Meshcheryakov Laboratory of Information Technologies

Mathematical Problems in Quantum Information Technologies

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Electrostatic ion traps in Triangles

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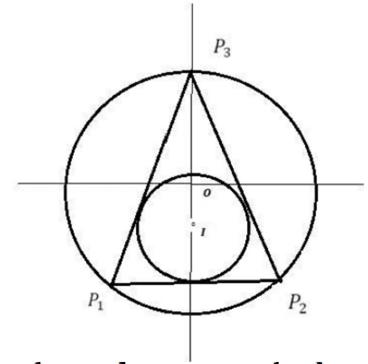
Abstract



In the talk we discuss on the equilibrium points (critical points) of the electrostatic (Coulomb) potential of three mutually repelling point charges placed at fixed points. This topics are closely related to the Maxwell conjecture for three point charges and linear electrostatic ion traps.

We show that the incenter of an isosceles triangle is a stable equilibrium point of the electrostatic potential of certain point charges placed at its vertices. To this end, explicit formulas for these charges are given and the hessian of their electrostatic potential is computed. The behaviour of this hessian in a family of triangles with the given inscribed and circumscribed circles is investigated and its extremal values are computed. As an application we prove that each point in the unit disc is a stable equilibrium point of a certain triple of point charges on its boundary, which yields an explicit scenario of robust electrostatic control in Euclidean discs.

References

[1] Giorgadze G. and Khimshiashvili G. Incenter of triangle as a stationary point. Georgian Mathematical Journal, vol. 29, no. 4, 2022, pp. 515-525. https://doi.org/10.1515/gmj-2022-2155

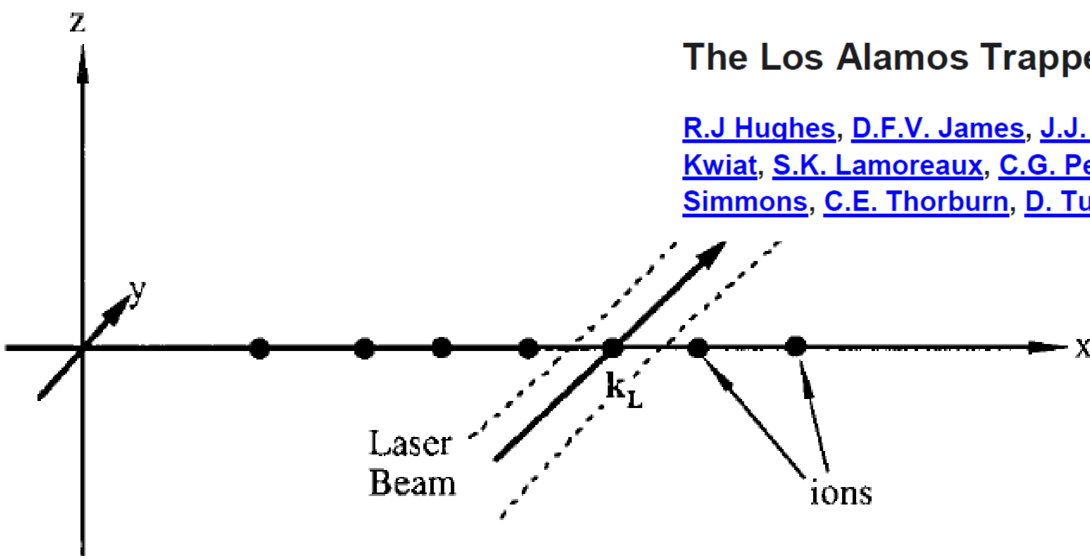
[2] Giorgadze G. and Khimshiashvili G. Triangles and electrostatic ion traps. J. Math. Phys. 62, 053501, 2021. https://doi.org/10.1063/5.0040735

The topics discussed in this presentation are closely related to the Maxwell conjecture for three point charges and linear electrostatic ion traps.

Let us consider n ≥ 2 point charges with Coulomb interaction in three-dimensional Euclidean space, which are kept in fixed positions.

The Maxwell conjecture states that if the set of equilibrium points of the given configuration of point charges is finite, then the number of equilibrium points does not exceed $(n-1)^2$.

