Particular notes in Black Hole shadow's modelling when spinning is taken into account
Особенности моделирования теней черных дыр при учете вращения

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Обсуждаются особенности улучшенного алгоритма Ньюмена-Яниса и его применение для получения метрик черных дыр в расширенных теориях гравитации. Рассмотрены теории расширяют ОТО различными способами: петлевая квантовая гравитация и $f(Q)$ гравитация (симметричная телепараллельная гравитация STEGR).

We discuss specific aspects of the improved Newman-Janis algorithm and its application to generate rotating black hole metrics in the extended theories of gravity. We consider here different theories extending general relativity in various ways: loop quantum gravity and $f(Q)$ gravity (symmetric teleparallel gravity STEGR).

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Introduction

Since the pioneering works on black hole shadows [1] it has been realized that accounting of the tidal charge in the Reissner-Nordström metric may allow to measure the contribution of new physics extending the general relativity (GR) [2]. To proceed, one must apply the new observational black hole images. So, comparing the results of shadow modeling in various gravity models one could find the model describing the shadow, accretion disk,

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... better than the others. At the first stage only the spherically-symmetric solutions were used. However, both black holes photographed by the Event Horizon Telescope (EHT) (M87 and Sgr A) rotate [3, 4] (the angular velocity was measured quite recently [5]). Therefore, to increase the accuracy it is highly desirable to take into account the effects of rotation. That is why one has to make a transition from Reissner-Nordstrom-type metrics to Kerr-Newman-type ones. In general the direct solving Einstein's equations for axially symmetric spacetime leads to complex implicit partial differential equations. Thus, it is preferable to find alternative methods. For example one can generate new solutions by introducing rotation in the parametric space. Therefore, making such a turn from the case $a = 0$, where a is the rotational parameter of a black hole, to $a \neq 0$, the Newman-Janis algorithm [6] allows for obtaining rotating solutions in a simpler way. The algorithm was later refined to facilitate this transformation in a simpler way [7]. This is important for studying BH shadows in extended theories of gravity (for example, Horndeski, bumblebee) [8], where rotation significantly influences the shape of the shadow.

Rotating solutions

Previously, the Newman-Janis algorithm [6] in its improved version [7] has been used in [9, 10]. Thus, the process begins by considering the spherically symmetric metric in the form:

$$ds^2 = -G(r)dt^2 + \frac{1}{F(r)}dr^2 + H(r)d\Omega^2, \tag{1}$$

where $G(r)$, $F(r)$ and $H(r)$ are the metric functions. Such a representation makes possible the application of the wide class of spherically symmetric space-times. The components of the axially symmetric metric obtained from equation (1) in [7] are:

$$\begin{aligned}
g_{tt} &= -\frac{FH + a^2\cos^2\theta}{(K + a^2\cos^2\theta)^2}\Psi, \qquad g_{t\phi} = -a\sin^2\theta\frac{K - FH}{(K + a^2\cos^2\theta)^2}\Psi, \\
g_{\theta\theta} &= \Psi, \qquad g_{rr} = \frac{\Psi}{FH + a^2}, \\
g_{\phi\phi} &= \Psi\sin^2\theta(1 + a^2\sin^2\theta\frac{2K - FH + a^2\cos^2\theta}{(K + a^2\cos^2\theta)^2}),
\end{aligned} \tag{2}$$

where $K = H(r)\sqrt{F(r)/G(r)}$.

In the next step, one introduce the function $\Psi(r, y^2, a)(y \equiv \cos\theta)$ satisfying the conditions:

$$\begin{aligned}
&\lim_{a\to 0}\Psi(r, y^2, a) = H(r), \\
&(K + a^2y^2)^2(3\Psi_r\Psi_{y^2} - 2\Psi\Psi_{r,y^2}) = 3a^2K_r\Psi^2, \\
&\Psi\left[K_r^2 + K(2 - K_{rr}) - a^2y^2(2 + K_{rr})\right] + (K + a^2y^2)[(4y^2\Psi_{y^2} - K_r\Psi_r] = 0.
\end{aligned} \tag{3}$$

The first line in eq. (3) describes the behavior at $a \to 0$, being the reverse transition to the non-rotating metric with two representations (Ψ_n and Ψ_c) connected by the conformal transformation. So the initial metric is: $ds_c^2 = \Psi_c/\Psi_n ds_n^2$ and thus, one finds the solution of eq. (3) in the form:

$$\Psi_c = H(r)\exp\left[a^2 f(r, a^2y^2, a)\right] \approx H(r) + a^2 X(y^2, r) + o(a^2), \qquad (4)$$
$$X(y^2, r) = \frac{H^2(8K - K_r^2)y^2}{K^2(8H - H_r K_r)}.$$

f(Q) gravity

We consider the Symmetric Teleparallel Equivalent of General Relativity (STEGR) with a non-zero non-metricity scalar Q [11]. The metric extending GR (I^+) has the form [11, 12]:

$$G(r) = 1 - \frac{2M_{ren}}{r} + \alpha\frac{32}{r^2}, \qquad F(r) = 1 - \frac{2M_{ren}}{r} + \alpha\frac{96}{r^2}, \qquad H(r) = r^2, \ (5)$$

