National Research Nuclear University MEPhI (Moscow Engineering Physics Institute)

ATROPOS

Scientific & Educational Centre NEVOD

On the problem of defining initial conditions for relativistic strings for hadronization modeling

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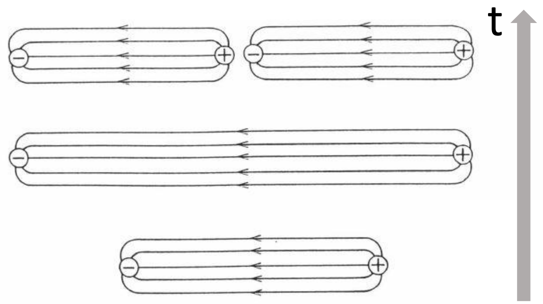
Introduction. String models of hadronization

- **Preconfinement** approach: partons after production stay bound together as colorless systems
- It is believed that QCD field between partons (quarks, gluons + diquarks) compresses into a flux **tube** due to gluons self-interaction
- To simplify the theoretical description we neglect the transverse size of the tube ➡ **strings**
- Strings fragment via **pair production** and light strings are identified as **hadrons**

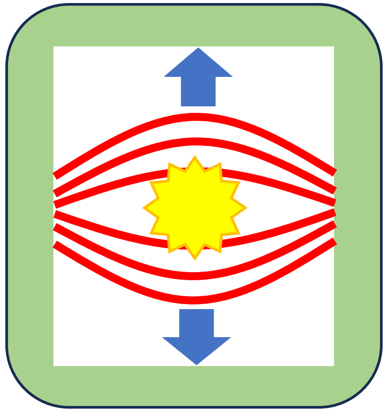
Field lines between a quark and an antiquark squeeze into a tube

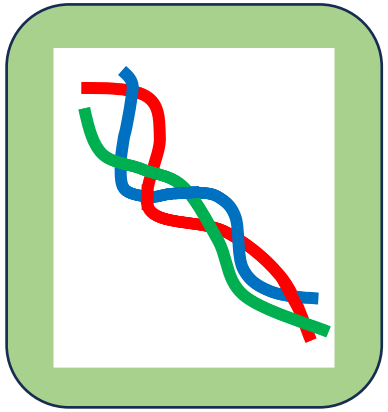


Meson production: q $-$ $\bar{q}$ pair
Baryon production: qq $-$ $\overline{qq}$ pair

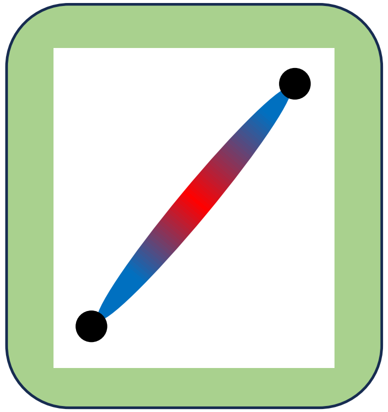
Challenges of modern string models

String interactions: shoving, rope hadronization, etc.

