Random surface growth in random environment: renormalization group analysis of infinite-dimensional model

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Physics of Elementary Particles and Statistical Physics

In this talk we deal with two areas of physic: statistical physics and high energy physics:

- Pavlik's model (extended Kardar-Parizi-Zhang model) describing the growth of the surface;
- Navier-Stokes equation describing the moving of the media;
- stochastic description of the system;
- functional integration and calculation of Feynman graphs;
- renormalization group (RG).

The problem under consideration is influence of a random environment on the dynamics of a fluctuating rough surface.

Plan of the talk

The main steps (general scheme) are following:

- stochastic formulation of the model;
- quantum field action and Feynman diagrams;
- divergences of the diagrams;
- renormalization, RG, RG flow and fixed points;
- critical dimensions at different fixed points.

Kardar-Parisi-Zhang equation

Famous (simplest non-linear) Kardar-Parisi-Zhang equation of surface growth reads

$$\partial_t h = \kappa_0 \partial^2 h + \frac{\lambda_0}{2} (\partial_i h) (\partial_i h) + \eta,$$

where $\eta = \eta(x)$ is a random Gaussian noise with zero mean $\langle \eta \rangle = 0$ and the pair correlator

$$\langle \eta(x) \, \eta(x') \rangle = B_0 \, \delta \left(t - t' \right) \, \delta^{(d)}(\mathbf{x} - \mathbf{x}').$$

Problem: the only non-trivial IR attractive fixed point has coordinate $g_*^2 = -16\pi\varepsilon$, so it is unattainable for RG flow.

Numerical simulations show that some "good" fixed point exists, so we believe that this point is non-perturbative.

Pavlik's model

One of the possible generalizations of the KPZ model was suggested by Pavlik:

$$\partial_t h = \kappa_0 \partial^2 h + U(h) + \eta$$

with $U(h) \simeq \partial^2 h^2/2 = (\partial h)^2 + h\partial^2 h$.

However, such model is not self-suficient and infinite number of non-linear terms $\partial^2 h^n$ with $n \geq 2$ must be included because all of them are equally relevant.

So, we deal with model that involve infinitely many coupling constants:

$$\partial_t h = \partial^2 V(h) + \eta, \quad V(h) = \sum_{n=1}^{\infty} \frac{1}{n!}\, \lambda_{n0}\, h^n.$$

Turbulence: Navier-Stokes equation

The advection by turbulent environment is introduced by the "minimal" replacement $\partial_t \to \nabla_t = \partial_t + (v\partial)$.

Turbulent environment is described by stochastic Navier-Stokes equation

$$\nabla_t v_i - \nu_0 \partial^2 v_i + \partial_i \wp - f_i = 0,$$

where $\wp$ is the pressure, f_i is random force and ν_0 is the kinematic coefficient of viscosity.

Navier-Stokes equation: local term

The force $\mathbf{f}$ is assumed to be Gaussian with zero mean and a given pair correlation function:

$$\langle f_i(x) f_j(x') \rangle_f = \delta(t-t') \int_{k>m} \frac{d\mathbf{k}}{(2\pi)^d} P_{ij}(\mathbf{k}) D_f(k) \exp\{\mathrm{i}\mathbf{k}(\mathbf{x}-\mathbf{x}')\},$$

where P_{ij} is the transverse projector.

The correlation function $D_f(k)$ for fully developed turbulence should be chosen in power-like form (large-scale stirring):

$$D_f(k) = D_0 k^{4-d-y}, \quad D_0 > 0.$$

Navier-Stokes equation: local term

However, Pavlik's model is renormalizable in $d = 2$, where Navier-Stokes equation has additional divergence in function $\langle v' v' \rangle$.

To absorb this divergence we should introduce local term:

$$D_f(k) = D_0 k^{4-d-y} + D_0' k^2, \qquad D_0' > 0.$$

This leads to appearance of one more coupling constant $g' = D_0'/\nu^3$ (together with $g = D_0/\nu^3$) and double y and $\varepsilon = 2 - d$ expansion.

This completes definition of the model.

Action functional: General rules

Theorem: any stochastic equation of the type

$$\partial_t \phi(x) = U(x, \phi) + f(x), \quad \langle f(x) f(x') \rangle = D(x, x'),$$

where $\phi(x) = \phi(t, \mathbf{x})$ is a random field, $U(x, \phi)$ is a t-local functional depending on the fields and their derivatives, $f(x)$ is a random force, **is equivalent to quantum field model** of the double set of fields $\widetilde{\phi} = \{\phi, \phi'\}$ and action functional

$$S[\varphi] = \underbrace{\frac{1}{2} \varphi' D \varphi'}_{\text{noise term}} + \varphi' \underbrace{[-\partial_t \varphi + U]}_{\text{dynamics}},$$

integration over t and $\mathbf{x}$ implied.

