Asymptotic method for modeling electromagnetic wave propagation in irregular waveguides

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Contents

Formulation of the problem

Consider Maxwell's equations for a case non-absorbing inhomogeneous isotropic media:

$$\nabla \times \vec{E} + \frac{\mu}{c}\frac{\partial \vec{H}}{\partial t} = \vec{0},$$
$$\nabla \times \vec{H} - \frac{\varepsilon}{c}\frac{\partial \vec{E}}{\partial t} = \vec{0}, \qquad (1)$$
$$\nabla \cdot \varepsilon \vec{E} = 0,$$
$$\nabla \cdot \mu \vec{H} = 0,$$

where c – speed of light in vacuum, ε – permittivity, μ – permeability.

Additional conditions

For multilayer waveguides the following conditions at the interface between dielectric media are satisfied:

$$\left[\vec{n}\times\vec{E}\right]_{(x,y,z)\in\Gamma}=\vec{0},\left[\vec{n}\times\vec{H}\right]_{(x,y,z)\in\Gamma}=\vec{0}, \qquad (2)$$

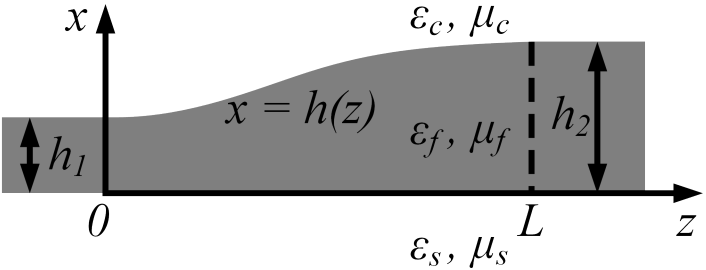
where $\vec{n}$ – normal to a surface Γ, $\left[\vec{h}\right]_{(x,y,z)\in\Gamma}$ represents jump in vector quantity $\vec{h}$ on the border $(x,y,z)\in\Gamma$. Asymptotic boundary conditions at infinity [1]:

$$\|\vec{E}\|\xrightarrow[\|x\|\to\infty]{}0,\|\vec{H}\|\xrightarrow[\|x\|\to\infty]{}0. \qquad (3)$$

Waveguide structure under consideration

We consider waveguides which are:

- *thin-film*: the thickness of waveguide layer h is comparable to the wavelength of light λ;
- *smoothly irregular*: geometry of waveguide layer satisfies the following constraints: $|\partial_y h| \ll 1, |\partial_z h| \ll 1$



Solution as asymptotic expansion

The solution to Maxwell's equations in proposed method is expressed as an asymptotic expansion with $(i\omega)^{-1}$ as a small parameter [4]:

$$\vec{E}(x, y, z, t) = \sum_{s=0}^{\infty} (i\omega)^{-s} \vec{E}_s(x, y, z) e^{i\omega t - ik\varphi(z)},$$
$$\vec{H}(x, y, z, t) = \sum_{s=0}^{\infty} (i\omega)^{-s} \vec{H}_s(x, y, z) e^{i\omega t - ik\varphi(z)}, \quad (4)$$

where s – asymptotic expansion index, $\vec{E}_s$ and $\vec{H}_s$ - the corresponding contributions to the electric and magnetic field strengths of the order s, $\omega \gg 1$ - angular frequency, k - wave number in vacuum, $\varphi(z)$ - phase.
For considered waveguides differential operators w.r.t. x, y, z give different values in order of magnitude: $\partial_y, \partial_z \sim (i\omega)^{-1} \partial_x$.

Applying computer algebra tools

The convenience of asymptotic methods is the ability to obtain intermediate results in symbolic form, therefore we use computer algebra tools, specifically Python's `SymPy` package [3], to reduce Maxwell's equations and construct symbolic representation of their solution.

Steps in symbolic investigation

The symbolic investigation of proposed method can be described in four steps:

1. Substitute in the system (1) $\vec{E}$ and $\vec{H}$ with their asymptotic expansion, respectively, then reduce the system;
2. Separate the reduced system into algebraic and differential parts; find solution to the system of differential equations;

And when considering a specific waveguide structure:

3. Construct the solution for each layer, taking into account asymptotic conditions at infinity (3);
4. From boundary conditions (2) we form a system of linear algebraic equations (SLAE) w.r.t. arbitrary constants and find its solution;

