Perturbative analysis of a five-parameter symmetrical teleparallel gravitational theory generalization

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Outline

Motivation

General Relativity Theory(**GR**) has a number of cosmological problems.
That is why we want

1) a theory **other than GR**
which would be **consistent** in reliably verified aspects

2) would **adequately describe** those phenomena
that are within the framework of GR requires the introduction of additional dark sectors or requires fine-tuning

Introduction

1) Our **goal** is to analyze the dynamic properties of the teleparallel generalization of Einstein's theory of relativity within the framework of **linear perturbation theory**.

2) Our **task** is to determine **the range of parameters** that would define a model that agrees well with GR, but is not identical to it.

Symmetric teleparallel framework

We can rewrite the Einstein-Gilbert action as:

$$S_{EH} = \int \sqrt{-g}\mathbb{R}d^4x = -\int \sqrt{-g}(\mathbb{Q}+\mathbb{B})d^4x$$

$$\mathbb{Q} = \frac{1}{4}Q_{\alpha\mu\nu}Q^{\alpha\mu\nu} - \frac{1}{2}Q_{\alpha\mu\nu}Q^{\mu\alpha\nu} - \frac{1}{4}Q_\mu Q^\mu + \frac{1}{2}Q_\mu \tilde{Q}^\mu$$

$$\mathbb{B} = \dot{\nabla}_\alpha(Q^\alpha - \tilde{Q}^\alpha)$$

$Q_{\alpha\mu\nu} = \partial_\alpha g_{\mu\nu}$ - the nonmetricity,

$$Q_\mu = Q_{\mu\alpha}^{\cdot\cdot\alpha}, \quad \tilde{Q}_\mu = Q_{\alpha\mu}^{\cdot\cdot\alpha}$$

We can modify the action to make it five-parameter:

$$S_{STGR} = -\int d^4x\sqrt{-g}\cdot$$

$$\cdot\left(\frac{a_1}{4}Q_{\alpha\mu\nu}Q^{\alpha\mu\nu} - \frac{a_2}{2}Q_{\alpha\mu\nu}Q^{\mu\alpha\nu} - \frac{a_3}{4}Q_\mu Q^\mu + +\frac{a_4}{2}\tilde{Q}_\mu\tilde{Q}^\mu + \frac{a_5}{2}Q_\mu\tilde{Q}^\mu\right)$$

Symmetric teleparallel framework

So, The action of our theory can be written as:

$$S_{STGR} = -\int d^4x\sqrt{-g}\mathfrak{Q}$$

$\mathfrak{Q}$ - a nonmetricity scalar of the most general kind

$$\mathfrak{Q} = \frac{1}{2}\mathfrak{B}^{\alpha\mu\nu}Q_{\alpha\mu\nu}$$

$\mathfrak{B}^{\alpha\mu\nu}$ - a five-parameter superpotential constructed from nonmetricity components

$$\mathfrak{B}_{\alpha\mu\nu} = \frac{a_1}{2}Q_{\alpha\mu\nu} - \frac{a_2}{2}\left(Q_{\mu\alpha\nu} + Q_{\nu\alpha\mu}\right) - \frac{a_3}{2}Q_\alpha g_{\mu\nu} +$$
$$+\frac{a_4}{2}\left(g_{\alpha\mu}\tilde{Q}_\nu + g_{\alpha\nu}\tilde{Q}_\mu\right) + \frac{a_5}{4}\left(2\tilde{Q}_\alpha g_{\mu\nu} + g_{\alpha\mu}Q_\nu + g_{\alpha\nu}Q_\mu\right)$$

The equations of motion can be represented as:

$$\mathfrak{G}^\mu_{\cdot\nu} \equiv -\partial_\alpha\mathfrak{B}^{\alpha\mu}_{\cdot\cdot\nu} - \frac{1}{2}Q_\alpha\mathfrak{B}^{\alpha\mu}_{\cdot\cdot\nu} - \frac{1}{2}\mathfrak{B}^{\mu\alpha\beta}Q_{\nu\alpha\beta} + \frac{1}{2}\mathfrak{Q}\delta^\mu_\nu = 0$$