The Los Alamos Trapped Ion Quantum Computer Experiment



The Los Alamos Trapped Ion Quantum Computer Experiment

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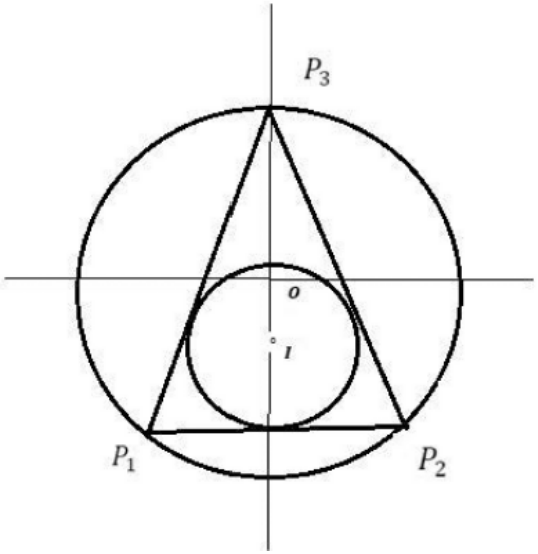
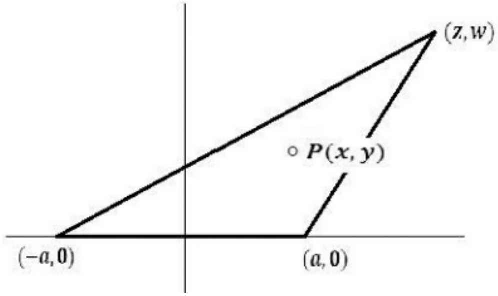
Figure 1: A schematic illustration of an idealized laser-ion interaction system; k_L is the wavevector of the single addressing laser.



Given a triple of positive real numbers $Q = (q_1, q_2, q_3)$, we interpret them as values of point charges with Coulomb interaction placed at vertices P_i of triangle $\Delta P_1P_2P_3$ and denote this system of point charges by S = Q@T.

We deal with the Coulomb potential E_S of system S = Q@T, considered as a function on the interior T of triangle $\Delta P_1P_2P_3$ given by

$$E_S(P) = \frac{q_1}{d_1} + \frac{q_2}{d_2} + \frac{q_3}{d_3}, \qquad (1)$$



where P is a point in T and d_i is the distance between points P_i and P.

STATIONARY CHARGES

In this setting, equilibrium points of system S are defined as critical points of Coulomb potential E_S considered as a function of point P.

Numbers q_i are called the vertex charges of system Q@T. If potential E_S has an isolated minimum at P $\in$ T, then P is called a stable equilibrium of system S.

Equilibrium points will sometimes be referred to as lacunas of E_S.

The following considerations form a framework for our approach.

Given a triangle T and a point P $\in$ T, let us try to find a system of charges Q such that P is a lacuna of S = Q@T.

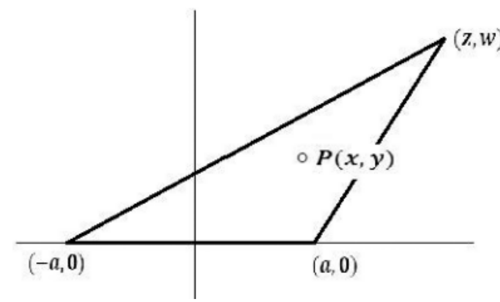
We always assume that the values of charges are strictly positive and normalized by requiring that their sum is equal to 1.

Theorem 1. For any triangle T and any point P $\in$ T, there exists exactly one system of normalized positive charges Q such that P is a lacuna of S = Q@T.

We prove this by giving explicit formulas for the values of stationary charges.

Without loss of generality, the first two vertices can be placed at the points (−1, 0) and (1, 0). Coordinates of the third vertex P3 will be denoted as (z,w) and coordinates of point P by (x, y).