0^{th} approximation: ODE problem

In 0^{th} approximation system (1) can be reduced into a system of two algebraic equations (5) and four differential equations (6)

$$\begin{gathered} E_0^x(x,z) = \varepsilon^{-1}\varphi'(z)H_0^y(x,z), \\ H_0^x(x,z) = -\mu^{-1}\varphi'(z)E_0^y(x,z) \end{gathered} \tag{5}$$

$$\begin{gathered} \partial_x E_0^y(x,z) + ik\mu H_0^z(x,z) = 0, \\ \partial_x E_0^z(x,z) + ik\varepsilon^{-1}\eta(z)H_0^y(x,z) = 0, \\ \partial_x H_0^y(x,z) - i\varepsilon k E_0^z(x,z) = 0, \\ \partial_x H_0^z(x,z) - ik\mu^{-1}\eta(z)E_0^y(x,z) = 0. \end{gathered} \tag{6}$$

where $\eta(z) = \varphi'(z)^2 - \varepsilon\mu$

0^{th} approximation: General solution

General solution of system (6) is obtained symbolically:

$$\vec{U}_0 = C_1 \vec{v}_1 e^{-\gamma x} + C_2 \vec{v}_2 e^{\gamma x} + C_3 \vec{v}_3 e^{-\gamma x} + C_4 \vec{v}_4 e^{\gamma x}, \qquad (7)$$

where $\gamma(z) = k\sqrt{\eta(z)}$, $\vec{U}_0 = \left(E_0^y, E_0^z, H_0^y, H_0^z\right)^T$ and $\vec{v}_j$:

$$\vec{v}_1 = \begin{pmatrix} i\mu \\ 0 \\ 0 \\ \sqrt{\eta(z)} \end{pmatrix}, \vec{v}_2 = \begin{pmatrix} -i\mu \\ 0 \\ 0 \\ \sqrt{\eta(z)} \end{pmatrix},$$

$$\vec{v}_3 = \begin{pmatrix} 0 \\ i\sqrt{\eta(z)} \\ \varepsilon \\ 0 \end{pmatrix}, \vec{v}_4 = \begin{pmatrix} 0 \\ -i\sqrt{\eta(z)} \\ \varepsilon \\ 0 \end{pmatrix}.$$

0^{th} approximation: Solution in each layer

For considered waveguide structure solution in each layer takes form:

$$\vec{U}_0^c = A_0^c \vec{v}_1^c e^{-\gamma_c (x - h(z))} + B_0^c \vec{v}_3^c e^{-\gamma_c (x - h(z))},$$
$$\vec{U}_0^f = A_0^f \vec{v}_1^f e^{-\gamma_f x} + B_0^f \vec{v}_2^f e^{\gamma_f x} + C_0^f \vec{v}_3^f e^{-\gamma_f x} + D_0^f \vec{v}_4^f e^{\gamma_f x}, \qquad (8)$$
$$\vec{U}_0^s = A_0^s \vec{v}_2^s e^{\gamma_s x} + B_0^s \vec{v}_4^s e^{\gamma_s x},$$

$$\vec{v}_1^\alpha = \begin{pmatrix} i\mu_\alpha \\ 0 \\ 0 \\ \sqrt{\eta_\alpha} \end{pmatrix}, \vec{v}_2^\alpha = \begin{pmatrix} -i\mu_\alpha \\ 0 \\ 0 \\ \sqrt{\eta_\alpha} \end{pmatrix}, \vec{v}_3^\alpha = \begin{pmatrix} 0 \\ i\sqrt{\eta_\alpha} \\ \varepsilon_\alpha \\ 0 \end{pmatrix}, \vec{v}_4^\alpha = \begin{pmatrix} 0 \\ -i\sqrt{\eta_\alpha} \\ \varepsilon_\alpha \\ 0 \end{pmatrix},$$
$$\eta_\alpha = {\varphi'}^2 - \varepsilon_\alpha \mu_\alpha, \gamma_\alpha = k\sqrt{\eta_\alpha}, \alpha = \{c, f, s\}.$$

0^{th} approximation: Arbitrary constants

Symbolic form of arbitrary constants is found from substituting solution for each layer (8) in boundary conditions (2):

$$M\vec{\xi^0} = \vec{0}, \tag{9}$$

$$M = \begin{pmatrix} & & & & 0 & 0 & 0 & 0 \\ & M_1 & & & 0 & 0 & 0 & 0 \\ & & & & 0 & 0 & 0 & 0 \\ & & & & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & & & & \\ 0 & 0 & 0 & 0 & & & & \\ 0 & 0 & 0 & 0 & & M_2 & & \\ 0 & 0 & 0 & 0 & & & & \end{pmatrix}, \vec{\xi^0} = \begin{pmatrix} A_0^c \\ A_0^f \\ B_0^f \\ A_0^s \\ B_0^c \\ C_0^f \\ D_0^f \\ B_0^s \end{pmatrix}$$