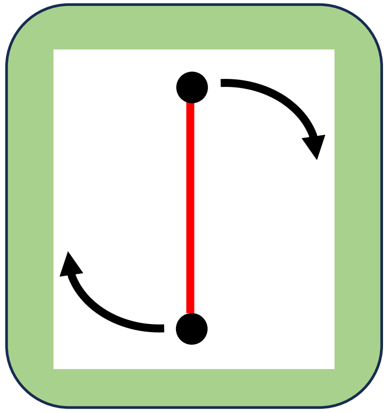




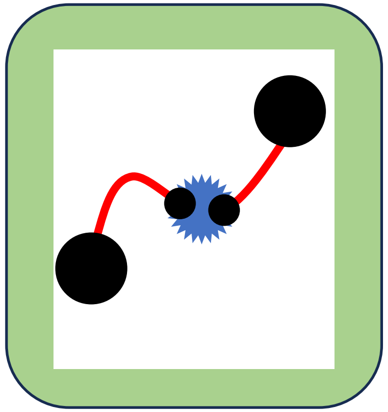
Non-constant string tension



Angular momentum conservation



Heavy quarks fragmentation

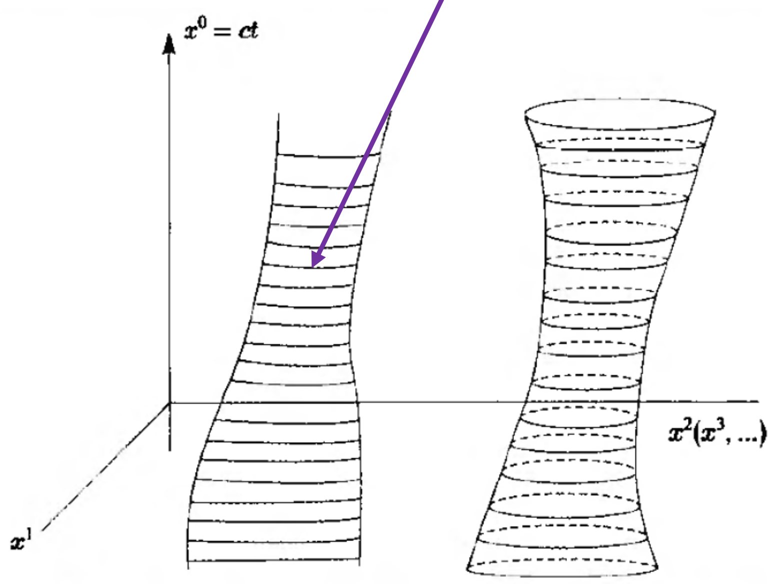


… and many more!

ATROPOS-v1.0 string model of hadronization

- PYTHIA: use LUND fragmentation model
  - Only the simplest case of the string is derived from the initial postulates
  - Fragmentation apparatus (de-facto) not applicable for any non-symmetric picture: non-zero quark masses, initial extension of the string, multi-gluon string, non-zero angular momentum…
  - String dynamics is not calculated microscopically

World sheet of the string
Use its area A as an invariant



ATROPOS:

- Use **detailed** Monte-Carlo simulation of relativistic string fragmentation to produce hadrons
- Derive string equations of motion directly from action
- Govern fragmentation process by Artru-Mennessier Area Decay Law:

$$\frac{dP_{\text{break}}}{dA} = P_0 = \text{const}$$

- Define the string **properly:** conserve energy, momentum and angular momentum + more …
- Use the exact analytical solutions to calculate invariant area and string characteristics at the break point
- Presented at XXXVI International Workshop HEP&FT and at ICPPA-2024

Relativistic string with free ends

$x_\mu(\tau, \sigma)$ is a 2-parameter definition of the string world sheet in time and space

$$\dot{x}_\mu \equiv \frac{\partial x_\mu(\tau,\sigma)}{\partial \tau}$$
$$x'_\mu \equiv \frac{\partial x_\mu(\tau,\sigma)}{\partial \sigma}$$

If we consider the usual string action

$$S_{\text{string}} = -\kappa \int_{\sigma_1}^{\sigma_2} d\sigma \int_{\tau_1(\sigma)}^{\tau_2(\sigma)} d\tau \sqrt{(x'\dot{x})^2 - x'^2\dot{x}^2}$$

the equations of motion will take the form

$$\frac{\partial}{\partial \tau}\left(\frac{(\dot{x}x')x'_\mu - {x'}^2\dot{x}_\mu}{\sqrt{(\dot{x}x')^2 - \dot{x}^2{x'}^2}}\right) + \frac{\partial}{\partial \sigma}\left(\frac{(\dot{x}x')\dot{x}_\mu - \dot{x}^2 x'_\mu}{\sqrt{(\dot{x}x')^2 - \dot{x}^2{x'}^2}}\right) = 0.$$

The Nambu-Goto action is reparameterization-invariant, so a specific relation between τ and σ can be chosen:

$$\boldsymbol{\dot{x}^2 + {x'}^2 = 0,\ \dot{x}x' = 0.}$$

This is called orthonormal gauge. Only with that we get the simple wave equation to describe string movement:

$$\ddot{x}_\mu - x''_\mu = 0.$$

Define the string: Conservation laws

- Conserve energy and momentum:

$$\kappa \int_0^\pi d\sigma\, v_\mu(\sigma) = \kappa \int_0^\pi d\sigma\, \dot{x}_\mu(0,\sigma) = P_\mu$$

- **Conserve angular momentum**:

$$\kappa \int_0^\pi d\sigma \left[\rho_\mu(\sigma) v_\nu(\sigma) - \rho_\nu(\sigma) v_\mu(\sigma)\right] = \kappa \int_0^\pi d\sigma \left[x_\mu(0,\sigma)\dot{x}_\nu(0,\sigma) - x_\nu(0,\sigma)\dot{x}_\mu(0,\sigma)\right] = M_{\mu\nu}$$