where α is the expansion parameter, c_1 is the integration constant, M_{ren} is the re-normalized mass. Note that a far observer cannot detect the difference between re-normalized and ordinary Schwarzschild masses. So we use $M_{ren} = M$ everywhere, normalizing all values to it. In [13] the restrictions on α ($-0.008 < \alpha < 0.005$) were obtained at $M = 1$. Therefore, we assume that $|\alpha| \ll M$. Also note that in our case $r > 2M$ (event horizon in GR). We begin the calculations with $\sqrt{F/G}$:

$$\sqrt{\frac{F}{G}} = \frac{1 - \frac{2M_{ren}}{r} + \alpha\frac{32}{r^2} + \alpha\frac{64}{r^2}}{1 - \frac{2M_{ren}}{r} + \alpha\frac{32}{r^2}} = 1 + \frac{64\alpha}{r^2 - 2Mr + 32\alpha} \approx 1. \qquad (6)$$

So $K = r^2$ and $K_r = 2r$. The metric in such approximation is symmetric if $F = G$ with the last term $96\alpha/r^2$. In this case $X(y^2, r) = y^2 = \cos^2\theta$ and $\Psi = r^2 + a^2\cos^2\theta$. As a result, we denote $A^2 = -2Mr + 96\alpha$, $\rho^2 = r^2 + a^2\cos^2\theta$ obtaining:

$$g_{tt} = -\frac{\rho^2 + A^2}{\rho^2}, \qquad g_{t\phi} = a\sin^2\theta\frac{A^2}{\rho^2}, \qquad g_{\theta\theta} = \rho^2,$$
$$g_{rr} = \frac{\rho^2}{r^2 + a^2 + A^2}, \qquad g_{\phi\phi} = \sin^2\theta\left(\rho^2 + a^2\sin^2\theta\frac{\rho^2 - A^2}{\rho^2}\right). \qquad (7)$$

Loop Quantum Gravity

We consider the modified Hayward metric without central singularity [12, 14] where:

$$G(r) = \left(1 - \frac{2Mr^2}{r^3 + 2Ml^2}\right)\left(1 - \frac{M\alpha\beta}{\alpha r^3 + \beta M}\right),$$
$$F(r) = 1 - \frac{2Mr^2}{r^3 + 2Ml^2}, \qquad H(r) = r^2, \qquad (8)$$

where the parameter l measures the central energy density with $3/8\pi^2$, α is the time delay between the center and infinity, and β is associated with single-loop quantum corrections to the Newtonian potential. In [14] the parameters were constrained as follows: $0 \leq \alpha < 1$, $\beta_{max} = 41/(10\pi) \approx 1.305$. For $l > \sqrt{16/27}M \approx 0.7698M$ the object has no horizon. In [12] it was shown that these values fall under restrictions from EHT data. Therefore, the additional parameters in this model cannot be small relative to M. Next:

$$\sqrt{\frac{F}{G}} = \left(1 - \frac{M\alpha\beta}{\alpha r^3 + \beta M}\right)^{-1/2} = A \neq 1. \tag{9}$$

From this expression one concludes that it cannot be equal to 1. Therefore:

$$K = r^2 A \neq r^2, \qquad K_r = A\left(2r - \frac{3r^4 M\alpha^2\beta}{2(\alpha^3 + \beta M}\right). \tag{10}$$

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Further X ($\rho^2 = r^2 + a^2\cos^2\theta$): $X = y^2/A = \cos^2\theta/A$. $\Psi = r^2 + a^2\cos^2\theta/A$. Let $B = r^2A + a^2\cos^2\theta$ and $C = 2Mr^4(r^3 + 2Ml^2)^{-1}$ so the new components are:

$$\begin{aligned} g_{tt} &= -\frac{\rho^2 - C}{AB}, \qquad g_{t\phi} = a\sin^2\theta\frac{r^2(A-1)+C}{AB}, \\ g_{\theta\theta} &= \frac{B}{A}, \qquad g_{rr} = \frac{B}{A(r^2 + a^2 - C)}, \\ g_{\phi\phi} &= \frac{B}{A}\sin^2\theta\left(1 + a^2\sin^2\theta\frac{r^2(2A-1) + C + a^2\cos^2\theta}{B^2}\right). \end{aligned} \tag{11}$$

In opposite when $a = 0$, the following static metric is realized:

$$g_{tt} = -G, \qquad g_{t\phi} = 0, \qquad g_{\theta\theta} = r^2, \qquad g_{rr} = \frac{1}{F}, \qquad g_{\phi\phi} = r^2\sin^2\theta. \tag{12}$$

Therefore, here even at $\Psi \neq r^2 + a^2\cos^2\theta$ the exact transition leads to the original static metric at $a = 0$.

Conclusion

We discuss the specifics of the improved Newman-Janis algorithm and show how to apply it to obtain rotating solutions for $f(Q)$ gravity and loop quantum gravity. For $f(Q)$ gravity the rotating metric transits to the symmetric one at $a = 0$ (although initially the metric was not such). In contrast, for loop quantum gravity the rotating metric at $a = 0$ turns into the original asymmetric one.

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