Action functional: General rules

What does it mean:

- statistical average is equivalent to functional integration with weight $\exp S[\phi]$;
- classical random field $\rightarrow$ quantum field;
- we may use all techniques from quantum field theory: Feynman graphs, renormalization group, operator product expansion, *etc.*

Actions functional

Quantum field action is $\mathcal{S}(\Phi) = \mathcal{S}_h(\Phi) + \mathcal{S}_v(\Phi)$, where

$$\mathcal{S}_h(\Phi) = \frac{1}{2} h' B_0 h' + h' \left[-\nabla_t h + \partial^2 V(h) \right],$$

$$\mathcal{S}_v(\Phi) = \frac{1}{2} {v_i}' D_f P_{ij} {v_j}' + {v_j}' \left[-\nabla_t v_j + \nu_0 \partial^2 v_j \right].$$

All integrations are implied:

$$h' \nabla_t h = \int dt \int d\mathbf{x} h'(t, \mathbf{x}) \nabla_t h(t, \mathbf{x}).$$

Renormalization

To renormalize our model we should introduce remormalization constants Z:

$$\lambda_{n0} = \lambda_n Z_n, \quad \nu_0 = \nu Z_\nu, \quad g'_0 = g' \mu^\varepsilon Z_{g'}, \quad g_{n0} = g_n \mu^{(n-1)\varepsilon/2} Z_{g_n},$$

$$g_0 = g \mu^y Z_g, \quad Z_h = Z_{h'} = Z_v = Z_{v'} = 1,$$

where

$g_{n0} = \lambda_{n0}/\nu_0^{(n+1)/2}$, $g_0 = D_0/\nu_0^3$ and $g'_0 = D'_0/\nu_0^3$ are couplings.

Renormalization

Renormalized action functional has form

$$\mathcal{S}_{hR}(\Phi) = \frac{1}{2} h' h' + h' \left\{ -\nabla_t h + \partial^2 V_R\left(h\right) \right\},$$

$$\mathcal{S}_{vR} = \frac{1}{2} v' \left[g \nu^3 \mu^y k^{2+\varepsilon-y} + Z_{ii} g' \nu^3 \mu^\varepsilon k^2 \right] v' + v' \left[-\nabla_t v + Z_i \nu \partial^2 v \right],$$

where

$$V_R(h) = \sum_{n=1}^{\infty} \frac{1}{n!} Z_n \lambda_n h^n \quad \text{and}$$

$$Z_\nu = Z_i, \quad Z_{g_n} = Z_n Z_i^{-(n+1)/2}, \quad Z_g = Z_i^{-3}, \quad Z_{g'} = Z_{ii} Z_i^{-3}.$$

Z_i and Z_{ii} (Navier-Stokes term)

Renormalization constants Z_i and Z_{ii} are calculated from two graphs by usual way:

Result is

$$Z_i = 1 - \frac{g}{32\pi y} - \frac{g'}{32\pi\varepsilon} + \dots$$

$$Z_{ii} = 1 - \frac{g^2}{32\pi g'\,(2y-\varepsilon)} - \frac{g}{16\pi y} - \frac{g'}{32\pi\varepsilon} + \dots$$

Infinite set of Z_n

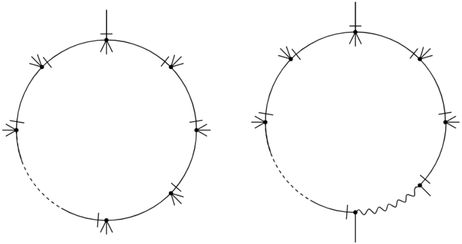
To calculate Z_n we used loop expansion of generating functional of 1-irreducible functions:

$$\Gamma_R(\Phi) = \sum_{p=0}^{\infty} \Gamma^{(p)}(\Phi), \quad \Gamma^{(0)}(\Phi) = \mathcal{S}_R(\Phi), \quad \Gamma^{(1)}(\Phi) = -\frac{1}{2}\left\{ \ln \frac{W}{W_0} \right\},$$

where

$$W(x, x') = -\frac{\delta^2 \mathcal{S}_{hR}(\Phi)}{\delta\Phi(x)\delta\Phi(x')} \, .$$

There are two types of graphs with divergent terms:



Infinite set of Z_n

From analysis of infinite set of these diagrams it follows that

$$\Gamma^{(1)}(\Phi) \simeq \frac{a_1}{\varepsilon} \left(\frac{\mu}{m}\right)^{\varepsilon} \int dx\, h'(x)\, \partial^2 F_1\left(h(x)\right) +$$

$$+ a_2 \left\{ \frac{g}{y} \left(\frac{\mu}{m}\right)^{y} + \frac{g'}{\varepsilon} \left(\frac{\mu}{m}\right)^{\varepsilon} \right\} \int dx\, h'(x)\, \partial^2 F_2\left(h(x)\right),$$

where

$$F_1\left(h\right) = \mu^{-\varepsilon} \frac{V''(h)}{V'(h)}, \quad F_2\left(h\right) = \int_0^h d\tilde{h}\, \frac{\nu^2}{\nu + V'\left(\tilde{h}\right)}$$

and

$$a_1 = \frac{S_d}{4(2\pi)^d}; \qquad a_2 = \frac{d-1}{2d} \cdot \frac{S_d}{(2\pi)^d}.$$

Infinite set of Z_n

Since we use perturbative RG, next step is to expand these relations in powers of the field h:

$$F_1(h) = \sum_{n=0}^{\infty} \frac{1}{n!}\, \mu^{\varepsilon(n-1)/2}\, \nu^{(n+1)/2}\, r_n\, h^n,$$

$$F_2(h) = \sum_{n=0}^{\infty} \frac{1}{n!}\, \mu^{\varepsilon(n-1)/2}\, \nu^{(n+1)/2}\, q_n\, h^n$$

with known dimensionless coefficients r_n and q_n.