- Imposes $4 + 6 = 10$ independent conditions
  - ➢ At least 10 parameters should be used to define a string between two partons
- Note: conservation of angular momentum **requires** non-zero initial extension of the string!
- Can potentially influence particle production, as presents additional restrictions on string-hadron transitions
- But there is more…

Virasoro conditions for the initial data

Lets substitute the solution to the free string Cauchy problem into the orthonormal gauge:

$$x_\mu(\tau,\sigma) = Q_\mu + P_\mu \frac{\tau}{\pi\kappa} + \frac{i}{\sqrt{\pi\kappa}} \sum_{\substack{n=-\infty \\ n\neq 0}}^{+\infty} e^{-in\tau} \frac{\alpha_{n\mu}}{n} \cos(n\sigma)$$

$$\begin{cases} \dot{x}^2 + x'^2 = 0 \\ \dot{x}x' = 0. \end{cases}$$

- Here Q_μ are the coordinates of the string center-of-mass in the initial moment of time, P_μ is a 4-vector of total string momentum

What we get are the Virasoro conditions:

$$\sum_{m=-\infty}^{+\infty} \boldsymbol{\alpha_{n-m}\,\alpha_m = 0}, \qquad \boldsymbol{n = 0, \pm 1, \pm 2, \ldots}$$

Here $\alpha_{n\mu}$ are the Fourier amplitudes:

$$\alpha_{n\mu} = \sqrt{\frac{\kappa}{\pi}} \int_0^\pi d\sigma \cos(n\sigma)\left(v_\mu(\sigma) - in\rho_\mu(\sigma)\right), \qquad n \neq 0,$$

$$\alpha_{0\mu} = \frac{P_\mu}{\sqrt{\kappa\pi}}.$$

$$\rho_\mu(\sigma) \equiv x_\mu(0,\sigma), \quad v_\mu(\sigma) \equiv \dot{x}_\mu(0,\sigma)$$

Functions $\boldsymbol{v_\mu(\sigma)}$ and $\boldsymbol{\rho_\mu(\sigma)}$ define the velocity and form of the string in the initial moment of time. So, Virasoro conditions restrict the way this functions may be defined.

Restrictions imposed by the Virasoro conditions (1)

Express the Fourier amplitudes as follows

$$\alpha_{n\mu} = 2\sqrt{\frac{\kappa}{\pi}}\int_0^\pi d\sigma \cos(n\sigma)\left(v_\mu(\sigma) - in\rho_\mu(\sigma)\right) = f_{n\mu} - ing_{n\mu}$$

$$f_{n\mu} \equiv 2\sqrt{\frac{\kappa}{\pi}}\int_0^\pi d\sigma \cos(n\sigma)\, v_\mu(\sigma), \qquad g_{n\mu} \equiv 2\sqrt{\frac{\kappa}{\pi}}\int_0^\pi d\sigma \cos(n\sigma)\, \rho_\mu(\sigma)$$

The Virasoro conditions take the form

$$\sum_{m=-\infty}^{+\infty} \alpha_{n-m}\alpha_m = 0, n = 0, \pm 1, \pm 2, \dots$$

Both real and imaginary parts must be zero:

$$\Rightarrow \begin{cases} \displaystyle\sum_{m=-\infty}^{+\infty} \left(f_{n-m}f_m - m(n-m)g_{n-m}g_m\right) = 0 \\ \displaystyle\sum_{m=-\infty}^{+\infty} \left(mf_{n-m}g_m + (n-m)f_m g_{n-m}\right) = 0 \end{cases}$$

Restrictions imposed by the Virasoro conditions (2)

$$\begin{cases} \sum_{m=-\infty}^{+\infty} (f_{n-m} f_m - m(n-m) g_{n-m} g_m) = 0 \\ \sum_{m=-\infty}^{+\infty} (m f_{n-m} g_m + (n-m) f_m g_{n-m}) = 0 \end{cases}$$

Three ways to satisfy the Virasoro conditions:

1. Define $v_\mu(\sigma)$, $\rho_\mu(\sigma)$ in the way that

$$f_n f_m = g_n g_m = f_n g_m = 0$$

- ➢ Imposes very strict restrictions on the strings parameters
  Example: the ansatz used in Caltech-II, NEXUS and EPOS models:

$$\rho_\mu \equiv 0, v_\mu = const = E/p$$

Oversimplified definition of the string: no angular momentum, point-like form…

leads to the requirement $\boldsymbol{M^2 = 0}$!

