Special Bohr - Sommerfeld geometry

Nikolay Tyurin
BLTPh JINR (Dubna)

based on
"Special Bohr - Sommerfeld lagrangian submanifolds", Izvestiya RAS: Mathematics, 80: 6 (2016);
"Special Bohr - Sommerfeld lagrangian submanifolds in algebraic varieties", Izvestiya RAS: Mathematics, 82: 3 (2018).
"Special Bohr - Sommerfeld geometry: variations", Izvestiya RAS: Mathematics, 87: 3 (2023).

"Mathematics in the constellation of sciences", 2.4.24, Dubna

Geometric Quantization:

(M, ω) — compact symplectic manifold ($=$ phase space of Classical Mechanical system),

ω — symplectic form (locally $\omega = \sum_{i=1}^{n} dp_i \wedge dx_i$ — famous **Darboux Lemma**)

$+$ **Arnold condition:** $[\omega]$ is integer

$$\Updownarrow$$

$\exists \quad (L, a)$ — **prequantization pair**

where L is a hermitian line bundle, a is a hermitian connection such that $F_a = 2\pi i \omega$ (therefore the first Chern class $c_1(L) = [\omega]$)
if M is simply connected $\Rightarrow$ such a is unique up to the gauge transformation

Quantization: find a Hilbert space $\mathcal{H}$ + representation of the Poisson algebra $C^\infty(M, \mathbb{R}, \{;\}_\omega)$ in $\mathrm{Op}(\mathcal{H})$ — self adjoint operators.

Geometric means that $\mathcal{H}$ is constructed in terms of *entire geometry* of (M, ω, L, a)

First example: Souriou - Kostant Quantization.

the Hilber space $\mathcal{H} = \Gamma(M, L)$ — smooth sections of L

$$< s_1, s_2 >_q = \int_M < s_1, s_2 >_h \mathrm{d}\mu_{\mathrm{L}} - \text{ hermitian scalar product}$$

correspondence "classical observable $\leftrightarrow$ quantum observables"

$$f \in C^\infty(M, \mathbb{R}) \mapsto Q_f s = \nabla_{X_f} s + 2\pi i f \cdot s$$

Problem: too big space $\Rightarrow$ **Reductions**

Polarizations: Complex or Real

Complex Polarization: (M, ω) admits an integrable complex structure I, compatible with ω

$\Rightarrow$ (M, ω, I) is a **Kahler manifold**

For the **Kahler case:**

the prequantization connection a induces a holomorphic structure on L:

$$s \in H^0(M_I, L) \quad \Leftrightarrow \quad \bar{\partial}_a s = 0$$

Therefore we have a finite dimensional subspace $H^0(M_I, L) \subset \Gamma(M, L)$

Then we have the following reductions of Souriou - Kostant method:

Berezin - Rawnsley: one quantizes the functions $f \in C^\infty(M, \mathbb{R})$ which satisfy

$$\mathcal{L}_{X_f} I \equiv 0 \qquad \Leftrightarrow \quad f - \quad \text{Berezin} \quad \text{symbol}$$

Berezin - Töplitz: via Szöge projector

$$\mathrm{Pr} : \Gamma(\mathrm{M}, \mathrm{L}) \to \mathrm{H}^0(\mathrm{M_I}, \mathrm{L}) - \quad \text{orthogonal} \quad \text{projector},$$

then

$$Q_f s = \mathrm{Pr}(\mathrm{f} \cdot \mathrm{s}) \in \mathrm{H}^0(\mathrm{M_I}, \mathrm{L})$$

Another way in Geometric Quantization was called **Lagrangian approach**:

Submanifold $S \subset M$ is called **Lagrangian** if $\omega|_S \equiv 0$ and $\mathrm{dim S = n}$

Lagrangian submanifold S is called **Bohr - Sommerfeld** if $(L, a)|_S$ admits covariantly constant section σ_S (unique up to const)

If M is simply connected $\Rightarrow$ BS - condition does not depend on the choice of a

Fix topological type S and homology class $[S] \in H_n(M, \mathbb{Z})$ $\Rightarrow$ $\mathcal{B}_S$ - **moduli space of Bohr - Sommerfeld lagrangian submanifolds**:

infinite dimensional real analytic manifold

— can be exploited in the Quantization story due to the following geometrical remark:

For a function $f \in C^\infty(M, \mathbb{R})$ take its Hamiltonian vector field $X_f = \omega^{-1}(\mathrm{dF})$ and then consider the flow $\Phi^t_{X_f}$ generated by the field:

since the BS - condition is **stable** with respect to Hamiltonian deformation $\Rightarrow$ it generates the corresponding evolution of $\mathcal{B}_S$

$$S \mapsto S_t = \Phi^t_{X_f}(S)$$

Fact: S is stationary point for the flow so

$$\Phi^t_{X_f}(S) = S \quad \text{if and only if} \quad f|_S = const$$

thus for this case we have the exact result for a "measure process"!