Perturbations of a flat background

the metric:

$$g_{\mu\nu} = \eta_{\mu\nu} + \delta g_{\mu\nu}$$

$\eta_{\mu\nu}$ -the Minkowski metric, $\delta g_{\mu\nu}$ - perturbations

$$g_{00} = 1 + 2\phi, \quad g_{0i} = \partial_i V + V_i, \quad g_{ij} = -\delta_{ij} + 2\psi\delta_{ij} - 2\partial^2_{ij}\sigma - \partial_i c_j - \partial_j c_i - h_{ij}$$

where the vectors V_i и c_i are non-divergent
The matrix h_{ij} is symmetric, traceless and non-divergent.
Perturbations for the inverse metric:

$$g^{\mu\nu} = \eta^{\mu\nu} + \delta g^{\mu\nu}$$

$$g^{00} = 1 - 2\phi, \quad g^{0i} = \partial_i V + V_i, \quad g^{ij} = -\delta_{ij} - 2\psi\delta_{ij} + 2\partial^2_{ij}\sigma + \partial_i c_j + \partial_j c_i + h_{ij}$$

Tensor perturbations

The metric in tensor sector:

$$g_{00} = 1, \quad g_{0i} = 0, \quad g_{ij} = -\delta_{ij} - h_{ij},$$

The inverse metric: $g^{00} = 1, \quad g^{0i} = 0, \quad g^{ij} = -\delta_{ij} + h_{ij}$
Nonzero components of the nonmetricity tensor:

$$Q_{0ij} = -\dot{h}_{ij}, \quad Q_{ijk} = -\partial_i h_{jk}$$

the nonzero components of the superpotential:
$\mathfrak{B}_{0ij} = -\frac{a_1}{2}\dot{h}_{ij}$, $\mathfrak{B}_{i0j} = \frac{a_2}{2}\dot{h}_{ij}$, $\mathfrak{B}_{ijk} = -\frac{1}{2}\left(a_1\partial_i h_{jk} - a_2\partial_j h_{ik} - a_2\partial_k h_{ij}\right)$
So the only nonzero component of the equations of motion is

$$\mathfrak{G}^{j}_{\cdot k} = -\frac{a_1}{2}\Box h_{jk} = 0$$

$\Box = \partial_t^2 - \Delta$ - the Dalembert operator. The first natural restriction on our model to have gravitational wave is $a_1 \neq 0$

Analysis of the nature of degrees of freedom

The simple dependence of the resulting equation on the parameters leads to only two options:

1.$a_1 \neq 0$ tensor modes are dynamic.

2.$a_1 = 0$ the tensor sector is fully gauge.

Vector perturbations

The metric in vector sector:

$$g_{00} = 1, \quad g_{0i} = V_i, \quad g_{ij} = -\delta_{ij} - \partial_i c_j - \partial_j c_i,$$

The inverse metric: $g^{00} = 1, \quad g^{0i} = V_i, \quad g^{ij} = -\delta_{ij} + \partial_i c_j + \partial_j c_i$

Nonzero components of the nonmetricity tensor:

$$Q_{00i} = \dot{V}_i, \quad Q_{0ij} = -\partial_i \dot{c}_j - \partial_j \dot{c}_i, \quad Q_{i0j} = \partial_i V_j, \quad Q_{ijk} = -\partial^2_{ij} c_k - \partial^2_{ik} c_j$$

Nonzero components of nonmetricity vectors:

$$\tilde{Q}_i = \dot{V}_i + \Delta c_i$$

Nonzero components of the equations of motion

$$\mathfrak{G}^0_{\cdot i} = \frac{a_1}{2} \Delta \left(V_i + \dot{c}_i \right) - \left(\frac{a_1 - a_2 + a_4}{2} \right) \left(\ddot{V}_i + \Delta \dot{c}_i \right) = 0,$$

$$\mathfrak{G}^i_{\cdot j} = -\frac{1}{2} \left[(a_2 - a_4) \left(\partial_i \dot{V}_j + \partial_j \dot{V}_i \right) + a_1 \left(\partial_i \ddot{c}_j + \partial_j \ddot{c}_i \right) \right] + \\ + \left(\frac{a_1 - a_2 + a_4}{2} \right) \Delta \left(\partial_i c_j + \partial_j c_i \right) = 0$$

Vector sector

In the case of STEGR, the equations will take the form

$$G^0_i = \frac{1}{2}\Delta\left(V_i + \dot{c}_i\right) = 0, \quad G^i_j = -\frac{1}{2}\left[\partial_i\left(\dot{V}_j + \ddot{c}_j\right) + \partial_j\left(\dot{V}_i + \ddot{c}_i\right)\right] = 0$$

We can see that the vector sector depends only on their combination $(V_i + \dot{c}_i)$, which means the restoration of calibration freedom and that only one of the vector variables is physical

