Rehearsal
Lectures 1-6:
(Betatron Motion)

Valeri Lebedev
JINR & Fermilab

US Particle Accelerator School,
February 2022

Lectures in JINR, 2023-2024

U.S. DEPARTMENT OF ENERGY

Uncoupled Motion

- Linearized equation of motion

$$x'' + \left(K_x^{\;2} + k\right)x = 0 \qquad \text{where: } K_x(s) \equiv K_x = eB_y(s)/Pc\,, \quad k(s) \equiv k = eG(s)/Pc$$

- Solution in matrix form

$$\begin{bmatrix} x(s) \\ \theta(s) \end{bmatrix} = \begin{bmatrix} M_{11}(s) & M_{12}(s) \\ M_{21}(s) & M_{22}(s) \end{bmatrix} \begin{bmatrix} x(0) \\ \theta(0) \end{bmatrix} \quad \text{or} \quad \boxed{\mathbf{x}(s) = \mathbf{M}(s)\mathbf{x}(0)}$$

- Conservation of the Phase Space Volume – Liouville theorem
  - ⇨ The phase space volume is conserved in the course of motion => $|\mathbf{M}| = 1$
  - ⇨ The conservation of the phase space volume is also justified for multidimensional motion. **It is called Liouville theorem**
- Eigen vectors: $\mathbf{M}\mathbf{v}_k = \lambda_k \mathbf{v}_k\,, \quad k = 1,2$
- For arbitrary stable turn-by-turn motion: $\mathbf{v}_2 = \mathbf{v}_1^{\,*} \Rightarrow \mathbf{v}_1 = \mathbf{v}, \quad \mathbf{v}_2 = \mathbf{v}^*$
- Then, at a given place, the position at n-th turn:

$$\mathbf{x}_n = \mathrm{Re}\left(\lambda^n\left(A_1\mathbf{v}\right) + \lambda^{*n}\left(A_2\mathbf{v}^*\right)\right) \equiv \mathrm{Re}\left(\lambda^n\left(C\,\mathbf{v}\right)\right), \quad C = A_1 + A_2^{\,*}$$

  - ♦ The betatron tune: $\lambda = e^{-i\mu} = e^{-2\pi i Q}$

X-Y Coupled Motion

- Linearized equations of motion

$$\begin{cases} x'' + \left(K_x^{\ 2} + k\right)x + \left(N - \frac{1}{2}R'\right)y - Ry' = 0 \\ y'' + \left(K_y^{\ 2} - k\right)y + \left(N + \frac{1}{2}R'\right)x + Rx' = 0 \end{cases}$$

where: $K_{x,y}(s) \equiv K_{x,y} = eB_{y,x}(s)/Pc$, $k(s) \equiv k = eG(s)/Pc$, $N = eG_s/Pc$, $R = eB_s/Pc$

- Canonical variables

canonical momenta are:
$$\begin{cases} p_x = x' - \frac{R}{2}y \\ p_y = y' + \frac{R}{2}x \end{cases}$$

In matrix form: $\hat{\mathbf{x}} = \mathbf{R}\mathbf{x}$

$$\hat{\mathbf{x}} \equiv \begin{bmatrix} x \\ p_x \\ y \\ p_y \end{bmatrix} \quad , \quad \mathbf{x} \equiv \begin{bmatrix} x \\ \theta_x \\ y \\ \theta_y \end{bmatrix} \quad , \quad \mathbf{R} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -R/2 & 0 \\ 0 & 0 & 1 & 0 \\ R/2 & 0 & 0 & 1 \end{bmatrix}$$

Matrix Form of Equations for X-Y Coupled Motion

$$H = \frac{1}{2}\hat{\mathbf{x}}^T \mathbf{H}\hat{\mathbf{x}}$$ where

$$\mathbf{H} = \begin{bmatrix} K_x^2 + k + \frac{R^2}{4} & 0 & N & -R/2 \\ 0 & 1 & R/2 & 0 \\ N & R/2 & K_y^2 - k + \frac{R^2}{4} & 0 \\ -R/2 & 0 & 0 & 1 \end{bmatrix}$$

Then the motion equations are

$$\frac{d\hat{\mathbf{x}}}{ds} = \mathbf{U}\mathbf{H}\hat{\mathbf{x}}$$

$$\mathbf{U} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

- Similar to 1D-motion we introduce 4-dimensional transfer matrix for the 2-dimensional motion:

$$\hat{\mathbf{x}} = \hat{\mathbf{M}}(0,s)\hat{\mathbf{x}}_0 ,$$

The cap denotes that the canonical variables (momenta) are used

- Motion Symplecticity

$$\hat{\mathbf{M}}^T \mathbf{U}\hat{\mathbf{M}} = \mathbf{U}$$ or alternative form $$\hat{\mathbf{M}}\mathbf{U}\hat{\mathbf{M}}^T = \mathbf{U}$$

- Thus, out of 16 matrix elements of matrix **M** the motion symplecticity leaves only 10 elements being linearly independent

Symplecticity of Eigen-Vectors

$$\hat{\mathbf{M}}\hat{\mathbf{v}}_k = \lambda_k \hat{\mathbf{v}}_k \ , \quad k = 1,..,4$$

- For stable motion the eigen-vectors always appear in two reciprocal pairs, and, consequently, the four eigen-values split into two complex conjugate pairs: λ_1, λ_1^* and λ_2, λ_2^* (since real **M**)
  For $\lambda_1 \neq \lambda_2$ (non-degenerate case) we obtain the orthogonality condition

$$\hat{\mathbf{v}}_1^{\ +}\mathbf{U}\hat{\mathbf{v}}_1 \neq 0 \ ,$$
$$\hat{\mathbf{v}}_2^{\ +}\mathbf{U}\hat{\mathbf{v}}_2 \neq 0 \ ,$$
$$\hat{\mathbf{v}}_i^{\ +}\mathbf{U}\hat{\mathbf{v}}_j = 0 \qquad \text{if } i \neq j,$$
$$\hat{\mathbf{v}}_i^{\ T}\mathbf{U}\hat{\mathbf{v}}_j = 0 \ ,$$