But models still use massive strings, as it is required for particle production

- ➢ A contradiction (+ non-physical configurations?)

2. Every term like $f_{n-m} f_m - m(n-m) g_{n-m} g_m = 0$
   - ➢ Needs a very special form for functions $v_\mu(\sigma)$, $\rho_\mu(\sigma)$:

$$g_{n\mu} \equiv \frac{1}{n} g_\mu \quad \text{or} \quad f_{n\mu} \equiv n f_\mu$$

   - ➢ Maybe solve integral equations to find $v_\mu(\sigma)$, $\rho_\mu(\sigma)$?

The FOEE-method to define the initial conditions of the string

3. Define $v_\mu(\sigma)$, $\rho_\mu(\sigma)$ so that only the value of entire sum is zero
   - The least strict conditions for the string
   - Starting at some order N the functions should yield $f_{n\mu} = 0$, $g_{n\mu} = 0$, $|n| > N$, so that the system produced is of finite size

$$\begin{cases} \sum_{m=-\infty}^{+\infty} (f_{n-m} f_m - m(n-m) g_{n-m} g_m) = 0 \\ \sum_{m=-\infty}^{+\infty} (m f_{n-m} g_m + (n-m) f_m g_{n-m}) = 0 \end{cases}$$

- How to define the initial data functions in such a peculiar way?

Final-Order Eigenfunction Expansion (FOEE) Method:

$$v_\mu(\sigma) = a_{0\mu} + \sum_{k\neq 0} a_{k\mu} \cos(k\sigma)$$

$$\rho_\mu(\sigma) = b_{0\mu} + \sum_{k\neq 0} b_{k\mu} \cos(k\sigma)$$

(In general case)

$$v_\mu(\sigma) = a_{0\mu} u_0(\sigma) + \sum_{k\neq 0} a_{k\mu} u_k(\sigma)$$

$$\rho_\mu(\sigma) = b_{0\mu} u_0(\sigma) + \sum_{k\neq 0} b_{k\mu} u_k(\sigma)$$

$u_k(\sigma)$ are the eigenfunctions of Sturm-Liouville problem for the string

The FOEE-method

- Use eigenfunctions $u_k(\sigma)$ to construct the sum
  - Eigenfunctions are orthogonal to each other:

$$\int_0^{\pi} d\sigma\, u_n(\sigma) u_m(\sigma) = \|u_n\|^2 \delta_{nm}, \qquad n, m = 0, \pm 1, \pm 2, \dots$$

so if such expansion is **finite**,

$$v_\mu(\sigma) = a_{0\mu} u_0(\sigma) + \sum_{\substack{k=-N, \\ k\neq 0}}^{N} a_{k\mu} u_k(\sigma)\,, \qquad \rho_\mu(\sigma) = b_{0\mu} u_0(\sigma) + \sum_{\substack{k=-N, \\ k\neq 0}}^{N} b_{k\mu} u_k(\sigma)$$

Fourier-amplitudes will be zero for $|n| > N$: $\alpha_{n\mu} = 0$

- Finite size of the system achieved!
- How many orders of expansion are enough?

Constructing the FOEE-system

- After substituting the FOEE-defined functions into the Virasoro conditions we obtain

$$\begin{cases} \sum\limits_{\substack{m=-N \\ m\neq 0, m\neq n}}^{N} (a_{n-m}a_m - m(n-m)b_{n-m}b_m) + \frac{2}{\kappa\pi} P a_n = 0 \\ \sum\limits_{\substack{m=-N \\ m\neq 0, m\neq n}}^{N} (m a_{n-m} b_m + (n-m) a_m b_{n-m}) - \frac{2n}{\kappa\pi} P b_n = 0, \end{cases} \quad n \neq 0 \qquad \begin{cases} \sum\limits_{\substack{m=-N \\ m\neq 0}}^{N} (a_{-m}a_m + m^2 b_{-m} b_m) = -\frac{P^2}{(\kappa\pi)^2} \\ \sum\limits_{\substack{m=-N \\ m\neq 0}}^{N} m(a_{-m}b_m - a_m b_{-m}) = 0, \end{cases} \quad n = 0$$