Therefore one can interpret certain "dressed" version of $\mathcal{B}_S$ as the quantum phase space of certain **non linear** version of Quantization procedure.

It leads to ALG(a) - quantization procedure for compact classical phase space (N. Tyurin, 2000), which naturally generalizes V. Maslov quasi classical approximation.

Thus we have two alternative approaches to GQ problem:

$\Gamma(M, L)$	*versus*	$\mathcal{B}_S$
sections of vector bundle	*or*	lagrangian submanifolds

But Quantization procedure must be universal, which assumes:

the answer must be independent on the way one has exploited to get it

therefore it is reasonable to study the problem:

Find in the direct product $\Gamma(M, L) \times \mathcal{B}_S$ **some universal object** such that one could transport geometrical ingredients of different approaches to GQ

Special Bohr - Sommerfeld geometry was invented to solve — at list partially — this problem.

Digression: **Mirror Symmetry** in the broadest context can be characterized (or even defined?) as a **duality** between

complex geometry | *symplectic geometry*

of Kahler manifolds: from $\mathbb{R}$ - geometry point of view Kahler manifold $= (M, I, \omega)$ where I is a *complex structure* and ω is a Kahler form $=$ *symplectic form*.
Thus every Kahler manifold carries two geometries — complex and symplectic — therefore must be studied from these two different viewpoints.
Main interest: compact algebraic variety, by the very definition admits Kahler form of the Hodge type ($[\omega] \in H^2(M, \mathbb{Z})$), which is not unique of course.
Duality means that for mirror partners M, W certain derivation from complex geometry of M is equivalent to the corresponding derivation from symplectic geometry of W and *vice versa*.
Example: Homological Mirror Symmetry by M. Kontsevich says that derived category of coheret sheaves $D^b(CohX)$ and Fukaya -Floer catogery $FF(W)$ are equivalent.

But if we would like to study geometrical objects:

complex geometry	|	**symplectic geometry**
$\Downarrow$	|	$\Downarrow$
complex submanifolds	|	symplectic submanifolds
holomorphic vector bundles	|	lagrangian submanifolds
$\Downarrow$	|	$\Downarrow$
linear systems	|	?
(semi) stable vector bundles	|	?
	main difference	
finitness	|	infinitness

Example: Special Lagrangian Geometry for Calabi - Yau manifolds, proposed by N. Hitchin and J. McLean: finite dimemsional moduli space formed by lagrangian submanifolds which satisfy some *speciality* condition.

Main Question: *Is it possible to construct a finite dimensional moduli space of certain "special" lagrangian submanifolds for arbitrary Kahler (or algebraic) variety?*

Special Bohr - Sommerfeld geometry appears as an elaboration of ALAG - programme, proposed by Andrey Tyurin and Alexey Gorodentsev in 1999: for a given compact simply connected symplectic manifold with integer symplectic form

$$\mathrm{ALAG} : (M, \omega, L, a) \to \mathcal{B}_S$$

the *moduli space of Bohr - Sommerfeld lagrangian cycles*, infinite dimensional Frechet smooth real manifold.
Recall the main definition: (M, ω) — compact simply connected symplectic manifold of $\dim = 2\mathrm{n}$, $[\omega] \in H^2(M, \mathbb{Z})$;
(L, a) — prequantization pair, $L \to M$, $a \in \mathcal{A}_h(L)$ s.t. $F_a = 2\pi i \omega$.

Def 0: *n - dimensional $S \subset M$ is lagrangian iff $\omega|_S \equiv 0$.*

Def 1: *Lagrangian $S \subset M$ is Bohr - Sommerfeld iff $(L, a)|_S$ admits covariantly constant section σ_S.*

Remark. *Bohr - Sommerfeld property for lagrangian submanifold $S \subset M$ does not depend on the choice of prequantization connection a,* **BUT** *σ_S does depend: $\nabla_a \mapsto \nabla_a + idF \implies \sigma_S \mapsto e^{-iF} \sigma_S$.*

Fix a class $[S] \in H_n(M, \mathbb{Z})$ and a smooth compact orientable n - dimensional source manifold S_0. Then

$$\mathcal{B}_S = \{\phi : S_0 \hookrightarrow M | [\phi(S_0)] = [S], \phi(S_0) \text{is B-S}\} / \{\mathrm{Diff} S_0\}$$

is the **moduli space of B-S cycles** of fixed topological type, locally modelled by $C^\infty(S_0, \mathbb{R})$ modulo constants; it is an infinite dimensional real Frechet smooth manifold (ALAG programme).
Now we can add a new definition in the story:

Def 2: *B-S lagrangian $S \subset M$ is special w.r.t. a smooth section $\alpha \in \Gamma(M, L)$ (or α - SBS for short) iff $\alpha|_S = fe^{it}\sigma_S$, where f is a real strictly positive function and $t \in \mathbb{R}$.*

In particular $S \cap D_\alpha = \emptyset$ where $D_\alpha = \{\alpha = 0\} \subset M$.
Reformulation: every smooth section $\alpha \in \Gamma(M, L)$ generates complex 1 - form $\rho_\alpha = \frac{\nabla_a \alpha}{\alpha} = \frac{<\nabla_a \alpha, \alpha>_h}{<\alpha, \alpha>_h}$ on $M \backslash D_\alpha$.
It satisfies

$$\mathrm{Re}\rho_\alpha = \mathrm{d}\ln|\alpha|_{\mathrm{h}}, \qquad \mathrm{dIm}\rho_\alpha = 2\pi\omega$$

.

In this terms

Def 2': *An* n *-dimensional* $S \subset M \backslash D_\alpha$ *is* α *- SBS iff* $\mathrm{Im} \rho_\alpha|_{\mathrm{S}} \equiv 0$.

Main Corollary: for a fixed smooth section $\alpha \in \Gamma(M, L)$ **no local deformations** for α - SBS lagrangian submanifolds! Therefore:

$$\begin{array}{ccc} \mathcal{B}_S & \times & \mathbb{P}\Gamma(M, L) \\ p_1 \nwarrow & \cup & \nearrow p_2 \\ & \mathcal{U}_{SBS}(a) & \end{array}$$

where $\mathcal{U}_{SBS}(a)$ is the *incidence cycle* for the relation "S is α - SBS".

Theorem: *The map* $p_2 : \mathcal{U}_{SBS}(a) \to \mathbb{P}\Gamma(M, L)$ *has discrete fibers; the image* $\mathrm{Im} p_2$ *is an open set in* $\mathbb{P}\Gamma(M, L)$*; the differential* $\mathrm{d} p_2$ *has trivial kernel at each point; no ramification appears in this covering picture.*

Corollary: *The set* $\mathcal{U}_{SBS}(a)$ *is weakly Kahler manifold.*

Now add to the story complex structure: (M, ω, I), where I is compatible with ω and integrable. It means that we are speaking about an **algebraic variety**: the Kahler form ω is integer, so the corresponding Kahler metric is of the **Hodge type**.

In the presence of I: the bundle (L, a) is equipped with a *holomorphic structure* since $F_a \in \Omega^{1,1}_M$ thus

$$p_2 : \mathcal{U}_{SBS}(a) \to \mathbb{P}\Gamma(M, L) \supset \mathbb{P}H^0(M_I, L) - \text{finite subspace}$$

and we are almost done since:

the total preimage $p_2^{-1}(\mathbb{P}H^0(M_I, L)_\delta) \subset \mathcal{U}_{SBS}(a)$ where $\mathbb{P}H^0(M_I, L)_\delta$ is a slightly deformed $\mathbb{P}H^0(M_I, L)$ by a generic sufficiently small variation δ — a finite dimensional discrete covering of the finite dimensional projective space!

Theorem. *The total preimage* $p_2^{-1}(\mathbb{P}H^0(M_I, L)_\delta) \subset \mathcal{U}_{SBS}(a)$ *does not depend on the variation* δ.

therefore we can define a finite dimensional object

$$\mathcal{M}_{SBS}(c_1(L), \mathrm{topS}, [\mathrm{S}]) = \mathrm{p}_2^{-1}(\mathbb{P}\mathrm{H}^0(\mathrm{M_I}, \mathrm{L})_\delta) \subset \mathcal{U}_{\mathrm{SBS}}(\mathrm{a})$$

Thus we get correspondence:

$$(M, \omega, I, L, a) \mapsto \mathcal{M}_{SBS}(c_1(L), \mathrm{topS}, [\mathrm{S}])$$

in terms of **lagrangian geometry** of M.

Moreover, known examples show that we have the following natural
Conjecture. *The moduli space* $\mathcal{M}_{SBS}(c_1(L), \mathrm{topS}, [\mathrm{S}])$ *admits the structure of open algebraic variety.*

Coming back to Geometric Quantization:
the moduli space $\mathcal{M}_{SBS}(c_1(L), \mathrm{topS}, [\mathrm{S}])$ covers the quantum phase space $\mathbb{P}H^0(M_I, L)$ of the Berezin - Töplitz method as well as Berezin - Rawnsley method therefore one could expect that it exists non linear versions of these methods in terms of Special Bohr - Sommerfeld geometry.

---

work in progress...