Strong fluctuation effects for annihilating/coalescing Brownian particles

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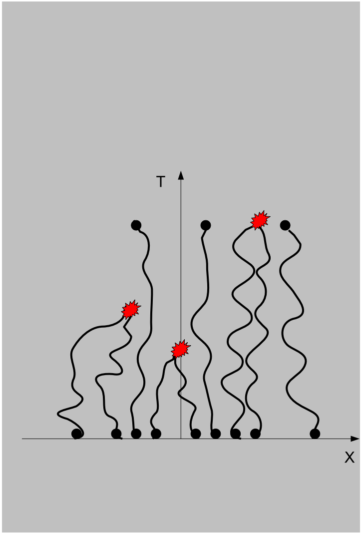
Outline

Annihilating/Coalescing Brownian motions



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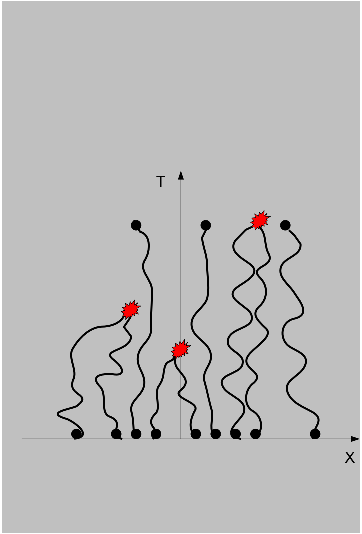

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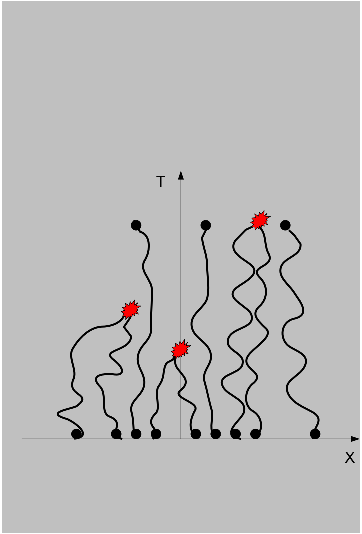

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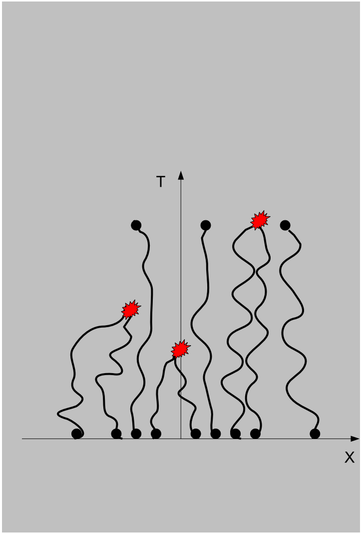

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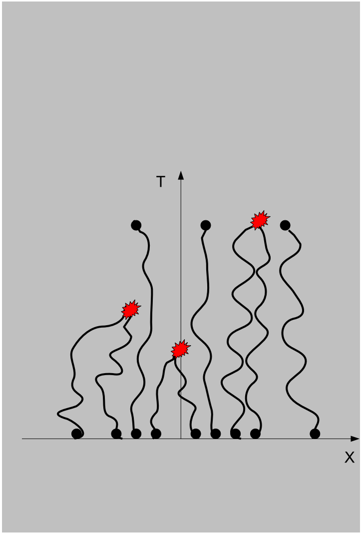

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- **Correlation functions**: $\rho_t^{(n)}(x_1, \ldots, x_n) d^d x_1 \ldots d^d x_n$

Statistics of ABM's, CBM's

- *Some* contributors: Smoluchowski, Glauber, Bramson, Lebowitz, Griffeath, Doi, Zeldovich, Ovchinnikov, Peliti, Droz, Lee, Cardy, Kesten, Derrida, Hakim, Pasquier, ben Avraham, Masser

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  - $d=1$: $\rho_t^{(n)} \sim t^{-\frac{n}{2}-\frac{n(n-1)}{4}}$
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- **Aim**: confirm $\gamma_n = \frac{n(n-1)}{2}$ - the spectrum of anomalous dimensions - rigorously

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- Exact equations: $\dot{\rho}^{(n)} = F_n[\rho^{(n)}, \rho^{(n+1)}]$ (Hopf chain)

Exact solvability (coalescing case)

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  3. $\rho_t^{(n)}(x_1, x_3, \ldots, x^{2n-1}) = \left(\prod_{k=1}^{n}(-\partial_{2k})\Phi_t^{(2n)}\right)|_{(x_{2m}=x_{2m-1}, m=1,2,\ldots,n)}$

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- Distributions of particles of this type are called *Pfaffian point processes*

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- Compare $\Phi_t^{(2n)}$ with similar observables for ASEP (Borodin, Corwin, Sasamoto)

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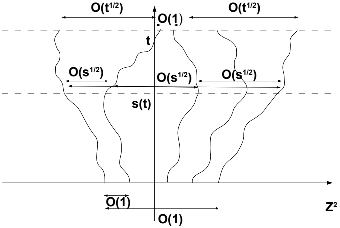
Non-coalescing Brownian disks in $2d$



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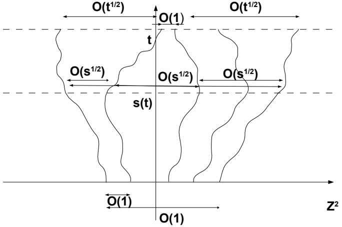

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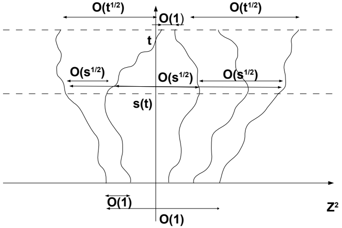

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- **Conclusion:** $p_{NC}^{(n)}(\mathbf{x}, t) = c^{(n)}(\mathbf{x}) \log(t)^{-n(n-1)/2}$ as $t \to \infty$

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- Bootstrapping to this:
  1. If there are n particles at time t at positions $x_1, x_2, \ldots, x_n$, there must exist n particles at time $t - s(t)$ at well separated positions, which do not meet before t

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- Bootstrapping to this:
  1. If there are n particles at time t at positions $x_1, x_2, \ldots, x_n$, there must exist n particles at time $t - s(t)$ at well separated positions, which do not meet before t
  2. $s(t) \sim t/\log(t)^\alpha$, $s(t) \to \infty$ as $t \to \infty$; $s(t) << t$

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- Asymptotically exact non-linear scaling can be established both for $d = 1$ and $d = 2$.

References

1. *Multi-Scaling of the n-Point Density Function for Coalescing Brownian Motions* , CMP Vol. **268**, No. **3**, (2006)
2. *Pfaffian formulae for one dimensional coalescing and annihilating systems*, EJP, vol. **16**, Article 76 (2011)
3. *Interacting particle systems on* $\mathbf{Z}$ *as Pfaffian point processes I - annihilating and coalescing random walks*, arXiv:1507.01843
4. *Interacting particle systems on* $\mathbf{Z}$ *as Pfaffian point processes II - coalescing branching random walks and annihilating random walks with immigration*, arXiv:1605.09668
5. *Multi-point correlations for two dimensional coalescing random walks*, arXiv:1707.06250