Finally, the answer is

$$Z_n = 1 - \frac{1}{8\pi\varepsilon}\,\frac{r_n}{g_n} - g\,\frac{1}{8\pi y}\,\frac{q_n}{g_n} - g'\,\frac{1}{8\pi\varepsilon}\,\frac{q_n}{g_n}.$$

Just for the case

$$
\begin{aligned}
r_1 &= \frac{g_3}{g_1} - \left(\frac{g_2}{g_1}\right)^2, \\
r_2 &= \frac{g_4}{g_1} - 3\,\frac{g_3\,g_2}{g_1^2} + 2\left(\frac{g_2}{g_1}\right)^3, \\
r_3 &= \frac{g_5}{g_1} - 4\,\frac{g_4\,g_2}{g_1^2} - 3\left(\frac{g_3}{g_1}\right)^2 + 12\,\frac{g_3\,g_2^2}{g_1^3} - 6\left(\frac{g_2}{g_1}\right)^4;
\end{aligned}
$$

$$
q_1 = \frac{1}{(g_1+1)}, \quad q_2 = \frac{-g_2}{(g_1+1)^2}, \quad q_3 = \frac{-g_3}{(g_1+1)^2} + \frac{2g_2^2}{(g_1+1)^3}
$$

and so on.

RG functions and RG equation

RG equation reads

$$\left\{ \mathcal{D}_\mu - \beta_g \partial_g - \beta_{g'} \partial_{g'} - \sum_{n=1}^{\infty} \beta_n \partial_{g_n} - \gamma_\nu \right\} G(e; \dots) = 0,$$

where

$$\gamma_F = \widetilde{\mathcal{D}}_\mu \ln Z_F \quad \text{for any } F \ , \quad \beta_g = \widetilde{\mathcal{D}}_\mu \, g, \quad \beta_{g'} = \widetilde{\mathcal{D}}_\mu \, g', \quad \beta_n = \widetilde{\mathcal{D}}_\mu \, g_n \, .$$

RG functions in our model reads

$$\beta_g = g \left[-y + 3\gamma_i \right], \quad \beta_{g'} = g' \left[-\varepsilon + 3\gamma_i - \gamma_{ii} \right],$$

$$\beta_n - g_n \left[-(n-1)\varepsilon/2 - \gamma_n + (n+1)\gamma_i/2 \right].$$

$$\gamma_n = \frac{1}{8\pi} \cdot \frac{r_n}{g_n} + \frac{1}{8\pi} \cdot \frac{q_n(g+g')}{g_n}$$

$$\gamma_i = \frac{1}{32\pi} \left(g + g' \right), \quad \gamma_{ii} = \frac{1}{32\pi g'} \left(g + g' \right)^2 .$$

Fixed points

Fixed points (attractors of the RG equation) are governed by the requirement

$$\beta_g\left(g^*, g'^*\right) = 0, \quad \beta_{g'}\left(g^*, g'^*\right) = 0, \quad \beta_n\left(g^*, g'^*, {g_n}^*\right) = 0 \quad (n > 0).$$

For Navier-Stokes part (couplings g and g') three points exists:

$$(1) \quad g^* = 0\,, \quad g'^* = 0.$$

$$(2) \quad g^* = 0\,, \quad g'^* = 16\pi\varepsilon.$$

$$(3) \quad g^* = \frac{32\pi}{9}\frac{y(2y-3\varepsilon)}{(y-\varepsilon)}\,, \quad g'^* = \frac{32\pi}{9}\frac{y^2}{(y-\varepsilon)}.$$

Each of them produces regime for full model, which together with expressions for r_i and q_i give two-dimensional surface (g_1^*, g_2^*) in full space of couplings.

Conclusion

We applied methods of **quantum field theory** to the Pavlik's model of surface growth together with turbulent moving of the media described by Navier-Stokes equation.

- The key point is the possibility to reformulate initial stochastic problem into some quantum field theory.
- Feynman graphs are divergent. Renormalization group allows us to work with these objects and, moreover, provides critical dimensions of measurable quantities.
- Instead original KPZ model, Pavlik's model has three "fixed points" (two-dimensional surfaces) which looks to be reachable by RG flow.
- For first (trivial) point NS and Pavlik's model decouples, therefore there is loss of universality (Δ_ω are different for both models). Othere two regimes are universal.

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For more details see arXiv:2407.13783

Thank you for your attention!