For computational reasons, it is convenient to put the charge at P_3 equal to 1. From the condition that point P is stationary for charges Q(P) = (r, s, 1), we get a system of linear equations for values of r and s:



$$\frac{r(x+1)}{\left((x+1)^2+y^2\right)^{3/2}}+\frac{s(x-1)}{\left((x-1)^2+y^2\right)^{3/2}}+\frac{x-z}{\left((x-z)^2+(y-w)^2\right)^{3/2}}=0,$$

(2)

$$\frac{ry}{\left((x+1)^2+y^2\right)^{3/2}}+\frac{sy}{\left((x-1)^2+y^2\right)^{3/2}}+\frac{y-w}{\left((x-z)^2+(y-w)^2\right)^{3/2}}=0.$$

Explicit formulas for solutions to this system:

$$r = \frac{(zy - wx + w - y)\left((x+1)^2 + y^2\right)^{\frac{3}{2}}}{xy\left[(x-z)^2 + (y-w)^2)\right]^{\frac{3}{2}}},$$

$$s = \frac{(xw - yz + w - y)\left((x-1)^2 + y^2\right)^{\frac{3}{2}}}{2y\left[(x-z)^2 + (y-w)^2)\right]^{\frac{3}{2}}}. \qquad (3)$$

It follows that the sought normalized charges are uniquely determined by

$$q_1 = \frac{r}{r+s+1}, q_2 = \frac{s}{r+s+1}, q_3 = \frac{1}{r+s+1}. \qquad (4)$$

Definitions

The charges q_i computed above will be called the normalized stationary charges for P in T, and the triple (q_1, q_2, q_3) will be denoted as $Q_T(P)$.

Let us denote by T the regular triangle in R^3 defined by

$$\mathbb{T} = \{x \geq 0, y \geq 0, z \geq 0; x + y + z - 1 = 0\}. \qquad (5)$$

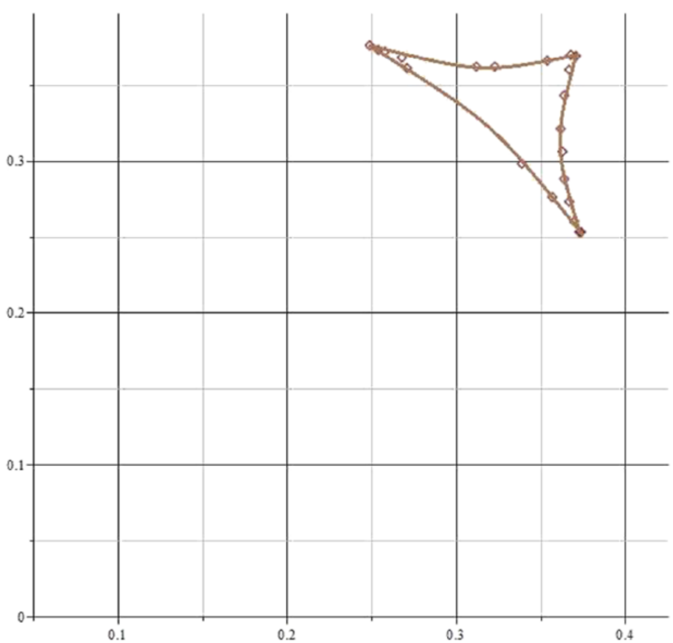
Theorem 1 implies that for any triangle T, one has a natural map

$$Q_T : T \rightarrow \mathbb{T}.$$

For a fixed triangle T and point P ∈ T, a natural problem is to find the number of lacunas of $Q_P = Q_T(P)$ and determine the type of P as a critical point of Q_P.

Computer experiments suggest that mapping Q_T should be stable in the sense of singularity theory. In the case of a regular triangle we have

Theorem 2. For a regular triangle T, the singular set B(T) of mapping Q_T is a closed curve of fold points with three points of $\Sigma^{1,1}$ type and the caustic $Q_T(B(T))$ is a deltoid curve with three ordinary cusps.



Graph of electrostatic caustic for a regular triangle with side 2.