Normalizing eigen-vectors we obtain:

$$\hat{\mathbf{v}}_1^{\ +}\mathbf{U}\hat{\mathbf{v}}_1 = -2i \quad , \quad \hat{\mathbf{v}}_2^{\ +}\mathbf{U}\hat{\mathbf{v}}_2 = -2i \quad ,$$
$$\hat{\mathbf{v}}_1^{\ T}\mathbf{U}\hat{\mathbf{v}}_1 = 0 \quad , \quad \hat{\mathbf{v}}_2^{\ T}\mathbf{U}\hat{\mathbf{v}}_2 = 0 \quad ,$$
$$\hat{\mathbf{v}}_2^{\ T}\mathbf{U}\hat{\mathbf{v}}_1 = 0 \quad , \quad \hat{\mathbf{v}}_2^{\ +}\mathbf{U}\hat{\mathbf{v}}_1 = 0 \quad .$$

  Out of 2 complex conjugated vectors we choose one which satisfies the normalization condition. Normalization of CC vector has different sign.
  - Stable betatron motion requires $\left|\lambda_k\right| = 1$ => $\lambda_3 = \lambda_1^*, \quad \lambda_4 = \lambda_2^*$
  - Then the motion is described

$$\mathbf{x}_n = \mathrm{Re}\left(\lambda_1^{\ n}\left(A_1\mathbf{v}_1\right) + \lambda_2^{\ n}\left(A_2\mathbf{v}_2\right)\right) = \mathrm{Re}\left(\lambda_1^{\ n}\left(e^{i\psi_1}\sqrt{2I_1}\mathbf{v}_1\right) + \lambda_2^{\ n}\left(e^{i\psi_2}\sqrt{2I_2}\mathbf{v}_2\right)\right)$$

  - Introduce betatron frequencies so that $\lambda_{1,2} = e^{-i\mu_{1,2}} = e^{-2\pi i Q_{1,2}}$

The Eigen-vector Parameterization

- For uncoupled motion the normalized eigen-vectors are

$$\mathbf{v} \equiv \mathbf{v}(s) = \begin{bmatrix} \sqrt{\beta(s)} \\ -\dfrac{i+\alpha(s)}{\sqrt{\beta(s)}} \end{bmatrix} , \quad \begin{cases} \mathbf{v}_1 = \mathbf{v} \\ \mathbf{v}_2 = \mathbf{v}^* \end{cases}$$

  - we define that $\mathrm{Im}(\mathrm{v}_1(s)) = 0$

$$\mathbf{S} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$\begin{cases} \mathbf{v}^+\mathbf{S}\mathbf{v} == -2i \,, \\ \mathbf{v}^T\mathbf{S}\mathbf{v} = 0 \quad \text{and} \quad \mathbf{v}_2^+\mathbf{S}\mathbf{v}_1 = 0 \,, \end{cases} \qquad \mathbf{S} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

- For coupled motion the normalized eigen-vectors are in extended Mais-Ripken parameterization is

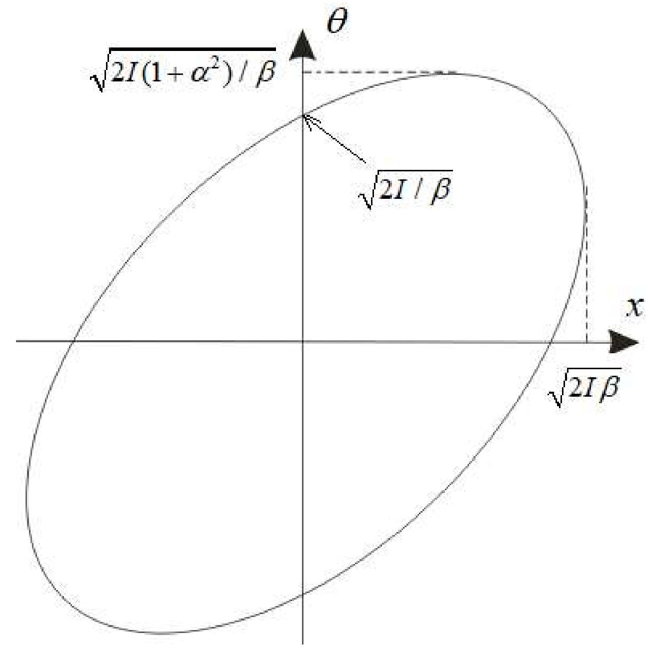
$$\hat{\mathbf{v}}_1 = \begin{bmatrix} \sqrt{\beta_{1x}} \\ -\dfrac{i(1-u)+\alpha_{1x}}{\sqrt{\beta_{1x}}} \\ \sqrt{\beta_{1y}}e^{i\nu_1} \\ -\dfrac{iu+\alpha_{1y}}{\sqrt{\beta_{1y}}}e^{i\nu_1} \end{bmatrix}, \quad \hat{\mathbf{v}}_2 = \begin{bmatrix} \sqrt{\beta_{2x}}e^{i\nu_2} \\ -\dfrac{iu+\alpha_{2x}}{\sqrt{\beta_{2x}}}e^{i\nu_2} \\ \sqrt{\beta_{2y}} \\ -\dfrac{i(1-u)+\alpha_{2y}}{\sqrt{\beta_{2y}}} \end{bmatrix}$$

- The betatron motion is described by 10 linearly independent functions: 4 β-functions, 4 α-functions, and 2 betatron phase advances
- Symplecticity yields u, ν_1 & ν_2 from known α's & β's.
  - However, there are 4 solutions & additional info is required to choose α's & β's. In practice, for a ring we, first, find the eigen-vectors from known transfer matrix, and, then unique solutions for all 4D-Twiss functions

Courant-Snider Invariant

- In uncoupled motion
  - Betatron amplitude
  - Maximum angle
  - Local angular spread
- Courant-Snyder invariant for uncoupled motion

$$2I = \left|\mathbf{v}^{+}\mathbf{S}\mathbf{x}\right|^2 = \beta\theta^2 + 2\alpha x\theta + \frac{1+\alpha^2}{\beta}x^2$$



- In coupled motion $2I_{1,2} = \left|\mathbf{v}_{1,2}{}^{+}\mathbf{S}\mathbf{x}\right|^2$