Analysis of the nature of degrees of freedom

We can obtain the equations of the vector sector

$$a_1 \Delta \left(V_i + \dot{c}_i \right) - \left(a_1 - a_2 + a_4 \right) \left(\ddot{V}_i + \Delta \dot{c}_i \right) = 0$$

$$(a_2 - a_4) \dot{V}_i + a_1 \ddot{c}_i - (a_1 - a_2 + a_4) \Delta c_i = 0$$

vary any combinations of parameters $a_1, \quad a_2 - a_4, \quad a_1 - a_2 + a_4$ to zero:

1. When none of the combinations vanishes, both vectors V_i and c_i are dynamic.

2. When $a_1 = 0, a_2 - a_4 \neq 0$ the equations become

$$\ddot{V}_i + \Delta \dot{c}_i = 0, \dot{V}_i + \Delta c_i = 0$$

linearly dependent, so the vector c_i is a gauge vector and the vector V_i is half dynamic and half limited, since the equation is only of the first order in time

Analysis of the nature of degrees of freedom

3. When $a_2 - a_4 = 0, a_1 \neq 0$ the equations become:

$$\Box V_i = 0, \quad \Box c_i = 0$$

from where it is obvious that both vectors are dynamic.

4. When $a_1 - a_2 + a_4 = 0, a_1 \neq 0, a_2 - a_4 \neq 0$ the equations become

$$V_i + \dot{c}_i = 0, \quad a_1 \left(\dot{V}_i + \ddot{c}_i \right) = 0,$$

where the second is done automatically.Therefore, the vector V_i is limited by the connection $V_i = -\dot{c}_i$, and c_i is a gauge vector

5. When $a_1 = a_2 - a_4 = 0$
The equations are performed identically, which means that both vectors become gauge vectors

Scalar perturbations

The metric in scalar sector:

$$g_{00} = 1 + 2\phi,\, g_{0i} = \partial_i U,\, g_{ij} = -\delta_{ij} + 2\psi\delta_{ij} - 2\partial^2_{ij}\sigma$$

The inverse metric: $g^{00} = 1 - 2\phi,\, g^{0i} = \partial_i,\, g^{ij} = -\delta_{ij} - 2\psi\delta_{ij} + 2\partial^2_{ij}\sigma$
Nonzero components of the nonmetricity tensor

$$Q_{000} = 2\dot{\phi}, \quad Q_{00i} = \partial_i \dot{U}, \quad Q_{0ij} = 2\left(\delta_{ij}\dot{\psi} - \partial^2_{ij}\dot{\sigma}\right),$$

$$Q_{i00} = 2\partial_i\phi, \quad Q_{i0j} = \partial^2_{ij}U, \quad Q_{ijk} = 2\left(\delta_{jk}\partial_i\psi - \partial^3_{ijk}\sigma\right)$$

nonzero components of the traces of the nonmetricity tensor

$$Q_0 = 2\left(\dot{\phi} - 3\dot{\psi} + \Delta\dot{\sigma}\right), \quad Q_i = 2\partial_i\left(\phi - 3\psi + \Delta\sigma\right), \quad \tilde{Q}_0 = 2\dot{\phi} - \Delta U,$$

$$\tilde{Q}_i = \partial_i\left(\dot{U} - 2\psi + 2\Delta\sigma\right)$$

Scalar perturbations

the components of the equations of motion:

$$\mathfrak{G}^{0}_{\cdot 0} = 2a_5\Delta\psi - \Box\Big[(a_1-a_3)\phi + (a_3-a_5)(3\psi - \Delta\sigma)\Big] -$$

$$-(a_5-a_2+a_4)\left[2\ddot{\phi} - \Delta\dot{U}\right] = 0,$$

$$\mathfrak{G}^{0}_{\cdot i} = \partial_i\left[2a_5\dot{\psi} - \left(\frac{a_1-a_2+a_4}{2}\right)\Box U - (a_5-a_2+a_4)\left(\dot{\phi} - \dot{\psi} + \Delta\dot{\sigma}\right)\right] = 0,$$

$$\mathfrak{G}^{i}_{\cdot j} = -\Big[(3a_3-a_1)\ddot{\psi} + (a_5-a_3)\ddot{\phi} + a_3\Delta\phi - (3a_3-a_1-a_5)\Delta\psi +$$

$$+(a_3-a_5)\Delta^2\sigma - \Delta\left(a_3\ddot{\sigma} + a_5\dot{U}\right)\Big]\delta_{ij} + \partial^2_{ij}\Big[a_5\phi - (3a_5-2a_2+2a_4)\psi -$$

$$-a_1\ddot{\sigma} - (a_2-a_4)\dot{U} - (2a_2-a_1-a_5-2a_4)\Delta\sigma\Big] = 0$$