0^{th} approximation: Arbitrary constants

System (9) is split into two subsystems: $\begin{cases} M_1\vec{\xi}_1^0 = \vec{0} \\ M_2\vec{\xi}_2^0 = \vec{0} \end{cases}$

$$M_1 = \begin{pmatrix} i\mu_c & -i\mu_f e^{-\gamma_f h} & i\mu_f e^{\gamma_f h} & 0 \\ -\sqrt{\eta_c} + ih'\varphi' & \left(\sqrt{\eta_f} - ih'\varphi'\right) e^{-\gamma_f h} & \left(\sqrt{\eta_f} + ih'\varphi'\right) e^{\gamma_f h} & 0 \\ 0 & i\mu_f & -i\mu_f & i\mu_s \\ 0 & -\sqrt{\eta_f} + ih'\varphi' & -\sqrt{\eta_f} - ih'\varphi' & \sqrt{\eta_s} + ih'\varphi' \end{pmatrix}$$

$$M_2 = \begin{pmatrix} -i\sqrt{\eta_c} - h'\varphi' & \left(i\sqrt{\eta_f} + h'\varphi'\right) e^{-\gamma_f h} & \left(-i\sqrt{\eta_f} + h'\varphi'\right) e^{\gamma_f h} & 0 \\ \varepsilon_c & -\varepsilon_f e^{-\gamma_f h} & -\varepsilon_f e^{\gamma_f h} & 0 \\ 0 & -i\sqrt{\eta_f} - h'\varphi' & i\sqrt{\eta_f} - h'\varphi' & -i\sqrt{\eta_s} + h'\varphi' \\ 0 & \varepsilon_f & \varepsilon_f & -\varepsilon_s \end{pmatrix}$$

$$\vec{\xi}_1^0 = \begin{pmatrix} A_0^c \\ A_0^f \\ B_0^f \\ A_0^s \end{pmatrix}, \vec{\xi}_2^0 = \begin{pmatrix} B_0^c \\ C_0^f \\ D_0^f \\ B_0^s \end{pmatrix}$$

0^{th} approximation: Solution of SLAE

We obtain symbolic solution of system (9) by applying previously developed method [2]:

$$
\begin{gathered}
A_0^c = -2i\mu_f\sqrt{\eta_f}, \\
A_0^f = \left(-i\mu_c\sqrt{\eta_f} - i\mu_f\sqrt{\eta_c} + (\mu_c - \mu_f)\, h'\varphi'\right) e^{\gamma_f h}, \\
B_0^f = \left(i\mu_c\sqrt{\eta_f} - i\mu_f\sqrt{\eta_c} + (\mu_c - \mu_f)\, h'\varphi'\right) e^{-\gamma_f h}, \\
A_0^s = -\mu_f\mu_s^{-1}\left(-i\mu_c\sqrt{\eta_f} - i\mu_f\sqrt{\eta_c} + (\mu_c - \mu_f)\, h'\varphi'\right) e^{\gamma_f h} + \\
+\mu_f\mu_s^{-1}\left(i\mu_c\sqrt{\eta_f} - i\mu_f\sqrt{\eta_c} + (\mu_c - \mu_f)\, h'\varphi'\right) e^{-\gamma_f h}, \\
B_0^c = -2i\varepsilon_f\sqrt{\eta_f}, \\
C_0^f = \left(-i\varepsilon_c\sqrt{\eta_f} - i\varepsilon_f\sqrt{\eta_c} + (\varepsilon_c - \varepsilon_f)\, h'\varphi'\right) e^{\gamma_f h}, \\
D_0^f = \left(-i\varepsilon_c\sqrt{\eta_f} + i\varepsilon_f\sqrt{\eta_c} + (\varepsilon_f - \varepsilon_c)\, h'\varphi'\right) e^{-\gamma_f h}, \\
B_0^s = \varepsilon_f\varepsilon_s^{-1}\left(-i\varepsilon_c\sqrt{\eta_f} - i\varepsilon_f\sqrt{\eta_c} + (\varepsilon_c - \varepsilon_f)\, h'\varphi'\right) e^{\gamma_f h} - \\
-\varepsilon_f\varepsilon_s^{-1}\left(i\varepsilon_c\sqrt{\eta_f} - i\varepsilon_f\sqrt{\eta_c} + (\varepsilon_c - \varepsilon_f)\, h'\varphi'\right) e^{-\gamma_f h}.
\end{gathered}
\tag{10}
$$

1^{st} approximation: ODE problem

In the 1^{st} approximation system (1) also consists of two algebraic equations (11) and four differential equations (12), but the system is now inhomogeneous:

$$
\begin{aligned}
H_1^x &= -\mu^{-1}(\tilde{\varphi}'(z)E_1^y + c\partial_z E_0^y),\\
E_1^x &= -\varepsilon^{-1}(\tilde{\varphi}'(z)H_1^y + c\partial_z H_0^y),
\end{aligned}
\tag{11}
$$

$$
\begin{gathered}
\partial_x E_1^z + ik\varepsilon^{-1}\left(\tilde{\varphi}'^2 - \varepsilon\mu\right)H_1^y = \boxed{i\omega\varepsilon^{-1}\left(\varepsilon\partial_z E_0^x + \partial_z H_0^y \tilde{\varphi}'\right)},\\
\partial_x E_1^y(x) + ik\mu H_1^z(x) = 0,\\
\partial_x H_1^z - ik\mu^{-1}\left(\tilde{\varphi}'^2 - \varepsilon\mu\right)E_1^y = \boxed{i\omega\mu^{-1}\left(\mu\partial_z H_0^x - \partial_z E_0^y \tilde{\varphi}'\right)},\\
\partial_x H_1^y(x) - i\varepsilon k E_1^z(x) = 0.
\end{gathered}
\tag{12}
$$