- Beam boundary in 4D phase space

$$\hat{\mathbf{x}}^T\hat{\mathbf{\Xi}}\hat{\mathbf{x}} = 1\,, \quad \hat{\mathbf{\Xi}} = \mathbf{U}\hat{\mathbf{V}}\hat{\mathbf{\Xi}}'\hat{\mathbf{V}}^T\mathbf{U}^T$$

where: $\hat{\mathbf{V}} = \left[\hat{\mathbf{v}}_1{}', -\hat{\mathbf{v}}_1{}'', \hat{\mathbf{v}}_2{}', -\hat{\mathbf{v}}_2{}''\right]$

$$\hat{\mathbf{\Xi}}' = \begin{bmatrix} 1/\varepsilon_1 & 0 & 0 & 0 \\ 0 & 1/\varepsilon_1 & 0 & 0 \\ 0 & 0 & 1/\varepsilon_2 & 0 \\ 0 & 0 & 0 & 1/\varepsilon_2 \end{bmatrix}$$

ε_1 and ε_2 are invariants of motion

1D and 2D Emittances

- We define the beam emittance as a product of the ellipsoid semi-axes (omitting the factor $\pi^2/2$ correcting for the real 4D volume of the ellipsoid):

$$\varepsilon_{4D} = \frac{1}{\sqrt{\hat{\Xi}'_{11}\hat{\Xi}'_{22}\hat{\Xi}'_{33}\hat{\Xi}'_{44}}} = \frac{1}{\sqrt{\det(\hat{\mathbf{\Xi}}')}}$$

- Consequently:

$$\varepsilon_1\varepsilon_2 = \varepsilon_{4D} \ , \quad \hat{\mathbf{\Xi}}' = \begin{bmatrix} 1/\varepsilon_1 & 0 & 0 & 0 \\ 0 & 1/\varepsilon_1 & 0 & 0 \\ 0 & 0 & 1/\varepsilon_2 & 0 \\ 0 & 0 & 0 & 1/\varepsilon_2 \end{bmatrix}$$

- Gaussian distribution: $f(\hat{\mathbf{x}}) = \dfrac{1}{4\pi^2\varepsilon_1\varepsilon_2}\exp\left(-\dfrac{1}{2}\hat{\mathbf{x}}^T\hat{\mathbf{\Xi}}\hat{\mathbf{x}}\right)$

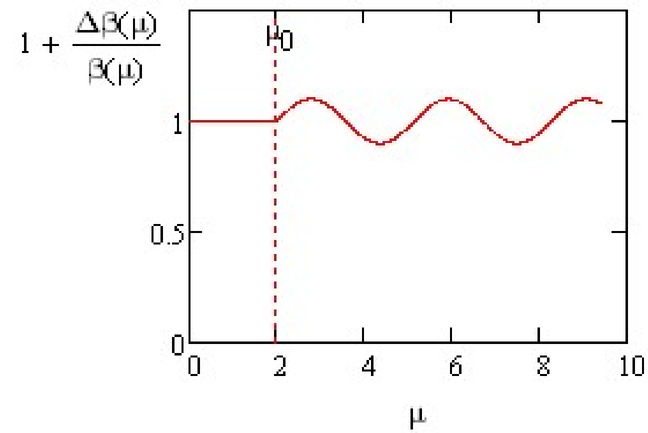
- Second order moments

$$\hat{\Sigma}_{ij} \equiv \overline{\hat{x}_i\hat{x}_j} = \int \hat{x}_i\hat{x}_j f(\hat{\mathbf{x}})d\hat{x}^4 = \frac{1}{4\pi^2\varepsilon_1\varepsilon_2}\int \hat{x}_i\hat{x}_j \exp\left(-\frac{1}{2}\hat{\mathbf{x}}^T\hat{\mathbf{\Xi}}\hat{\mathbf{x}}\right)d\hat{x}^4$$

Perturbed Betatron Motion (Uncoupled Case)

- In transfer line

$$\frac{\beta'(\mu)}{\beta(\mu)} \approx \begin{cases} 1 \quad , & \mu < \mu_0 , \\ 1 - \Phi \beta_0 \sin\left(2\left(\mu - \mu_0\right)\right), & \mu > \mu_0 , \end{cases} \qquad \Phi \equiv \frac{1}{F}$$



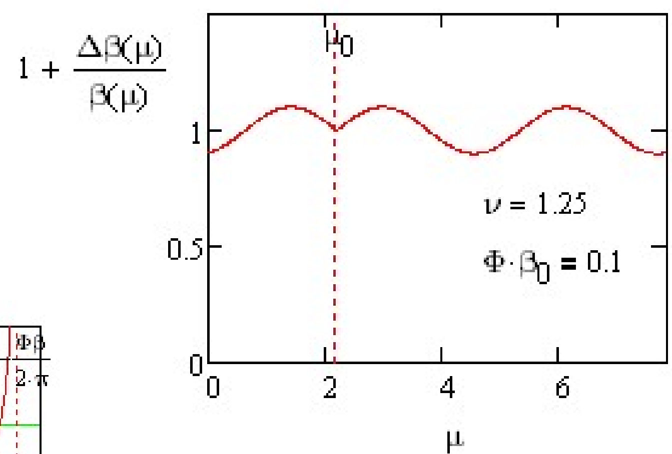
- In a ring

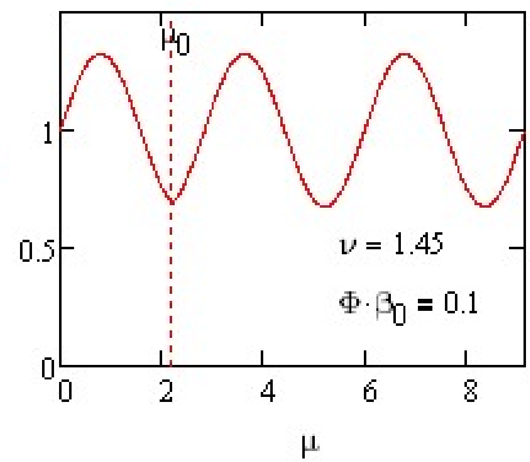
$$\frac{\beta'(\mu)}{\beta(\mu)} \approx 1 + \frac{\Phi \beta}{\sin \mu_0} \cos\left(\mu_0 - 2\mu\right)$$

