Evading Quantum Mechanics à la Sudarshan: quantum-mechanics-free subsystem as a realization of Koopman-von Neumann mechanics

Zurab K. Silagadze

Outline of the talk

The talk is based on the paper Z.K. Silagadze, Evading Quantum Mechanics à la Sudarshan: quantum-mechanics-free subsystem as a realization of Koopman-von Neumann mechanics, https://arxiv.org/abs/2308.08919. Published in Foundations of Physics 53 (2023), 92.

- Koopmann-von Neumann mechanics
- Quantum-mechanics-free subsystems
- Concluding remarks: KvN mechanics is realized in QMFS

Koopman-von Neumann mechanics

- Through the Liouville equation
- Through the correspondence principle/Ehrenfest's theorem
- Via Wigner function

$$\frac{\partial \rho(q,p,t)}{\partial t} = \frac{\partial H_{cl}}{\partial q}\frac{\partial \rho}{\partial p} - \frac{\partial H_{cl}}{\partial p}\frac{\partial \rho}{\partial q}.$$

The classical wave function $\psi(q,p,t) = \sqrt{\rho(q,p,t)}$ obeys the same Liouville equation, which can be rewritten in Schrödinger-type form

$$i\frac{\partial \psi(q,p,t)}{\partial t} = \hat{L}\psi, \quad \hat{L} = i\left(\frac{\partial H_{cl}}{\partial q}\frac{\partial}{\partial p} - \frac{\partial H_{cl}}{\partial p}\frac{\partial}{\partial q}\right).$$

- It is possible to develop a formulation of classical mechanics in Hilbert space that completely resembles the quantum formalism, except that, of course, all interference effects are absent. Koopman 1931, von Neumann 1932.

D. Mauro, Topics in Koopman-von Neumann Theory, https://doi.org/10.48550/arXiv.quant-ph/0301172

Through the correspondence principle/Ehrenfest's theorem

- Ordinary axioms of quantum mechanics.
- $|\Psi(t)\rangle = \hat{U}(t)|\Psi(0)\rangle$: unitary representation of a group of time shifts. According to Stone's theorem, there must exist a Hermitian generating operator with $i\frac{d|\Psi\rangle}{dt} = \hat{L}|\Psi\rangle$.
- Ehrenfest's theorem $\frac{d}{dt}\langle\hat{q}\rangle = \langle\frac{\hat{p}}{m}\rangle$, $\frac{d}{dt}\langle\hat{p}\rangle = -\langle\frac{d}{dq}U(\hat{q})\rangle$ requires

$$i[\hat{L},\hat{q}] = \frac{\hat{p}}{m}, \quad i[\hat{L},\hat{p}] = -\frac{d}{dq}U(\hat{q}).$$

- $[\hat{q},\hat{p}] = i\hbar$ -> quantum mechanics: $\hbar\hat{L} = \hat{H} = \frac{\hat{p}^2}{2m} + U(\hat{q})$.
- $[\hat{q},\hat{p}] = 0$ -> we cannot construct $\hat{L}$ from only dynamic variables $\hat{q},\hat{p}$. To correct the situation, we introduce two additional Hermitian operators $\hat{\lambda}_q, \hat{\lambda}_p$, satisfying the conditions $[\hat{q},\hat{\lambda}_q] = i$, $[\hat{p},\hat{\lambda}_p] = i$. Then $\hat{L} = \frac{\hat{p}}{m}\hat{\lambda}_x - \frac{dU(\hat{q})}{dq}\hat{\lambda}_p$.

F. Wilczek, Notes on Koopman von Neumann Mechanics, and a Step Beyond. https://frankwilczek.com/2015/koopmanVonNeumann02.pdf

Via Wigner function

$$W(q,p) = \frac{1}{\sqrt{2\pi\hbar}} \int e^{\frac{i}{\hbar}py} \Psi^*(q+y/2,t)\Psi(q-y/2,t)dy.$$

$\hbar \to k\hbar,\ y = k\hbar\lambda_p,\ u = q - \frac{k\hbar\lambda_p}{2},\ v = q + \frac{k\hbar\lambda_p}{2}$:

$$W(q,p) = \sqrt{\frac{k\hbar}{2\pi}} \int e^{ip\lambda_p}\rho(u,v,t)d\lambda_p, \quad ;\rho(u,v,t) = \Psi^*(v)\Psi(u).$$

$$ik\hbar\frac{\partial\rho}{\partial t} = \left[\hat{H}_u - \hat{H}_v\right]\rho, \quad \hat{H}_u = \frac{(k\hbar)^2}{2m}\frac{\partial^2}{\partial u^2} + U(u).$$