1^{st} approximation: ODE solution

For inhomogenous systems of differential equations the solution can be represented as $\vec{U}_1 = \vec{U}_g + \vec{U}_p$, where:

$$\vec{U}_g = C_1 \vec{v}_1 e^{-\tilde{\gamma}x} + C_2 \vec{v}_2 e^{\tilde{\gamma}x} + +C_3 \vec{v}_3 e^{-\tilde{\gamma}x} + C_4 \vec{v}_4 e^{\tilde{\gamma}x}, \tag{13}$$

$$\vec{U}_p = \begin{pmatrix} -a_1 P_2(x,z) \\ a_2 P_4(x,z) \\ -a_1 P_4(x,z) \\ -a_3 P_2(x,z) \end{pmatrix} e^{-\tilde{\gamma}x} + \begin{pmatrix} a_1 P_1(x,z) \\ a_2 P_3(x,z) \\ a_1 P_3(x,z) \\ -a_3 P_1(x,z) \end{pmatrix} e^{\tilde{\gamma}x},$$

$$a_1 = \frac{\omega}{2\sqrt{\eta(z)}},\ a_2 = \frac{i\omega}{2\varepsilon},\ a_3 = \frac{i\omega}{2\mu},$$

$$\begin{aligned} P_1(x,z) &= \int \left(-\mu \partial_z H_0^x(x,z) + \partial_z E_0^y(x,z)\tilde{\varphi}'(z)\right) e^{\tilde{\gamma}x}\, dx, \\ P_2(x,z) &= \int \left(-\mu \partial_z H_0^x(x,z) + \partial_z E_0^y(x,z)\tilde{\varphi}'(z)\right) e^{-\tilde{\gamma}x}\, dx, \\ P_3(x,z) &= \int \left(\varepsilon \partial_z E_0^x(x,z) + \partial_z H_0^y(x,z)\tilde{\varphi}'(z)\right) e^{\tilde{\gamma}x}\, dx, \\ P_4(x,z) &= \int \left(\varepsilon \partial_z E_0^x(x,z) + \partial_z H_0^y(x,z)\tilde{\varphi}'(z)\right) e^{-\tilde{\gamma}x}\, dx \end{aligned} \tag{14}$$

1^{st} approximation: Solution in each layer

Solution in each layer takes form as $\vec{U}_1^{\alpha} = \vec{U}_g^{\alpha} + \vec{U}_p^{\alpha}$ for $\alpha = \{c, f, s\}$, where:

$$\begin{gathered}
\vec{U}_g^c = B_1^c \vec{v}_1^c e^{-\gamma_c (x - h(z))} + A_1^c \vec{v}_3^c e^{-\gamma_c (x - h(z))}, \\
\vec{U}_g^f = B_1^f \vec{v}_1^f e^{-\gamma_f x} + D_1^f \vec{v}_2^f e^{\gamma_f x} + A_1^f \vec{v}_3^f e^{-\gamma_f x} + C_1^f \vec{v}_4^f e^{\gamma_f x}, \\
\vec{U}_g^s = B_1^s \vec{v}_2^s e^{\gamma_s x} + A_1^s \vec{v}_4^s e^{\gamma_s x},
\end{gathered} \tag{15}$$

$\vec{U}_p^{\alpha}$ has the same symbolic structure as $\vec{U}_p$ with a difference in values of permittivity and permeability ε_α and μ_α for each layer α.

1^{st} approximation: Arbitrary constants

Symbolic form of arbitrary constants is found from substituting solution for each layer (15) in boundary conditions (2):

$$M\vec{\xi^1} = \vec{q}, \tag{16}$$

$$\vec{\xi^1} = \left(A_1^c, A_1^f, B_1^f, A_1^s, B_1^c, C_1^f, D_1^f, B_1^s\right)^T,$$

$$\vec{q} = \begin{pmatrix} -E_p^{y,c}(h(z),z) + E_p^{y,f}(h(z),z) \\ \left(H_p^{x,c}(h(z),z) - H_p^{x,f}(h(z),z)\right) h'(z) + H_p^{z,c}(h(z),z) - H_p^{z,f}(h(z),z) \\ -E_p^{y,f}(0,z) + E_p^{y,s}(0,z) \\ \left(H_p^{x,f}(0,z) - H_p^{x,s}(0,z)\right) h'(z) + H_p^{z,f}(0,z) - H_p^{z,s}(0,z) \\ \left(E_p^{x,c}(h(z),z) - E_p^{x,f}(h(z),z)\right) h'(z) + E_p^{z,c}(h(z),z) - E_p^{z,f}(h(z),z) \\ -H_p^{y,c}(h(z),z) + H_p^{y,f}(h(z),z) \\ \left(E_p^{x,f}(0,z) - E_p^{x,s}(0,z)\right) h'(z) + E_p^{z,f}(0,z) - E_p^{z,s}(0,z) \\ -H_p^{y,f}(0,z) + H_p^{y,s}(0,z) \end{pmatrix}$$