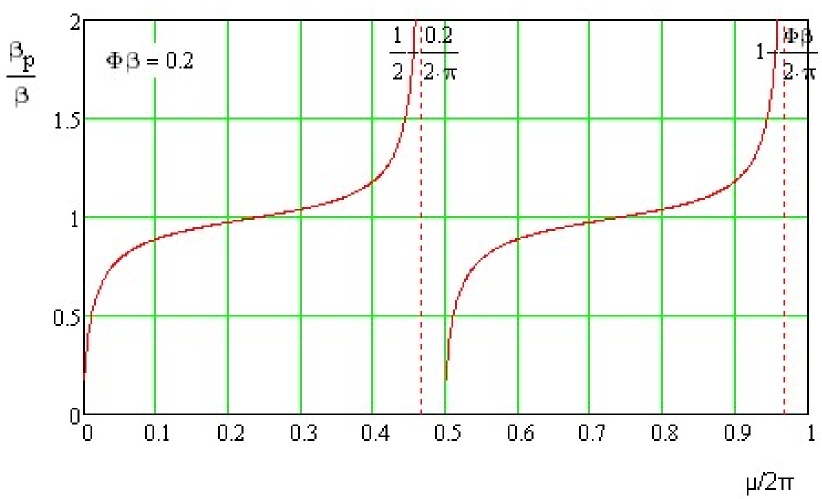
$$\Delta \mu = \frac{1}{2} \Phi \beta$$

The stopband width

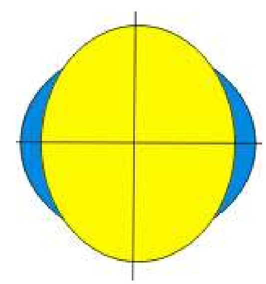
$$\Delta \mu_b = -\Phi \beta$$







- Oscillations are happening at double betatron frequency



Perturbation Theory for Symplectic Motion

- The symplecticity enables to build an effective perturbation theory

$$\hat{\mathbf{M}}^T \mathbf{U} \hat{\mathbf{M}} = \mathbf{U} \quad \Rightarrow \quad \hat{\mathbf{M}} = \mathbf{U}^T \hat{\mathbf{M}}^T \mathbf{U}$$

- For the perturbed motion one can write: $\left(\mathbf{I} + \Delta\mathbf{M}\right)\mathbf{M}\tilde{\mathbf{v}}_j = \left(\lambda_j + \Delta\lambda_j\right)\tilde{\mathbf{v}}_j$
  - $\mathbf{M}$ – symplectic
  - transfer matrix, $(\mathbf{I} + \Delta\mathbf{M})\mathbf{M}$, is not required to be symplectic
- Account relationship between eigenvalue corrections and the tune shifts

$$\Delta Q_n = i/(4\pi)\left(\Delta\lambda_n / \lambda_n\right)$$

- We obtain corrections for betatron tunes

$$\begin{cases} \Delta Q_1 = -\dfrac{1}{4\pi}\,\mathbf{v}_1^{\;+}\mathbf{U}\,\Delta\mathbf{M}\,\mathbf{v}_1 \\ \Delta Q_2 = -\dfrac{1}{4\pi}\,\mathbf{v}_2^{\;+}\mathbf{U}\,\Delta\mathbf{M}\,\mathbf{v}_2 \end{cases}$$

Problems for Lectures 1&2

1. For uncoupled betatron motion prove that the normalization of eigen-vectors, $\hat{\mathbf{v}}_k^{+}\mathbf{S}\hat{\mathbf{v}}_k = -2i$, yields that $d\mu/ds = 1/\beta$ and $\alpha = -(1/2)d\beta/ds$ (For the proof use top Eq. of page 7)

From page 6: $\mathbf{x}(s) = \sqrt{2I}\,\mathrm{Re}\left(e^{i(\psi-\mu(s))}\mathbf{v}\right) = \sqrt{2I}\left(\dfrac{e^{i(\psi-\mu(s))}\mathbf{v} + e^{-i(\psi-\mu(s))}\mathbf{v}^*}{2}\right)$, $\mathbf{v} \equiv \begin{bmatrix} \mathrm{v}_1 \\ \mathrm{v}_2 \end{bmatrix} \equiv \mathbf{v}(s) = \begin{bmatrix} \sqrt{\beta(s)} \\ -\dfrac{i+\alpha(s)}{\sqrt{\beta(s)}} \end{bmatrix}$, $\begin{cases} \mathbf{v}_1 = \mathbf{v} \\ \mathbf{v}_2 = \mathbf{v}^* \end{cases}$

$$\Rightarrow \frac{d\mathbf{x}}{ds} = \sqrt{\frac{I}{2}}\frac{d}{ds}\left(e^{i(\psi-\mu(s))}\begin{bmatrix} \sqrt{\beta(s)} \\ -\dfrac{i+\alpha(s)}{\sqrt{\beta(s)}} \end{bmatrix} + CC\right) = \sqrt{\frac{I}{2}}e^{i(\psi-\mu(s))}\left(i\frac{d\mu}{ds}\begin{bmatrix} \sqrt{\beta(s)} \\ -\dfrac{i+\alpha(s)}{\sqrt{\beta(s)}} \end{bmatrix} + \begin{bmatrix} (d\beta/ds)\big/\left(2\sqrt{\beta(s)}\right) \\ \dfrac{d}{ds}\left(\dfrac{i+\alpha(s)}{\sqrt{\beta(s)}}\right) \end{bmatrix} + CC\right)$$

On other hand in the first order

$$\mathbf{x} + d\mathbf{x} = \begin{bmatrix} 1 & ds \\ -d\Phi & 1 \end{bmatrix}\mathbf{x} = \begin{bmatrix} 1 & ds \\ -d\Phi & 1 \end{bmatrix}\left(e^{i(\psi-\mu(s))}\begin{bmatrix} \sqrt{\beta(s)} \\ -\dfrac{i+\alpha(s)}{\sqrt{\beta(s)}} \end{bmatrix} + CC\right)$$

