Finite Difference Models of Dynamical Systems with Polynomial Right-Hand Side

MMCP'2024

Mikhail Malykh* **, Edic Ayryan**, Ilshat Dulatov*, and Leonid Sevastianov* **

* Peoples' Friendship University of Russia, Moscow, Russia
** JINR, Dubna,Russia

24 Oct. 2024, ver. October 24, 2024

This work is supported by the Russian Science Foundation (grant no. 20-11-20257).

Difference schemes

The finite difference method proposes replacing the system of differential equations

$$\frac{dx_i}{dt} = f_i(x_1, \dots, x_n), \quad i = 1, \dots, n,$$

or, for short,

$$\frac{d\mathfrak{x}}{dt} = \mathfrak{f}(\mathfrak{x}), \tag{1}$$

with a system of algebraic equations

$$g_i(\mathfrak{x}, \hat{\mathfrak{x}}, \Delta t) = 0, \quad i = 1, \dots, n, \tag{2}$$

relating the value $\mathfrak{x}$ of the solution at some moment in time t with the value $\hat{\mathfrak{x}}$ of the solution at the moment in time $t + \Delta t$.
The system of the algebraic equation (2) itself will be called a difference scheme for a system of the differential equation (1).

Discrete models

In mechanics, both old and new, the quantity dt has often been treated as a finite increment, and it was implied that Newton's equations were actually difference equations [Feynman].

Example

The explicit Euler scheme

$$\hat{\mathfrak{x}} - \mathfrak{x} = \mathfrak{f}(\mathfrak{x})\Delta t$$

for linear oscillator preserves the energy $H = x^2 + y^2$ only at $\Delta t \to 0$.

The problem is that classical difference schemes (explicit Runge-Kutta schemes) are not rich in algebraic properties. We describe properties of discrete models by looking back at continuous models.

Systems with quadratic Hamiltonian

The midpoint scheme perfectly imitates a Hamiltonian system

$$\frac{dx}{dt} = \frac{\partial H}{\partial y}, \quad \frac{dy}{dt} = -\frac{\partial H}{\partial x} \tag{3}$$

with a quadratic Hamiltonian H, for example, a harmonic oscillator with Hamiltonian $H = x^2 + y^2$.

- According to Cooper's theorem, the energy integral is preserved exactly on the scheme, and the approximate solution itself is a sequence of points $\mathfrak{x}_n = (x_n, y_n)$ of the circle $x^2 + y^2 = C$.
- Each step of the approximate solution is a rotation by an angle

$$\Delta u = \int\limits_{\mathfrak{x}_n}^{\mathfrak{x}_{n+1}} \frac{dx}{\sqrt{C - x^2}},$$

which does not depend on n.

Nonlinear systems

If $\mathfrak{f}$ is not linear function of $\mathfrak{x}$, than equations

$$\hat{\mathfrak{x}} - \mathfrak{x} = \mathfrak{f}\left(\frac{\hat{\mathfrak{x}} + \mathfrak{x}}{2}\right)\Delta t$$

define a multiple-valued correspondence between $\mathfrak{x}$ and $\hat{\mathfrak{x}}$ spaces. Multiple values of $\hat{\mathfrak{x}}$ correspond to the same value $\mathfrak{x}$ and vice versa. The geometric meaning of the extra roots is not clear. In numerical analysis, they are discarded.
They do not allow to investigate the algebraic properties of the midpoint scheme. This scheme is probably poor in algebraic properties.

Reversible schemes

Newton's equations must define a one-to-one correspondence between the initial and final positions of a dynamical system. Difference schemes define a correspondence between the initial and final positions of the system, which is described by algebraic equations. Such a correspondence will be one-to-one if and only if it is birational.

Definition

We call a difference scheme reversible if it specifies a birational map between an n-dimensional $\mathfrak{x}$-space and an n-dimensional $\hat{\mathfrak{x}}$-space.

We believe, that the «reversibility» is more significance than conservativity or symplectivity.

Construction of reversible schemes

Any dynamical system with a quadratic right-hand side

$$\frac{d\mathfrak{x}}{dt} = \mathfrak{f}(\mathfrak{x})$$

can be approximate by the equation

$$\hat{\mathfrak{x}} - \mathfrak{x} = \mathfrak{F}\Delta t,$$

which is linear with respect to $\mathfrak{x}$ and $\hat{\mathfrak{x}}$. Thus $\hat{\mathfrak{x}}$ is a rational function of $\mathfrak{x}$ and vice verse $\mathfrak{x}$ is a rational function of $\hat{\mathfrak{x}}$.

Example

$$\frac{dx}{dt} = 1 + x^2 \quad \rightarrow \quad \hat{x} - x = (1 + x \cdot \hat{x})\Delta t.$$

An unconventional integrator of W. Kahan

Firstly, indicated method to construct reversible schemes was presented by William "Velvel" Kahan in 1993 at conference in Ontario.

> *I have used these unconventional methods for 24 years without quite understanding why they work so well as they do, when they work. That is why I pray that some reader of these notes will some day explain the methods' behavior to me better than I can, and perhaps improve them.*

In 1994 Sanz-Serna applied the method to Volterra-Latka system and explain the successes of the method to the inheritance of the symplectic structure

$$\frac{dx \wedge dy}{xy}.$$

Ref.: J.M. Sanz-Serna // Applied Numerical Mathematics 16 (1994) 245-250.