1^{st} approximation: Solution of SLAE

Solution to (16) takes the following form:

$$
\begin{aligned}
|M_1| = \Big(&\mu_c\mu_f\sqrt{\eta_f}\sqrt{\eta_s} - \mu_c\mu_s\eta_f - \mu_f^2\sqrt{\eta_c}\sqrt{\eta_s} + \mu_f\mu_s\sqrt{\eta_c}\sqrt{\eta_f} + \\
&+ i\mu_f\Big(\mu_c\sqrt{\eta_f} - \mu_c\sqrt{\eta_s} - \mu_f\sqrt{\eta_c} + \mu_f\sqrt{\eta_s} + \mu_s\sqrt{\eta_c} - \mu_s\sqrt{\eta_f}\Big)h'\tilde{\varphi}' + \\
&+ \left(\mu_c\mu_f - \mu_c\mu_s - \mu_f^2 + \mu_f\mu_s\right)\left(h'\right)^2\left(\tilde{\varphi}'\right)^2\Big)e^{-\tilde{\gamma}_f h} + \\
&+ \Big(\mu_c\mu_f\sqrt{\eta_f}\sqrt{\eta_s} + \mu_c\mu_s\eta_f + \mu_f^2\sqrt{\eta_c}\sqrt{\eta_s} + \mu_f\mu_s\sqrt{\eta_c}\sqrt{\eta_f} + \\
&+ i\mu_f\left(\mu_c\sqrt{\eta_f} + \mu_c\sqrt{\eta_s} + \mu_f\sqrt{\eta_c} - \mu_f\sqrt{\eta_s} - \mu_s\sqrt{\eta_c} - \mu_s\sqrt{\eta_f}\right)h'\tilde{\varphi}' + \\
&+ \left(-\mu_c\mu_f + \mu_c\mu_s + \mu_f^2 - \mu_f\mu_s\right)\left(h'\right)^2\left(\tilde{\varphi}'\right)^2\Big)e^{\tilde{\gamma}_f h},
\end{aligned}
$$

1^{st} approximation: Solution of SLAE

$$
\begin{aligned}
|M_2| = &\Big(\varepsilon_c\varepsilon_f\sqrt{\eta_f}\sqrt{\eta_s} - \varepsilon_c\varepsilon_s\eta_f - \varepsilon_f^2\sqrt{\eta_c}\sqrt{\eta_s} + \varepsilon_f\varepsilon_s\sqrt{\eta_c}\sqrt{\eta_f} + \\
&+ i\varepsilon_f\left(\varepsilon_c\sqrt{\eta_f} - \varepsilon_c\sqrt{\eta_s} - \varepsilon_f\sqrt{\eta_c} + \varepsilon_f\sqrt{\eta_s} + \varepsilon_s\sqrt{\eta_c} - \varepsilon_s\sqrt{\eta_f}\right)h'\tilde{\varphi}' + \\
&+ \left(\varepsilon_c\varepsilon_f - \varepsilon_c\varepsilon_s - \varepsilon_f^2 + \varepsilon_f\varepsilon_s\right)\left(h'\right)^2\left(\tilde{\varphi}'\right)^2\Big)e^{-\tilde{\gamma}_f h} + \\
&+ \Big(\varepsilon_c\varepsilon_f\sqrt{\eta_f}\sqrt{\eta_s} + \varepsilon_c\varepsilon_s\eta_f + \varepsilon_f^2\sqrt{\eta_c}\sqrt{\eta_s} + \varepsilon_f\varepsilon_s\sqrt{\eta_c}\sqrt{\eta_f} + \\
&+ i\varepsilon_f\left(\varepsilon_c\sqrt{\eta_f} + \varepsilon_c\sqrt{\eta_s} + \varepsilon_f\sqrt{\eta_c} - \varepsilon_f\sqrt{\eta_s} - \varepsilon_s\sqrt{\eta_c} - \varepsilon_s\sqrt{\eta_f}\right)h'\tilde{\varphi}' + \\
&+ \left(-\varepsilon_c\varepsilon_f + \varepsilon_c\varepsilon_s + \varepsilon_f^2 - \varepsilon_f\varepsilon_s\right)\left(h'\right)^2\left(\tilde{\varphi}'\right)^2\Big)e^{\tilde{\gamma}_f h},
\end{aligned}
$$