This is reminiscent of the chiral decomposition method.
Generalized pseudo-differential Bopp operators:

$$\hat{u} = \hat{q} - \frac{k\hbar\hat{\lambda}_p}{2},\ \hat{v} = \hat{q} + \frac{k\hbar\hat{\lambda}_p}{2},\ \hat{p}_u = \hat{p} + \frac{k\hbar\hat{\lambda}_q}{2},\ \hat{p}_v = \hat{p} - \frac{k\hbar\hat{\lambda}_q}{2}.$$

$$[\hat{u},\hat{p}_u] = ik\hbar, \quad [\hat{v},\hat{p}_v] = -ik\hbar \quad k \to 0 \text{ means } [\hat{q},\hat{p}] = 0.$$

The difference of Hamiltonians of two uncoupled one-dimensinal oscillators yield an interesting non-commutative system in the plane:

P. D. Alvarez, J. Gomis, K. Kamimura, M. S. Plyushchay, Anisotropic harmonic oscillator, non-commutative Landau problem and exotic Newton-Hooke symmetry, Phys. Lett. **B 659**, 906-912 (2008).
https://arxiv.org/abs/0711.2644

P. D. Alvarez, J. Gomis, K. Kamimura, and M. S. Plyushchay, (2+1)D Exotic Newton-Hooke Symmetry, Duality and Projective Phase, Annals Phys. **322** (2007) 1556-1586.
https://arxiv.org/abs/hep-th/0702014

P.-M. Zhang, P. A. Horvathy, Chiral Decomposition in the Non-Commutative Landau Problem, Annals Phys. 327 (2012) 1730–1743.
https://arxiv.org/abs/1112.0409.

$k \to 0$ limit and Koopman-von Neumann equation

$$\hat{H}_u - \hat{H}_v = \frac{k\hat{p}\hat{P}}{m} + U\left(\hat{q} + \frac{k\hat{Q}}{2}\right) - U\left(\hat{q} - \frac{k\hat{Q}}{2}\right), \quad \hat{\lambda}_q = \frac{\hat{P}}{\hbar}, \; \hat{\lambda}_p = -\frac{\hat{Q}}{\hbar}.$$

$$i\hbar\frac{\partial \Psi_{KvN}}{\partial t} = \left[\frac{\hat{p}\hat{P}}{m} + \frac{1}{k}U\left(\hat{q} + \frac{k\hat{Q}}{2}\right) - \frac{1}{k}U\left(\hat{q} - \frac{k\hat{Q}}{2}\right)\right]\Psi_{KvN},$$

where

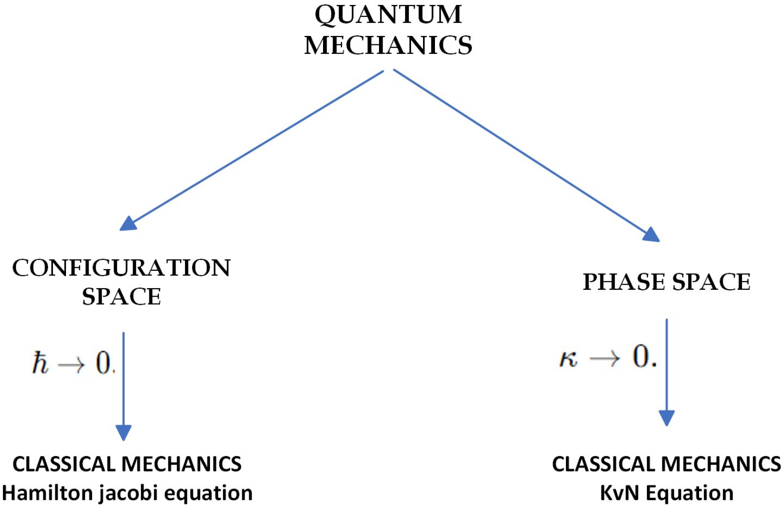
$$\Psi_{KvN}(q, Q, t) \sim \rho(u, v, t).$$

We have a well defined $k \to 0$ limit:

$$i\hbar\frac{\partial \Psi_{KvN}}{\partial t} = \left[\frac{\hat{p}\hat{P}}{m} + \frac{\partial U(q)}{\partial q}Q\right]\Psi_{KvN} = \hat{H}_{KvN}\Psi_{KvN}.$$

D.I. Bondar *et al.*, Operational dynamic modeling transcending quantum and classical mechanics, Phys. Rev. Lett. 109 (2012) 190403.
https://arxiv.org/abs/1105.4014