Compare the top rows of matrix equations: $d\mathbf{x} = \sqrt{\dfrac{I}{2}}e^{i(\psi-\mu(s))}\left(-\dfrac{i+\alpha(s)}{\sqrt{\beta(s)}}\right)ds + CC = \sqrt{\dfrac{I}{2}}e^{i(\psi-\mu(s))}\left(id\mu\sqrt{\beta(s)} + \dfrac{d\beta}{2\sqrt{\beta(s)}}\right)$

$$\Rightarrow -\frac{i+\alpha(s)}{\sqrt{\beta(s)}} = i\frac{d\mu}{ds}\sqrt{\beta(s)} + \frac{1}{2\sqrt{\beta(s)}}\frac{d\beta}{ds} \quad \Rightarrow \quad \begin{cases} \dfrac{d\mu}{ds} = \dfrac{1}{\beta(s)} \\ \alpha = -\dfrac{1}{2}\dfrac{d\beta}{ds} \end{cases}$$

2. Prove that if **v** is the eigen-vector for matrix **M** corresponding to the one turn matrix starting at s=0 (point 1) then the vector $\mathbf{M}_{12}\,\mathbf{v}$ will be the eigen vector of the transfer matrix corresponding to the point 2. Here $\mathbf{M}_{12}$ is the transfer matrix from point 1 to point 2.

$$\mathbf{M} = \mathbf{M}_2\mathbf{M}_1, \quad \mathbf{M}\mathbf{v} = \lambda\mathbf{v},$$

$$\mathbf{M}_1\left(\mathbf{M}_2\mathbf{M}_1\right)\mathbf{v} = \lambda\mathbf{M}_1\mathbf{v}, \quad \mathbf{M}_1\mathbf{M}_2\left(\mathbf{M}_1\mathbf{v}\right) = \lambda\left(\mathbf{M}_1\mathbf{v}\right),$$

3. Find 2D analog of Courant-Snyder invariant

$$\mathbf{x} = A_1\mathbf{v}_1 + A_2\mathbf{v}_2 + CC$$

$$\mathbf{x}^{+}\mathbf{U}\mathbf{v}_1 = \left(A_1\mathbf{v}_1 + A_2\mathbf{v}_2 + CC\right)^{+}\mathbf{U}\mathbf{v}_1 = A_1\mathbf{v}_1{}^{+}\mathbf{U}\mathbf{v}_1 = -2iA_1$$

$$\left|A_1\right|^2 = \frac{1}{4}\left|\mathbf{x}^{+}\mathbf{U}\mathbf{v}_1\right|^2 = \frac{1}{4}\left|\begin{bmatrix} x & \theta_x & y & \theta_y \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}\begin{bmatrix} \sqrt{\beta_{1x}} \\ -\dfrac{i(1-u)+\alpha_{1x}}{\sqrt{\beta_{1x}}} \\ \sqrt{\beta_{1y}}\,e^{i\nu_1} \\ -\dfrac{iu+\alpha_{1y}}{\sqrt{\beta_{1y}}}e^{i\nu_1} \end{bmatrix}\right|^2 = const$$

4. Prove that matrix $\hat{\mathbf{V}} = \left[\hat{\mathbf{v}}_1{}',-\hat{\mathbf{v}}_1{}'',\hat{\mathbf{v}}_2{}',-\hat{\mathbf{v}}_2{}''\right]$ is symplectic

$$\hat{\mathbf{V}} = \frac{1}{2}\left[\hat{\mathbf{v}}_1 + \hat{\mathbf{v}}_1{}^{*}, i\left(\hat{\mathbf{v}}_1 - \hat{\mathbf{v}}_1{}^{*}\right), \hat{\mathbf{v}}_2 + \hat{\mathbf{v}}_2{}^{*}, i\left(\hat{\mathbf{v}}_1 - \hat{\mathbf{v}}_1{}^{*}\right)\right]$$

$$\left[\mathbf{V}^T\mathbf{U}\mathbf{V}\right]_{11} = \frac{1}{4}\left(\hat{\mathbf{v}}_1 + \hat{\mathbf{v}}_1{}^{*}\right)^T\mathbf{U}\left(\hat{\mathbf{v}}_1 + \hat{\mathbf{v}}_1{}^{*}\right) = \frac{1}{4}\left(\hat{\mathbf{v}}_1{}^{+}\mathbf{U}\hat{\mathbf{v}}_1 + \left(\hat{\mathbf{v}}_1{}^{+}\mathbf{U}\hat{\mathbf{v}}_1\right)^{*}\right) = 0$$

$$\left[\mathbf{V}^T\mathbf{U}\mathbf{V}\right]_{12} = \frac{i}{4}\left(\hat{\mathbf{v}}_1 + \hat{\mathbf{v}}_1{}^{*}\right)^T\mathbf{U}\left(\hat{\mathbf{v}}_1 - \hat{\mathbf{v}}_1{}^{*}\right) = \frac{1}{4}\left(\hat{\mathbf{v}}_1{}^{+}\mathbf{U}\hat{\mathbf{v}}_1 - \left(\hat{\mathbf{v}}_1{}^{+}\mathbf{U}\hat{\mathbf{v}}_1\right)^{*}\right) = 1$$

…

5. Fill missed calculations in computation $\hat{\mathbf{\Sigma}} = \hat{\mathbf{V}}\hat{\mathbf{\Xi}}'^{-1}\hat{\mathbf{V}}^T = \hat{\mathbf{\Xi}}^{-1}$

$$\Sigma_{ij} \equiv \overline{\hat{x}_i \hat{x}_j} = \int x_i x_j f(\mathbf{x}) d\hat{x}^4 = \frac{1}{4\pi^2 \varepsilon_1 \varepsilon_2} \int \hat{x}_i \hat{x}_j \exp\left(-\frac{1}{2}\mathbf{x}^T \mathbf{\Xi} \mathbf{x}\right) d\hat{x}^4$$