1st approximation: Solution of SLAE

$$
\begin{aligned}
A_1^c = \frac{1}{|M_1|} \Bigg(& 2\mu_f q_3 \sqrt{\eta_f} h' \tilde{\varphi}' + 2\mu_f \left(-\mu_s q_4 - i q_3 \sqrt{\eta_s}\right) \sqrt{\eta_f} + \\
& + \Big(-\mu_f^2 q_2 \sqrt{\eta_s} - \mu_f \mu_s q_2 \sqrt{\eta_f} - i \mu_f q_1 \sqrt{\eta_f} \sqrt{\eta_s} + \\
& + \mu_f \left(-i \mu_f q_2 + i \mu_s q_2 + q_1 \sqrt{\eta_f} + q_1 \sqrt{\eta_s}\right) h' \tilde{\varphi}' - i \mu_s q_1 \eta_f + \\
& + i q_1 \left(\mu_f - \mu_s\right) \left(h'\right)^2 \left(\tilde{\varphi}'\right)^2 \Big) e^{\tilde{\gamma}_f h} + \\
& + \Big(\mu_f^2 q_2 \sqrt{\eta_s} - \mu_f \mu_s q_2 \sqrt{\eta_f} - i \mu_f q_1 \sqrt{\eta_f} \sqrt{\eta_s} + \mu_f (i \mu_f q_2 - i \mu_s q_2 + \\
& + q_1 \sqrt{\eta_f} - q_1 \sqrt{\eta_s}) h' \tilde{\varphi}' + i \mu_s q_1 \eta_f + i q_1 \left(-\mu_f + \mu_s\right) \left(h'\right)^2 \left(\tilde{\varphi}'\right)^2 \Big) e^{-\tilde{\gamma}_f h} \Bigg)
\end{aligned}
$$

1^{st} approximation: Solution of SLAE

$$A_1^f = \frac{1}{|M_1|}\Bigg(\mu_c\mu_f q_2\sqrt{\eta_s} - \mu_c\mu_s q_2\sqrt{\eta_f} - i\mu_f q_1\sqrt{\eta_c}\sqrt{\eta_s} + i\mu_s q_1\sqrt{\eta_c}\sqrt{\eta_f} +$$

$$+ iq_1\left(-\mu_f + \mu_s\right)\left(h'\right)^2\left(\tilde{\varphi}'\right)^2 + (i\mu_c\mu_f q_2 - i\mu_c\mu_s q_2 + \mu_f q_1\sqrt{\eta_c} - \mu_f q_1\sqrt{\eta_s} -$$

$$- \mu_s q_1\sqrt{\eta_c} + \mu_s q_1\sqrt{\eta_f})h'\tilde{\varphi}' + \Big(-\mu_c\mu_s q_4\sqrt{\eta_f} - i\mu_c q_3\sqrt{\eta_f}\sqrt{\eta_s} - \mu_f\mu_s q_4\sqrt{\eta_c} -$$

$$- i\mu_f q_3\sqrt{\eta_c}\sqrt{\eta_s} + iq_3\left(\mu_c - \mu_f\right)\left(h'\right)^2\left(\tilde{\varphi}'\right)^2 + (-i\mu_c\mu_s q_4 + \mu_c q_3\sqrt{\eta_f} +$$

$$+ \mu_c q_3\sqrt{\eta_s} + i\mu_f\mu_s q_4 + \mu_f q_3\sqrt{\eta_c} - \mu_f q_3\sqrt{\eta_s})h'\tilde{\varphi}'\Big)e^{\tilde{\gamma}_f h}\Bigg)$$

1^{st} approximation: Solution of SLAE

$$
\begin{aligned}
B_1^f = \frac{1}{|M_1|}\Bigg(&\mu_c\mu_f q_2\sqrt{\eta_s} + \mu_c\mu_s q_2\sqrt{\eta_f} - i\mu_f q_1\sqrt{\eta_c}\sqrt{\eta_s} - i\mu_s q_1\sqrt{\eta_c}\sqrt{\eta_f} + \\
&+ iq_1\left(-\mu_f + \mu_s\right)\left(h'\right)^2\left(\tilde{\varphi}'\right)^2 + (i\mu_c\mu_f q_2 - i\mu_c\mu_s q_2 + \mu_f q_1\sqrt{\eta_c} - \mu_f q_1\sqrt{\eta_s} - \\
&- \mu_s q_1\sqrt{\eta_c} - \mu_s q_1\sqrt{\eta_f})h'\tilde{\varphi}' + \Big(\mu_c\mu_s q_4\sqrt{\eta_f} + i\mu_c q_3\sqrt{\eta_f}\sqrt{\eta_s} - \mu_f\mu_s q_4\sqrt{\eta_c} - \\
&- i\mu_f q_3\sqrt{\eta_c}\sqrt{\eta_s} + iq_3\left(\mu_c - \mu_f\right)\left(h'\right)^2\left(\tilde{\varphi}'\right)^2 + (-i\mu_c\mu_s q_4 - \mu_c q_3\sqrt{\eta_f} + \\
&+ \mu_c q_3\sqrt{\eta_s} + i\mu_f\mu_s q_4 + \mu_f q_3\sqrt{\eta_c} - \mu_f q_3\sqrt{\eta_s})h'\tilde{\varphi}'\Big)e^{-\tilde{\gamma}_f h}\Bigg)
\end{aligned}
$$