Scalar sector

In the STEGR limit, the equations take the form:

$$G^0_0 = 2\Delta\psi = 0, \quad G^0_i = 2\partial_i\dot{\psi} = 0,$$

$$G^i_j = -2\left[\ddot{\psi} + \frac{1}{2}\Delta\left(\phi - \ddot{\sigma} - \dot{U} - \psi\right)\right]\delta_{ij} + \partial^2_{ij}\left[\phi - \ddot{\sigma} - \dot{U} - \psi\right] = 0.$$

Where the scalars ϕ, σ and U are included only as a combination of $(\phi - \ddot{\sigma} - \dot{U})$, which means the restoration of calibration freedom, and that only two scalars ψ and a combination of $\phi - \ddot{\sigma} - \dot{U}$ are physical so two scalars, for example σ and U can be arbitrary

Analysis of the nature of degrees of freedom

Here, three combinations of parameters are responsible for the presence or absence of time derivatives:

$$a_1, \quad a_3, \quad a_5, \quad a_1 - a_3,$$

$$a_3 - a_5, \quad a_5 - a_1, \quad a_2 - a_4,$$

$$3a_3 - a_1, \quad a_5 - a_2 + a_4, \quad a_1 - a_2 + a_4$$

If none of the equalities vanishes, then all four scalars: ϕ,ψ,σ,U are dynamic

The quadratic form of velocities

The velocity-quadratic part of the Lagrangian density:

$$\mathfrak{K} = \frac{1}{2}\mathfrak{B}^{0\mu\nu}Q_{0\mu\nu} = \frac{a_1}{4}\left(\dot{h}_{ij}^2 + 2\left(\partial_i \dot{c}_j\right)^2\right) + \left(\frac{a_2 - a_4 - a_1}{2}\right)\left(\dot{V}_i^2 + \left(\partial_i \dot{V}\right)^2\right) +$$

$$+ \left(a_1 - a_3 + 2a_5 - 2b_2\right)\dot{\phi}^2 + (a_1 - a_3)(\Delta\dot{\sigma})^2 + (3a_3 - a_1)\dot{\psi}(2\Delta\dot{\sigma} - 3\dot{\psi}) - 2(a_3 - a_5)\dot{\phi}\,(\Delta\dot{\sigma} - 3\dot{\psi})$$

where the quadratic form of the velocities of scalars

$$\mathrm{QF}(\dot{\phi}, \Delta\dot{\sigma}, \dot{\psi}) =$$

$$= \begin{pmatrix}\dot{\phi} & \Delta\dot{\sigma} & \dot{\psi}\end{pmatrix} \begin{pmatrix} a_1 - a_3 + 2a_5 - 2b_2 & a_5 - a_3 & 3(a_3 - a_5) \\ a_5 - a_3 & a_1 - a_3 & 3a_3 - a_1 \\ 3(a_3 - a_5) & 3a_3 - a_1 & -3(3a_3 - a_1) \end{pmatrix} \begin{pmatrix} \dot{\phi} \\ \Delta\dot{\sigma} \\ \dot{\psi} \end{pmatrix},$$

where $b_2 = a_2 - a_4$.

The quadratic form of velocities

Thus, the theory will not have ghosts degrees of freedom when $a_1 \geq 0$ and $a_2 - a_4 - a_1 \geq 0$ and also provided that the matrix of the quadratic form is positively defined. Which, according to the Sylvester criterion, means

$$2a_1 \left[(a_1 - 2a_3 + a_5 - b_2)^2 - 4\left(a_3 - \frac{a_5}{2} - \frac{b_2}{4} \right)^2 - 3\left(a_5 - \frac{b_2}{2} \right)^2 \right] > 0,$$

$$(a_1 - a_3 + a_5 - b_2)^2 - (a_3 - a_5)^2 - (a_5 - b_2)^2 > 0,$$

$$a_1 - a_3 + 2a_5 - 2b_2 > 0$$

Conclusion

We obtained the equations of linearized perturbations of the theory over a flat Minkowski space

We classified all variables in the tensor, vector and scalar sectors of the tetrad perturbations and

imposed additional restrictions on the parameters of the model under study

Thank you!

Questions?