- $2(4N+1)$ equations, $8(2N+1)$ variables
  - ➢ System can be resolved at any order, can use additional assumptions to simplify calculations
- Let the $a_{-n\mu} = a_{n\mu}$, $b_{-n\mu} = b_{n\mu}$ for any n
  - ➢ Kills $4N+1$ equations, removes $8N$ variables from the system
- In total, $4N+11$ equations (with conservation laws) and $8(N+1)$ variables
  - ➢ Should be solvable with $N \geq 1$

The FOEE system: 1 order

Take the initial conditions in the form:

$$v_\mu(\sigma) = a_\mu + b_\mu \cos(\sigma)$$
$$\rho_\mu(\sigma) = c_\mu + d_\mu \cos(\sigma)$$

$$\Rightarrow \begin{cases} \alpha_{0\mu} = \dfrac{P_\mu}{\sqrt{\kappa\pi}} \\ \alpha_{1\mu} = \sqrt{\kappa\pi}\left(b_\mu - i d_\mu\right) \\ \alpha_{-1\mu} = \sqrt{\kappa\pi}\left(b_\mu + i d_\mu\right) \end{cases}$$

Expressions for Fourier amplitudes

Virasoro conditions system:

$$\begin{cases} b^2 - d^2 = 0 \\ bd = 0 \\ bP = 0 \\ dP = 0 \\ b^2 + \dfrac{P^2}{4(\kappa\pi)^2} = 0 \end{cases}$$

4-momentum conservation: $a_\mu = \dfrac{P_\mu}{\kappa\pi}$

And angular momentum: $c_\mu P_\nu - c_\nu P_\mu + \dfrac{\kappa\pi}{2}\left(d_\mu b_\nu - d_\nu b_\mu\right) = M_{\mu\nu}$

15 equations, 16 variables

➢ Suppose $c_0 = 0$

- **Solution: work in progress…**
- **May not have enough free parameters for generalization over string with masses at its ends**
  - ➢ **A higher order of the system is worth considering**

The FOEE system: 2 order

- Define initial data in the form $$v_\mu(\sigma) = a_\mu + b_\mu \cos(\sigma) + c_\mu \cos(2\sigma)\,,$$ $$\rho_\mu(\sigma) = d_\mu + e_\mu \cos(\sigma) + f_\mu \cos(2\sigma)\,.$$

This yields the system of equations:

Virasoro conditions

Angular momentum conservation

$$\begin{cases} b^2 + c^2 + e^2 + 4f^2 + \dfrac{P^2}{2(\kappa\pi)^2} = 0 \\ bc + 2ef + \dfrac{bP}{\kappa\pi} = 0 \\ 2bf - ce + \dfrac{eP}{\kappa\pi} = 0 \\ b^2 - e^2 + \dfrac{2cP}{\kappa\pi} = 0 \\ be + \dfrac{2fP}{\kappa\pi} = 0 \\ bc - 2ef = 0 \\ ce + 2bf = 0 \\ c^2 - 4f^2 = 0 \\ cf = 0 \\ d_\mu P_\nu - d_\nu P_\mu + \dfrac{\kappa\pi}{2}\left(e_\mu b_\nu - e_\nu b_\mu + f_\mu c_\nu - f_\nu c_\mu\right) = M_{\mu\nu}. \end{cases}$$

20 variables, 15 equations
- ➢ Additional assumptions to be made
- ➢ Suppose $d_0 = b_3 = c_3 = e_3 = f_3 = 0$

No analytical solution is known:

- **Use numerical methods instead**
- **However, the complexity of the system does not allow usual methods to succeed**
- ➢ **Need for better method (or more brute force computation power)**

The FOEE system: 2 order – first solution candidates

String with $P_\mu = \{P_0, P_x, P_y, P_z\} = \{10, 1, 2, 3\}$ GeV, no rotation

Zero, as the string has no rotation

"Solution" 1 (accuracy 10^{-5}):

$$a_\mu = \begin{pmatrix} 15.92 \\ 1.59 \\ 3.18 \\ 4.77 \end{pmatrix}, \quad b_\mu = \begin{pmatrix} -0.63 \\ -9.48 \\ 1.6 \\ 0 \end{pmatrix}, \quad c_\mu = \begin{pmatrix} 1.95 \\ -0.2 \\ -1.94 \\ 0 \end{pmatrix}, \quad d_\mu = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad e_\mu = \begin{pmatrix} 0.27 \\ 4.06 \\ -0.69 \\ 0 \end{pmatrix}, \quad f_\mu = \begin{pmatrix} -1.02 \\ 0.1 \\ 1.02 \\ 0 \end{pmatrix}$$