Definitions

Denote by E_P the Coulomb potential

$$E_S(P) = \frac{q_1}{d_1} + \frac{q_2}{d_2} + \frac{q_3}{d_3}$$

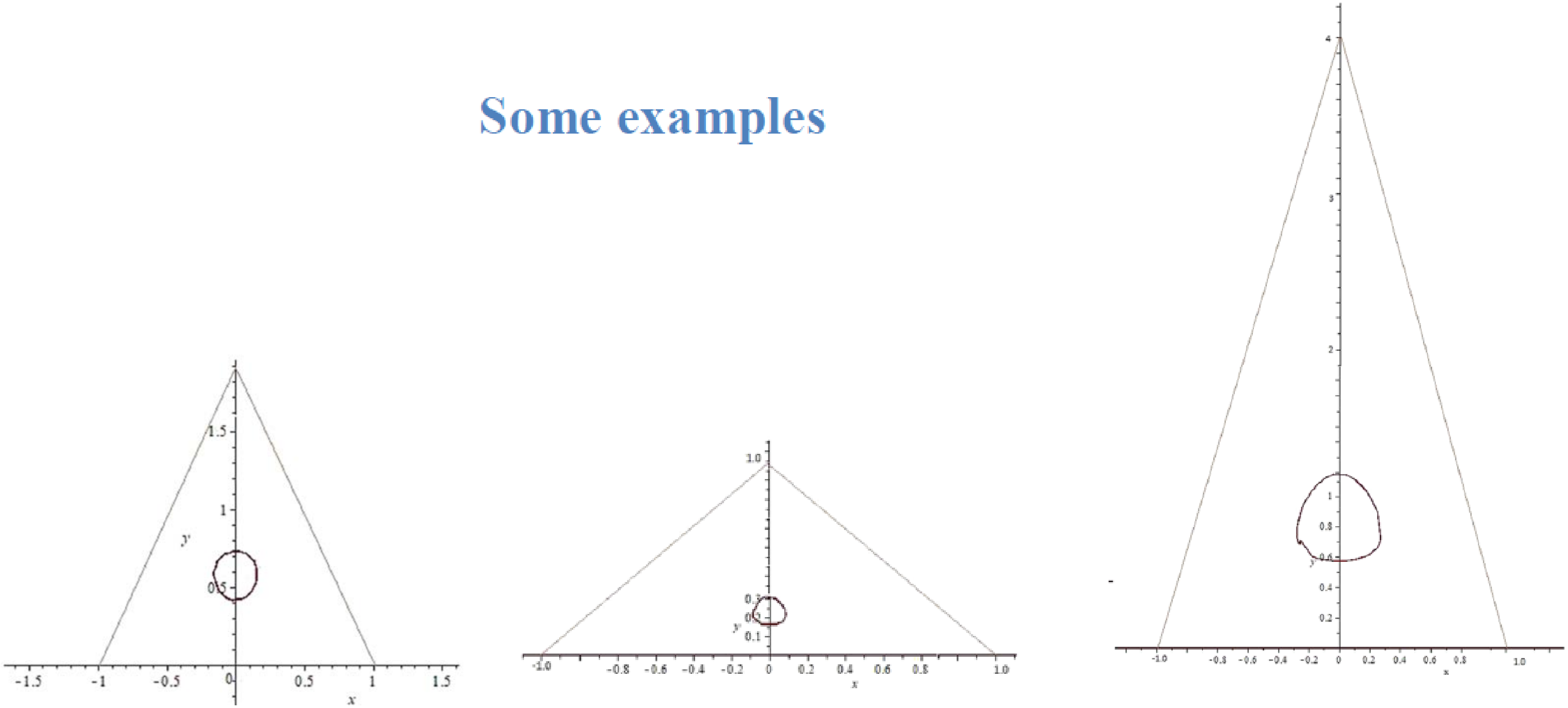
of system $Q_P@T$, by h_P the hessian of E_P, and by H_T the function of T defined by $H_T(P) = h_P(P)$.

Furthermore, we denote by S(T) the set of all points in T such that P is a point of minimum for $Q_T(P)$ and by B(T) the zero-set of function H_T.

By a way of analogy with the concept of an electrostatic ion trap, the set S(T) will be called the electrostatic trapping domain of T and B(T) will be called the trapping boundary of T.

Theorem 3. **If the shape of a triangle T is sufficiently close to the shape of a regular triangle, then the trapping set S(T) is non-empty and contains the incenter of T.**

Some examples



Trapping domain and its boundary for two isosceles triangles.

END

Thank you very much for your attention!