Systems with cubic Hamiltonian

If the Hamiltonian is a cubic polynomial, then the exact solution to the continuous model lies on a third degree curve

$$H(x, y) = c,$$

whose genus is 1. Thus the quadrature

$$\int \frac{dx}{H_y(x, y)} = t + C$$

on the curve H is elliptic integral of the 1st kind.
If the invariant curve is closed, the functions $x(t), y(t)$ are elliptic, one of the periods is real and we see periodic movement along the oval on the phase plane xy.

Systems with cubic Hamiltonian, external properties

Kahan's scheme perfectly imitates a Hamiltonian system with a cubic Hamiltonian H, for example, a elliptic $\wp$-oscillator.

- According to 1st Celledoni's theorem, the symplectic structure is inherit, i.e.

$$d\hat{x} \wedge d\hat{y} = (1 + O(\Delta t))dx \wedge dy.$$

- According to 2nd Celledoni's theorem, the energy integral is inherit, thus the approximate solution itself is a sequence of points $\mathfrak{x}_n = (x_n, y_n)$ of an elliptic curve $f(x, y, \Delta t) = c$.

Ref.: Suris et al. // Proc. R. Soc. A. 2019. 475: 20180761

Systems with cubic Hamiltonian, quadrature

Consider more closely the narrowing of Cremona map to the invariant curve $f(x, y, \Delta t) = c$.
Using constructions from Picard's theorem, we can prove that the difference scheme can be again represented using quadrature

$$\int_{\mathfrak{x}}^{\hat{\mathfrak{x}}} v(x, y, \Delta t) dx = \Delta u(\Delta t),$$

where vdx_1 is an elliptic integral of the 1st kind on invariant curve and, of course,

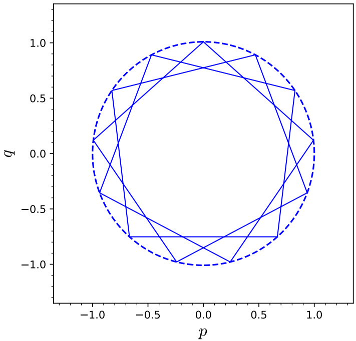
$$vdx \to \frac{dx}{H_y} \quad (\text{at} \Delta t \to 0).$$

Ref.: 1.) Malykh et al. // Mathematics 2024, 12 (1), 167; 2.) Malykh et al. // Zapiski sem. POMI. 2023

Systems with cubic Hamiltonian, internal properties

Consequences of quadrature representation:

1. The approximate solution can be represented using an elliptic function of a discrete argument.
2. We can pick a step Δt so that $O(\mathfrak{x})$ is a periodic sequence.



The reversible difference scheme imitates all the known properties of the system with cubic Hamiltonian.

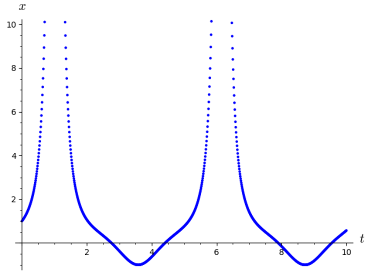
Points at infinity

If at some value k the denominator of the transformation becomes zero, then the point $\mathfrak{x}_{k+1}$ will be infinitely remote. Thus we consider $\mathfrak{x}$ as a point in the projective space $\mathbb{P}_n$.

Example ($\wp$-oscillator)

$$\begin{cases} \dfrac{dx}{dt} = y, \\ \dfrac{d}{dt} y = 6\,x^2 - 1, \\ x(0) = 1, \quad y(0) = 2 \end{cases}$$

The exact solution is periodic and has pols of 2nd degree.



The approximate solutions describe correct the behavior at infinity.

The main difference between discrete and continuous models

Newton's equations must define a one-to-one correspondence between the initial and final positions of a dynamical system.
The system with cubic Hamiltonian define birational transformation on the integral curve $H(x, y) = C$, which cannot be continued to the Cremona transformation of all planes xy (it's Hermite-Klein discussion!).
However, we can approximate the system so that transition from layer to layer is carried out by the Cremona transformation of the entire xy plane.

Further investigations

1. Using Appelroth's quadratization, we extended Kahan's approach to systems with polynomial right-hand sides[1].
2. For quadratic systems, approximate trajectory points are sometimes lined up and sometimes into more complex structures. We calculated their fractal dimension [2].
3. When discretizing the Navier-Stokes equations, dynamic systems with a quadratic right-hand side are obtained. We implemented Kahan's method for such systems in FreeFem++[3].

---

[1] Lapshenkova, L., RUDN, 2024; Malykh et al., Math. 2024. Vol. 12, no. 7. DOI: 10.3390/math12172725

[2] Kadrov, V., RUDN 2024

[3] Dulatov, I., RUDN, 2024.

The End

© 2024, Mikhail Malykh et al. Creative Commons Attribution-Share Alike 3.0 Unported.

Additional materials: `https://malykhmd.neocities.org`