Introduce new variable $\hat{\mathbf{y}} = \hat{\mathbf{V}}^{-1}\hat{\mathbf{x}}$ then

$$\Sigma_{ij} = \frac{1}{4\pi^2 \varepsilon_1 \varepsilon_2} \int V_{in} V_{im} y_n y_m \exp\left(-\frac{1}{2}\mathbf{y}^T \mathbf{V}^T \mathbf{\Xi} \mathbf{V} \mathbf{y}\right) dy^4$$

Account that $\hat{\mathbf{\Xi}}' = \hat{\mathbf{V}}^T \hat{\mathbf{\Xi}} \hat{\mathbf{V}}$ where $\hat{\mathbf{\Xi}}' = \begin{bmatrix} 1/\varepsilon_1 & 0 & 0 & 0 \\ 0 & 1/\varepsilon_1 & 0 & 0 \\ 0 & 0 & 1/\varepsilon_2 & 0 \\ 0 & 0 & 0 & 1/\varepsilon_2 \end{bmatrix}$

$$\Sigma_{ij} = \frac{1}{4\pi^2 \varepsilon_1 \varepsilon_2} V_{in} V_{im} \int y_n y_m \exp\left(-\frac{1}{2}\mathbf{y}^T \mathbf{\Xi}' \mathbf{y}\right) dy^4$$

Matrix $K_{nm} = \frac{1}{4\pi^2 \varepsilon_1 \varepsilon_2} \int y_n y_m \exp\left(-\frac{1}{2}\mathbf{y}^T \mathbf{\Xi}' \mathbf{y}\right) dy^4$ is diagonal $\mathbf{K} = \begin{bmatrix} \varepsilon_1 & 0 & 0 & 0 \\ 0 & \varepsilon_1 & 0 & 0 \\ 0 & 0 & \varepsilon_2 & 0 \\ 0 & 0 & 0 & \varepsilon_2 \end{bmatrix}$

⇨ $\hat{\mathbf{\Sigma}} = \hat{\mathbf{V}}\hat{\mathbf{\Xi}}'^{-1}\hat{\mathbf{V}}^T = \hat{\mathbf{\Xi}}^{-1}$

6. Prove that for a symplectic matrix, defined by the following equation $\hat{\mathbf{M}}^T\mathbf{U}\hat{\mathbf{M}} = \mathbf{U}$, (a) its determinant is $\left|\hat{\mathbf{M}}\right| = 1$, (b) $\hat{\mathbf{M}}^{-1} = \mathbf{U}^T\hat{\mathbf{M}}^T\mathbf{U}$ and (c) the matrix also satisfies to $\hat{\mathbf{M}}\mathbf{U}\hat{\mathbf{M}}^T = \mathbf{U}$.

a) $\det\left(\hat{\mathbf{M}}^T\mathbf{U}\hat{\mathbf{M}}\right) = \det\mathbf{U} \quad \Rightarrow \quad \left|\hat{\mathbf{M}}^T\right|\left|\hat{\mathbf{M}}\right| = 1 \quad \Rightarrow \quad \left|\hat{\mathbf{M}}\right|^2 = 1 \quad \Rightarrow \quad \left|\hat{\mathbf{M}}\right| = \pm 1$

Only sign + is right since the total matrix is a multiplication of matrices for infinitesimal displacement for which $\left|\hat{\mathbf{M}}_n\right| = 1$

b) $\left(\hat{\mathbf{M}}^T\mathbf{U}\hat{\mathbf{M}} = \mathbf{U}\right)\hat{\mathbf{M}}^{-1} \quad \Rightarrow \quad \mathbf{U}\hat{\mathbf{M}}^{-1} = \hat{\mathbf{M}}^T\mathbf{U} \quad \Rightarrow \quad \mathbf{U}^+\left(\mathbf{U}\hat{\mathbf{M}}^{-1} = \hat{\mathbf{M}}^T\mathbf{U}\right) \quad \Rightarrow \quad \hat{\mathbf{M}}^{-1} = \mathbf{U}^+\hat{\mathbf{M}}^T\mathbf{U}$

c)

$$\hat{\mathbf{M}}\mathbf{U}\left(\hat{\mathbf{M}}^T\mathbf{U}\hat{\mathbf{M}} = \mathbf{U}\right) \quad \Rightarrow \quad \left(\hat{\mathbf{M}}\mathbf{U}\hat{\mathbf{M}}^T\right)\mathbf{U}\hat{\mathbf{M}} = \hat{\mathbf{M}}\mathbf{U}\mathbf{U} = -\hat{\mathbf{M}}$$

$$\left(\left(\hat{\mathbf{M}}\mathbf{U}\hat{\mathbf{M}}^T\right)\mathbf{U}\hat{\mathbf{M}} = -\hat{\mathbf{M}}\right)\mathbf{M}^{-1} \quad \Rightarrow \quad \left(\left(\hat{\mathbf{M}}\mathbf{U}\hat{\mathbf{M}}^T\right)\mathbf{U} = -\mathbf{I}\right)\mathbf{U} \quad \Rightarrow \quad \hat{\mathbf{M}}\mathbf{U}\hat{\mathbf{M}}^T = \mathbf{U}$$

where we accounted that $\mathbf{U}\mathbf{U} = -\mathbf{I}$

7. Assuming that the motion after exit from KRION ion source is uncoupled and described uncoupled Twiss-parameters find equations describing the horizontal and vertical rms sizes in the downstream beam transport for two below cases.
(1) Ions exit at the axis of magnetic field. Beam parameters at the ion source center: magnetic field - B_0, ion rms beam size - σ, transverse temperature - T.
(2) Now add that the ions exit solenoid with offset r_0 directed at angle θ from the horizontal plane.