Comparison of $\hbar \to 0$ and $k \to 0$ limits



If we introduce $\hat{Q}$ and $\hat{P}$ operators as follows

$$\hat{Q} = i\hbar\frac{\partial}{\partial p}, \quad \hat{P} = -i\hbar\frac{\partial}{\partial q},$$

then the Liouville-Schrödinger equation takes the form

$$i\hbar\frac{\partial\psi(q,p,t)}{\partial t} = \hat{H}\psi, \quad \hat{H} = \frac{\partial H_{cl}}{\partial q}\hat{Q} + \frac{\partial H_{cl}}{\partial p}\hat{P},$$

and it can be interpreted as the Schrödinger equation in the (q,p)-representation (with diagonal operators q and p) of a genuine quantum system with two pairs of canonical variables (q,P) and (Q,p).

E. C. G. Sudarshan, Interaction between classical and quantum systems and the measurement of quantum observables, Pramana 6(3) (1976), 117.
https://link.springer.com/article/10.1007/BF02847120

Let us assume that the Hamiltonian of the quantum system is equal to

$$\hat{H} = f(q,p,t)\hat{P} + g(q,p,t)\hat{Q} + h(q,p,t),$$

where $f(q,p,t)$, $g(q,p,t)$, $h(q,p,t)$ are arbitrary functions, and q, P and Q, p represent are two pairs of quantum mechanical conjugate variables that obey canonical commutation relations. Then the Heisenberg equations of motion for the commuting variables q, p

$$\frac{dq}{dt} = \frac{\partial H}{\partial P} = f(q,p,t), \quad \frac{dp}{dt} = -\frac{\partial H}{\partial Q} = -g(q,p,t),$$

do not contain "hidden" variables $\hat{Q}, \hat{P}$ and will correspond to classical Hamiltonian dynamics if there exists a classical Hamiltonian function $H_{cl}(q,p,t)$ such that

$$f(q,p,t) = \frac{\partial H_{cl}}{\partial p}, \quad g(q,p,t) = \frac{\partial H_{cl}}{\partial q}.$$

Experimental implementation of QMFS

M. Tsang, C. M. Caves, Evading quantum mechanics: Engineering a classical subsystem within a quantum environment, Phys. Rev. X 2 (2012), 031016. https://arxiv.org/abs/1203.2317 A pair of positive and negative mass oscillators can be used for this purpose. The quantum Hamiltonian in this case has the form

$$H = \frac{p_1^2}{2m} + \frac{1}{2}m\omega^2 q_1^2 - \frac{p_2^2}{2m} - \frac{1}{2}m\omega^2 q_2^2.$$

In terms of new canonical variables

$$q = q_1 + q_2, \quad Q = \frac{1}{2}(q_1 - q_2), \quad p = p_1 - p_2, \quad P = \frac{1}{2}(p_1 + p_2),$$

The Hamiltonian takes the form $H = \frac{pP}{m} + m\omega^2 qQ$, and is a KvN-type Hamiltonian.
Sidney Coleman: "The career of a young theoretical physicist consists of treating the harmonic oscillator in ever-increasing levels of abstraction."

KvN mechanics is realized in QMFS

- Similarity of the Sudarshan interpretation of the KvN mechanics with the idea of QMFS is obvious.
- (q, p) subsystem of KvN mechanics is nothing more than QMFS.
- Resumption of interest in KvN mechanics was caused by the need to create suitable formalism for hybrid classical-quantum systems.
- The identification of quantum-mechanics-free subsystems with Sudarshan's interpretation of KvN mechanics, combined with the fact that such systems were actually implemented experimentally, makes the KvN mechanics, in a sense, engineering science.

Quantum gravity destroys classicality?

- Modification of quantum mechanics, expected from quantum gravity, can lead to deformation of classical mechanics (O.I Chashchina, A. Sen, Z.K. Silagadze, On deformations of classical mechanics due to Planck-scale physics, Int. J. Mod. Phys. D29 (2020), 2050070 https://arxiv.org/abs/1902.09728).
- This deformation actually destroys the classicality if Sudarshan's views on KvN mechanics are taken seriously.
- You are not required to accept the Sudarshan interpretation in order to develop the KvN mechanics.
- However, we now see that the existence of quantum-mechanics-free subsystems indicates that we should take Sudarshan's interpretation of KvN mechanics seriously.
- Therefore, we expect that, due to the universal nature of gravity, if the effects of quantum gravity do modify quantum mechanics, these effects will destroy the classical dynamics in QMFS.