1^{st} approximation: Solution of SLAE

$$A_1^s = \frac{1}{|M_1|}\Bigg(-2\mu_f q_1\sqrt{\eta_f}h'\tilde{\varphi}' + 2\mu_f\left(\mu_c q_2 - iq_1\sqrt{\eta_c}\right)\sqrt{\eta_f} + \Big(\mu_c\mu_f q_4\sqrt{\eta_f} -$$

$$- i\mu_c q_3\eta_f + \mu_f^2 q_4\sqrt{\eta_c} - i\mu_f q_3\sqrt{\eta_c}\sqrt{\eta_f} + \mu_f(i\mu_c q_4 - i\mu_f q_4 - q_3\sqrt{\eta_c} -$$

$$- q_3\sqrt{\eta_f})h'\tilde{\varphi}' + iq_3\left(-\mu_c + \mu_f\right)\left(h'\right)^2\left(\tilde{\varphi}'\right)^2\Big)e^{\tilde{\gamma}_f h} +$$

$$+ \Big(\mu_c\mu_f q_4\sqrt{\eta_f} + i\mu_c q_3\eta_f - \mu_f^2 q_4\sqrt{\eta_c} - i\mu_f q_3\sqrt{\eta_c}\sqrt{\eta_f} + \mu_f(-i\mu_c q_4 + i\mu_f q_4 +$$

$$+ q_3\sqrt{\eta_c} - q_3\sqrt{\eta_f})h'\tilde{\varphi}' + iq_3\left(\mu_c - \mu_f\right)\left(h'\right)^2\left(\tilde{\varphi}'\right)^2\Big)e^{-\tilde{\gamma}_f h}\Bigg)$$

1^{st} approximation: Solution of SLAE

$$B_1^c = \frac{1}{|M_2|}\Bigg(2i\varepsilon_f q_8\sqrt{\eta_f}h'\tilde{\varphi}' + 2\varepsilon_f\left(i\varepsilon_s q_7 + q_8\sqrt{\eta_s}\right)\sqrt{\eta_f} + \Big(-i\varepsilon_f^2 q_5\sqrt{\eta_s} +$$

$$+ i\varepsilon_f\varepsilon_s q_5\sqrt{\eta_f} + \varepsilon_f q_6\sqrt{\eta_f}\sqrt{\eta_s} + \varepsilon_f\left(\varepsilon_f q_5 - \varepsilon_s q_5 + iq_6\sqrt{\eta_f} - iq_6\sqrt{\eta_s}\right)h'\tilde{\varphi}' -$$

$$- \varepsilon_s q_6\eta_f + q_6\left(\varepsilon_f - \varepsilon_s\right)\left(h'\right)^2\left(\tilde{\varphi}'\right)^2\Big)e^{-\tilde{\gamma}_f h} +$$

$$+ \Big(i\varepsilon_f^2 q_5\sqrt{\eta_s} + i\varepsilon_f\varepsilon_s q_5\sqrt{\eta_f} + \varepsilon_f q_6\sqrt{\eta_f}\sqrt{\eta_s} + \varepsilon_f(-\varepsilon_f q_5 + \varepsilon_s q_5 + iq_6\sqrt{\eta_f} +$$

$$+ iq_6\sqrt{\eta_s})h'\tilde{\varphi}' + \varepsilon_s q_6\eta_f + q_6\left(-\varepsilon_f + \varepsilon_s\right)\left(h'\right)^2\left(\tilde{\varphi}'\right)^2\Big)e^{\tilde{\gamma}_f h}\Bigg)$$

1^{st} approximation: Solution of SLAE

$$C_1^f = \frac{1}{|M_2|}\Bigg(-i\varepsilon_c\varepsilon_f q_5\sqrt{\eta_s} + i\varepsilon_c\varepsilon_s q_5\sqrt{\eta_f} + \varepsilon_f q_6\sqrt{\eta_c}\sqrt{\eta_s} - \varepsilon_s q_6\sqrt{\eta_c}\sqrt{\eta_f} +$$

$$+ q_6\left(\varepsilon_f - \varepsilon_s\right)\left(h'\right)^2\left(\tilde{\varphi}'\right)^2 + (\varepsilon_c\varepsilon_f q_5 - \varepsilon_c\varepsilon_s q_5 + i\varepsilon_f q_6\sqrt{\eta_c} - i\varepsilon_f q_6\sqrt{\eta_s} -$$

$$- i\varepsilon_s q_6\sqrt{\eta_c} + i\varepsilon_s q_6\sqrt{\eta_f})h'\tilde{\varphi}' + \Big(i\varepsilon_c\varepsilon_s q_7\sqrt{\eta_f} + \varepsilon_c q_8\sqrt{\eta_f}\sqrt{\eta_s} + i\varepsilon_f\varepsilon_s q_7\sqrt{\eta_c} +$$

$$+ \varepsilon_f q_8\sqrt{\eta_c}\sqrt{\eta_s} + q_8\left(\varepsilon_f - \varepsilon_c\right)\left(h'\right)^2\left(\tilde{\varphi}'\right)^2 + (-\varepsilon_c\varepsilon_s q_7 + i\varepsilon_c q_8\sqrt{\eta_f} + i\varepsilon_c q_8\sqrt{\eta_s} +$$

$$+ \varepsilon_f\varepsilon_s q_7 + i\varepsilon_f q_8\sqrt{\eta_c} - i\varepsilon_f q_8\sqrt{\eta_s})h'\tilde{\varphi}'\Big)e^{\tilde{\gamma}_f h}\Bigg)$$