(1) First, we find bilinear form describing the distribution inside solenoid

The distribution function in the magnetic field:

$$f(x,\theta_x,y,\theta_y)=\exp\left(-\frac{x^2+y^2}{2\sigma^2}-\left(\theta_x{}^2+\theta_y{}^2\right)\frac{m\mathrm{v}_0}{T^2}\right)=\exp\left(-\frac{x^2+y^2}{2\sigma^2}-\frac{\theta_x{}^2+\theta_y{}^2}{2\sigma_\theta{}^2}\right)$$

Then transiting to canonical variables, we obtain for matrix Ξ ($\mathbf{x}\,\Xi\,\mathbf{x}=1$)

$$\boldsymbol{\Xi}=\begin{bmatrix}1&0&0&0\\0&1&-R/2&0\\0&0&1&0\\R/2&0&0&1\end{bmatrix}^T\begin{bmatrix}1/\sigma^2&0&0&0\\0&1/\sigma_\theta{}^2&0&0\\0&0&1/\sigma^2&0\\0&0&0&1/\sigma_\theta{}^2\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&1&-R/2&0\\0&0&1&0\\R/2&0&0&1\end{bmatrix}$$

$$\Xi=\begin{pmatrix}\frac{1}{\sigma^2}+\frac{R^2}{4\cdot\sigma_\theta{}^2}&0&0&\frac{R}{2\cdot\sigma_\theta{}^2}\\0&\frac{1}{\sigma_\theta{}^2}&-\frac{R}{2\cdot\sigma_\theta{}^2}&0\\0&-\frac{R}{2\cdot\sigma_\theta{}^2}&\frac{1}{\sigma^2}+\frac{R^2}{4\cdot\sigma_\theta{}^2}&0\\\frac{R}{2\cdot\sigma_\theta{}^2}&0&0&\frac{1}{\sigma_\theta{}^2}\end{pmatrix}$$

We could extract the emittances and Twiss parameters directly from Ξ using the following procedure suggested in <Lebedev, Bogacz>

the beam emittances ε_1 and ε_2 can be computed from matrix $\hat{\Sigma}$ as roots of its characteristic equation,

$$\det\left(\hat{\Sigma}\mathbf{U} + i\lambda\,\mathbf{I}\right) = 0 \quad , \qquad \varepsilon_l = \lambda_l \quad (2.27)$$

while the equations for the eigen-vectors are

$$\left(\hat{\Sigma}\mathbf{U} + i\varepsilon_l\mathbf{I}\right)\hat{\mathbf{v}}_l = 0 \quad . \qquad (2.28)$$

However, we can simplify the solution if we account symmetry of the problem.
We assume that the distribution is described by the circular modes with unknown beta-function and, additionally, we will need to find two mode emittances. The eigen vectors are:

$$\hat{\mathbf{v}}_1 = \begin{bmatrix} \sqrt{\beta} \\ \dfrac{i}{2\sqrt{\beta}} \\ i\sqrt{\beta} \\ \dfrac{1}{2\sqrt{\beta}} \end{bmatrix}, \quad \hat{\mathbf{v}}_2 = \begin{bmatrix} \sqrt{\beta} \\ \dfrac{i}{2\sqrt{\beta}} \\ i\sqrt{\beta} \\ \dfrac{1}{2\sqrt{\beta}} \end{bmatrix}, \quad \mathbf{V} = \begin{bmatrix} \sqrt{\beta} & 0 & 0 & -\sqrt{\beta} \\ 0 & \dfrac{1}{2\sqrt{\beta}} & \dfrac{1}{2\sqrt{\beta}} & 0 \\ 0 & -\sqrt{\beta} & \sqrt{\beta} & 0 \\ \dfrac{1}{2\sqrt{\beta}} & 0 & 0 & \dfrac{1}{2\sqrt{\beta}} \end{bmatrix}$$

Then matrix Ξ is:

$$\hat{\mathbf{\Xi}} = \mathbf{U}\hat{\mathbf{V}}\hat{\mathbf{\Xi}}'\hat{\mathbf{V}}^T\mathbf{U}^T, \quad \hat{\mathbf{\Xi}}' = diag(1/\varepsilon_1, 1/\varepsilon_1, 1/\varepsilon_2, 1/\varepsilon_2)$$

Substituting we have

$$\Xi = \begin{pmatrix} \frac{\varepsilon_1+\varepsilon_2}{4\cdot\beta\cdot\varepsilon_1\cdot\varepsilon_2} & 0 & 0 & \frac{1}{2\cdot\varepsilon_1}-\frac{1}{2\cdot\varepsilon_2} \\ 0 & \frac{\beta}{\varepsilon_1}+\frac{\beta}{\varepsilon_2} & \frac{1}{2\cdot\varepsilon_2}-\frac{1}{2\cdot\varepsilon_1} & 0 \\ 0 & \frac{1}{2\cdot\varepsilon_2}-\frac{1}{2\cdot\varepsilon_1} & \frac{\varepsilon_1+\varepsilon_2}{4\cdot\beta\cdot\varepsilon_1\cdot\varepsilon_2} & 0 \\ \frac{1}{2\cdot\varepsilon_1}-\frac{1}{2\cdot\varepsilon_2} & 0 & 0 & \frac{\beta}{\varepsilon_1}+\frac{\beta}{\varepsilon_2} \end{pmatrix}$$

Equalizing

$$\begin{pmatrix} \frac{\varepsilon_1+\varepsilon_2}{4\cdot\beta\cdot\varepsilon_1\cdot\varepsilon_2} & 0 & 0 & \frac{1}{2\cdot\varepsilon_1}-\frac{1}{2\cdot\varepsilon_2} \\ 0 & \frac{\beta}{\varepsilon_1}+\frac{\beta}{\varepsilon_2} & \frac{1}{2\cdot\varepsilon_2}-\frac{1}{2\cdot\varepsilon_1} & 0 \\ 0 & \frac{1}{2\cdot\varepsilon_2}-\frac{1}{2\cdot\varepsilon_1} & \frac{\varepsilon_1+\varepsilon_2}{4\cdot\beta\cdot\varepsilon_1\cdot\varepsilon_2} & 0 \\ \frac{1}{2\cdot\varepsilon_1}-\frac{1}{2\cdot\varepsilon_2} & 0 & 0 & \frac{\beta}{\varepsilon_1}+\frac{\beta}{\varepsilon_2} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sigma^2}+\frac{R^2}{4\cdot\sigma_\theta^{\,2}} & 0 & 0 & \frac{R}{2\cdot\sigma_\theta^{\,2}} \\ 0 & \frac{1}{\sigma_\theta^{\,2}} & -\frac{R}{2\cdot\sigma_\theta^{\,2}} & 0 \\ 0 & -\frac{R}{2\cdot\sigma_\theta^{\,2}} & \frac{1}{\sigma^2}+\frac{R^2}{4\cdot\sigma_\theta^{\,2}} & 0 \\ \frac{R}{2\cdot\sigma_\theta^{\,2}} & 0 & 0 & \frac{1}{\sigma_\theta^{\,2}} \end{pmatrix}$$