"Solution" 2 (accuracy 10^{-7}):

$$a_\mu = \begin{pmatrix} 15.92 \\ 1.59 \\ 3.18 \\ 4.77 \end{pmatrix}, \quad b_\mu = \begin{pmatrix} -0.07 \\ -3.08 \\ 1.91 \\ 0 \end{pmatrix}, \quad c_\mu = \begin{pmatrix} -3.34 \\ -1.7 \\ -2.88 \\ 0 \end{pmatrix}, \quad d_\mu = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad e_\mu = \begin{pmatrix} -0.2 \\ 8.32 \\ -5.16 \\ 0 \end{pmatrix}, \quad f_\mu = \begin{pmatrix} -1.43 \\ -0.73 \\ -1.23 \\ 0 \end{pmatrix}$$

"Solution" 3 (accuracy 10^{-12}):

$$a_\mu = \begin{pmatrix} 15.92 \\ 1.59 \\ 3.18 \\ 4.77 \end{pmatrix}, \quad b_\mu = \begin{pmatrix} 1.34 \\ 6.04 \\ 3.67 \\ 0 \end{pmatrix}, \quad c_\mu = \begin{pmatrix} -0.47 \\ 0.16 \\ -0.44 \\ 0 \end{pmatrix}, \quad d_\mu = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad e_\mu = \begin{pmatrix} 1.5 \\ 6.78 \\ 4.13 \\ 0 \end{pmatrix}, \quad f_\mu = \begin{pmatrix} 2.01 \\ -0.7 \\ 1.88 \\ 0 \end{pmatrix}$$

Note:
Even the simplest case of the massive relativistic string requires non-zero spatial extension coefficients!

No solutions for rotating strings found yet… urgent need for search algorithms optimization!

Summary

- For proper string fragmentation treatment in hadronization models, a series of conditions must be satisfied when defining the string in the initial moment of time
- Conservation laws alone impose 10 equations on the parameters of the initial data functions
- A crucial part is to take into account the Virasoro conditions
  - ➢ Non-trivial restrictions on initial data functions ➡ **FOEE method to define them!**
  - ➢ Massive strings can not be defined at ground state – higher oscillation modes required!
  - ➢ Virasoro + momentum conservation + angular momentum conservation seems to force the string to be defined as initially-stretched object…
- A challenge to find the exact solutions to the FOEE-system
  - ➢ But yet it is possible!

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Thank you for your attention!

Relativistic string with masses at its ends

Regular Nambu-Goto action

$$S_{\text{string}} = -\kappa \int_{\tau_1}^{\tau_2} d\tau \int_{\sigma_1(\tau)}^{\sigma_2(\tau)} d\sigma \sqrt{(\dot{x}x')^2 - \dot{x}^2 x'^2} - \sum_{i=1}^{2} m_i \int_{\tau_1}^{\tau_2} d\tau \sqrt{\left(\frac{dx_\mu(\tau,\sigma_i(\tau))}{d\tau}\right)^2}$$

Term to describe heavy quarks at string ends

From the action follow the equations of motion

$$\ddot{x}^\mu - x''^\mu = 0$$

and boundary conditions

$$\frac{m_1}{\kappa}\frac{d}{d\tau}\left(\frac{\dot{x}_\mu}{\sqrt{\dot{x}^2}}\right) = x'_\mu, \qquad \sigma = 0,$$
$$\frac{m_2}{\kappa}\frac{d}{d\tau}\left(\frac{\dot{x}_\mu}{\sqrt{\dot{x}^2}}\right) = -x'_\mu, \qquad \sigma = \pi.$$

linearization →

$$\ddot{x}_\mu(\tau,0) = q_1 x'_\mu(\tau,0),$$
$$\ddot{x}_\mu(\tau,\pi) = -q_2 x'_\mu(\tau,\pi),$$
$$q_1 = \frac{\kappa}{m_1^2}, \qquad q_2 = \frac{\kappa}{m_2^2}$$

Restriction on string movement:

$$\dot{x}^2(\tau,0) = m_1^{-2}, \qquad \dot{x}^2(\tau,\pi) = m_2^{-2}$$

Solution:

$$x^\mu(\tau,\sigma) = C_0^\mu \tau + D_0^\mu + \sum_{n=1}^{+\infty}\left[C_n^\mu \sin(\omega_n\tau) + D_n^\mu \cos(\omega_n\tau)\right]u_n(\sigma)$$