1^{st} approximation: Solution of SLAE

$$D_1^f = \frac{1}{|M_2|}\Bigg(i\varepsilon_c\varepsilon_f q_5\sqrt{\eta_s} + i\varepsilon_c\varepsilon_s q_5\sqrt{\eta_f} - \varepsilon_f q_6\sqrt{\eta_c}\sqrt{\eta_s} - \varepsilon_s q_6\sqrt{\eta_c}\sqrt{\eta_f} +$$

$$+ q_6\left(\varepsilon_s - \varepsilon_f\right)\left(h'\right)^2\left(\tilde{\varphi}'\right)^2 + (-\varepsilon_c\varepsilon_f q_5 + \varepsilon_c\varepsilon_s q_5 - i\varepsilon_f q_6\sqrt{\eta_c} + i\varepsilon_f q_6\sqrt{\eta_s} +$$

$$+ i\varepsilon_s q_6\sqrt{\eta_c} + i\varepsilon_s q_6\sqrt{\eta_f})h'\tilde{\varphi}' + \Big(i\varepsilon_c\varepsilon_s q_7\sqrt{\eta_f} + \varepsilon_c q_8\sqrt{\eta_f}\sqrt{\eta_s} - i\varepsilon_f\varepsilon_s q_7\sqrt{\eta_c} -$$

$$- \varepsilon_f q_8\sqrt{\eta_c}\sqrt{\eta_s} + q_8\left(\varepsilon_c - \varepsilon_f\right)\left(h'\right)^2\left(\tilde{\varphi}'\right)^2 + (\varepsilon_c\varepsilon_s q_7 + i\varepsilon_c q_8\sqrt{\eta_f} - i\varepsilon_c q_8\sqrt{\eta_s} -$$

$$- \varepsilon_f\varepsilon_s q_7 - i\varepsilon_f q_8\sqrt{\eta_c} + i\varepsilon_f q_8\sqrt{\eta_s})h'\tilde{\varphi}'\Big)e^{-\tilde{\gamma}_f h}\Bigg)$$

1st approximation: Solution of SLAE

$$B_1^s = \frac{1}{|M_2|}\Bigg(2i\varepsilon_f q_6\sqrt{\eta_f}h'\tilde{\varphi}' + 2\varepsilon_f\left(i\varepsilon_c q_5 - q_6\sqrt{\eta_c}\right)\sqrt{\eta_f} + \Big(i\varepsilon_c\varepsilon_f q_7\sqrt{\eta_f} -$$

$$-\varepsilon_c q_8\eta_f + i\varepsilon_f^2 q_7\sqrt{\eta_c} - \varepsilon_f q_8\sqrt{\eta_c}\sqrt{\eta_f} + \varepsilon_f(-\varepsilon_c q_7 + \varepsilon_f q_7 + iq_8\sqrt{\eta_c} +$$

$$+ iq_8\sqrt{\eta_f})h'\tilde{\varphi}' + q_8\left(-\varepsilon_c + \varepsilon_f\right)\left(h'\right)^2\left(\tilde{\varphi}'\right)^2\Big)e^{\tilde{\gamma}_f h} +$$

$$+\Big(i\varepsilon_c\varepsilon_f q_7\sqrt{\eta_f} + \varepsilon_c q_8\eta_f - i\varepsilon_f^2 q_7\sqrt{\eta_c} - \varepsilon_f q_8\sqrt{\eta_c}\sqrt{\eta_f} + \varepsilon_f(\varepsilon_c q_7 - \varepsilon_f q_7 -$$

$$- iq_8\sqrt{\eta_c} + iq_8\sqrt{\eta_f})h'\tilde{\varphi}' + q_8\left(\varepsilon_c - \varepsilon_f\right)\left(h'\right)^2\left(\tilde{\varphi}'\right)^2\Big)e^{-\tilde{\gamma}_f h}\Bigg)$$

Results

- Asymptotic method allows to formulate the problem of finding waveguide modes in symbolic form.
- Using computer algebra tools, namely `SymPy`, for the first time the system of inhomogeneous differential equations, corresponding to the first approximation of the proposed method, is solved in symbolic form.
- Linear system of boundary equations is also solved symbolically – we obtained the expressions for first contributions to the electric and magnetic field strengths.

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