We obtain 3 independent equations

$$\begin{cases} \dfrac{R}{2\cdot\sigma_\theta^{\,2}} = \dfrac{1}{2\cdot\varepsilon_1} - \dfrac{1}{2\cdot\varepsilon_2} \\[2ex] \dfrac{1}{\sigma^2} + \dfrac{R^2}{4\cdot\sigma_\theta^{\,2}} = \dfrac{\varepsilon_1+\varepsilon_2}{4\cdot\beta\cdot\varepsilon_1\cdot\varepsilon_2} \\[2ex] \dfrac{1}{\sigma_\theta^{\,2}} = \dfrac{\beta}{\varepsilon_1} + \dfrac{\beta}{\varepsilon_2} \end{cases}$$

Finally, the solution is:

$$\beta = \frac{\beta_0}{\sqrt{1+\dfrac{4\sigma_\theta{}^2\beta_0{}^2}{\sigma^2}}}\ , \quad \varepsilon_1 = \frac{\beta_0}{\sqrt{1+\dfrac{\sigma^2}{4\sigma_\theta{}^2\beta_0{}^2}}+\dfrac{\sigma}{2\sigma_\theta\beta_0}}\ , \quad \varepsilon_2 = \frac{\beta_0}{\sqrt{1+\dfrac{\sigma^2}{4\sigma_\theta{}^2\beta_0{}^2}}-\dfrac{\sigma}{2\sigma_\theta\beta_0}}$$

where we accounted that $\beta_0 = 1/R$ the is matched beta-function of the solenoid.

Note that this solution is not matched to the solenoid. This solution corresponds to the beam emission from the thermal cathode

Now we consider the solution matched to the solenoid (the case of ion source). Here we account that $\beta = 1/R$

$$\Xi_B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -\frac{1}{2\cdot\beta} & 0 \\ 0 & 0 & 1 & 0 \\ \frac{1}{2\cdot\beta} & 0 & 0 & 1 \end{pmatrix}^T \cdot \begin{pmatrix} \frac{\varepsilon_1+\varepsilon_2}{4\cdot\beta\cdot\varepsilon_1\cdot\varepsilon_2} & 0 & 0 & \frac{1}{2\cdot\varepsilon_1}-\frac{1}{2\cdot\varepsilon_2} \\ 0 & \frac{\beta}{\varepsilon_1}+\frac{\beta}{\varepsilon_2} & \frac{1}{2\cdot\varepsilon_2}-\frac{1}{2\cdot\varepsilon_1} & 0 \\ 0 & \frac{1}{2\cdot\varepsilon_2}-\frac{1}{2\cdot\varepsilon_1} & \frac{\varepsilon_1+\varepsilon_2}{4\cdot\beta\cdot\varepsilon_1\cdot\varepsilon_2} & 0 \\ \frac{1}{2\cdot\varepsilon_1}-\frac{1}{2\cdot\varepsilon_2} & 0 & 0 & \frac{\beta}{\varepsilon_1}+\frac{\beta}{\varepsilon_2} \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -\frac{1}{2\cdot\beta} & 0 \\ 0 & 0 & 1 & 0 \\ \frac{1}{2\cdot\beta} & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{\beta\cdot\varepsilon_1} & 0 & 0 & \frac{1}{\varepsilon_1} \\ 0 & \frac{\beta}{\varepsilon_1}+\frac{\beta}{\varepsilon_2} & -\frac{1}{\varepsilon_1} & 0 \\ 0 & -\frac{1}{\varepsilon_1} & \frac{1}{\beta\cdot\varepsilon_1} & 0 \\ \frac{1}{\varepsilon_1} & 0 & 0 & \frac{\beta}{\varepsilon_1}+\frac{\beta}{\varepsilon_2} \end{pmatrix}$$

The particle distribution is

$$f = C\cdot\exp\left[-\frac{1}{2}\cdot\left[\frac{x^2+y^2}{\beta\cdot\varepsilon_1}+\left(\frac{1}{\varepsilon_1}+\frac{1}{\varepsilon_2}\right)\cdot\beta\cdot\left(\theta_x{}^2+\theta_y{}^2\right)+2\cdot\frac{x\cdot\theta_y-y\cdot\theta_x}{\varepsilon_1}\right]\right]$$

Corresponding matrix of the second order moments is

$$\Sigma = \Xi_B{}^{-1} = \begin{bmatrix} \beta\cdot(\varepsilon_1+\varepsilon_2) & 0 & 0 & -\varepsilon_2 \\ 0 & \frac{\varepsilon_2}{\beta} & \varepsilon_2 & 0 \\ 0 & \varepsilon_2 & \beta\cdot(\varepsilon_1+\varepsilon_2) & 0 \\ -\varepsilon_2 & 0 & 0 & \frac{\varepsilon_2}{\beta} \end{bmatrix}$$

$$\Sigma = \Xi_B^{-1} = \begin{bmatrix} \beta\cdot(\varepsilon_1+\varepsilon_2) & 0 & 0 & -\varepsilon_2 \\ 0 & \frac{\varepsilon_2}{\beta} & \varepsilon_2 & 0 \\ 0 & \varepsilon_2 & \beta\cdot(\varepsilon_1+\varepsilon_2) & 0 \\ -\varepsilon_2 & 0 & 0 & \frac{\varepsilon_2}{\beta} \end{bmatrix}$$

That yields

$$\sigma^2 = \beta\cdot(\varepsilon_1+\varepsilon_2) \qquad \Rightarrow \qquad \begin{cases} \varepsilon_1 = \frac{\sigma^2}{\beta} - \sigma_\theta^{\,2}\cdot\beta \\ \varepsilon_2 = \sigma_\theta^{\,2}\cdot\beta \end{cases}$$

$$\sigma_\theta^{\,2} = \frac{\varepsilon_2}{\beta}$$