmetric ϕ^6-theory and ϵ-expansions of tricritical exponents up to ϵ^3".

Thank you!

Email: a.trenogin@spbu.ru

This work was performed at the Saint Petersburg Leonhard Euler International Mathematical Institute and supported by the Ministry of Science and Higher Education of the Russian Federation (agreement no. 075–15–2022–287).