Long-time Markovian dynamics of open quantum systems in all the orders of perturbation theory

Alexander Teretenkov

Department of Mathematical Methods for Quantum Technologies,
Steklov Mathematical Institute of Russian Academy of Sciences, Moscow

MPQIT-2024, MLIT of JINR, Dubna, Moscow region, Russia
May 28, 2024

Outline

Hilbert space

$\mathcal{H} = \mathbb{C}^2 \otimes \mathfrak{F}_b(L^2(\mathbb{R}))$, where $\mathfrak{F}_b(L^2(\mathbb{R}))$ is bosonic Fock space with creation and annihilation operators.

In $\mathbb{C}^2$ we define $\sigma_+ \equiv |1\rangle\langle 0|, \sigma_- \equiv |0\rangle\langle 1|$ and in $\mathfrak{F}_b(L^2(\mathbb{R}))$ we define the creation and annihilation operators $b_k, b_k^\dagger$ and vacuum $b_k|\text{vac}\rangle = 0$.

Liouville-von Neumann equation

Hamiltonian has the form

$$H(\lambda) = \int \omega(k) b_k^\dagger b_k dk + \Omega \sigma_+ \sigma_- + \lambda \int \left(g^*(k) \sigma_- b_k^\dagger + g(k) \sigma_+ b_k \right) dk.$$

Let us define $\rho_{SB}(t;\lambda)$ as a solution of the Cauchy problem for the Liouville-von Neumann equation

$$\frac{d}{dt} \rho_{SB}(t;\lambda) = -i[H(\lambda), \rho_{SB}(t;\lambda)], \qquad \rho_{SB}(0;\lambda) = \sigma \otimes |\mathrm{vac}\rangle\langle \mathrm{vac}|,$$

where σ is an arbitrary density matrix in $\mathbb{C}^2$.

Bogolubov-van Hove scailing

The reduced density matrix of the system in the interaction picture is defined by the formula

$$\rho_S(t;\lambda) \equiv e^{i\Omega\sigma_+\sigma_- t} \mathrm{Tr}_{\mathfrak{F}_b(L^2(\mathbb{R}))} \rho_{SB}(t;\lambda) e^{-i\Omega\sigma_+\sigma_- t}$$

Bogolubov-van Hove scailing

$$\rho(t;\lambda) \equiv \rho_S(\lambda^{-2}t;\lambda)$$

Bogolubov-van Hove limit

$$\lambda \to +0$$

From the physical point of view the scaling $t \to \lambda^{-2}t$ allows one to separate the time scale on which the Markovian behavior occurs from the time scale of order of reservoir correlation time.

Bogolubov-van Hove scailing

Reduced dynamics in the Bogolubov-van Hove limit
E. B. Davies, Commun. Math. Phys. 39, 91–110 (1974).
Stochastic (unitary) daynamics of both system and reservoir:
L. Accardi, Y.G. Lu, and I. Volovich, *Quantum theory and its stochastic limit* (Springer, Berlin, 2002).
Corrections:
A.N. Pechen and I.V. Volovich, Quant. Prob. and Rel. Top. 5 (4), 441–464 (2002).
A.N. Pechen, On an asymptotic expansion in quantum theory, Math. Notes, 75 (3), 426–429 (2004).
I.Y. Aref'eva, and I.V. Volovich, Quant. Prob. and Rel. Top. 3 (04), 453–482 (2000).

Gorini-Kossakowski-Sudarshan-Lindblad equation

$$\frac{d}{dt}\rho(t) = \mathcal{L}(\rho(t))$$

$$\mathcal{L}(\rho) \equiv -i[H,\rho] + \sum_j \left(C_j \rho C_j^\dagger - \frac{1}{2} C_j^\dagger C_j \rho - \frac{1}{2} \rho C_j^\dagger C_j \right)$$

V. Gorini, A. Kossakowski, E. C. D. Sudarshan, J. Math. Phys., 17 (5) , 821–825 (1976).
G. Lindblad, Comm. Math. Phys., 48(2), 119–130 (1976).

Main result

Theorem. Let the integral $G(t) \equiv \int |g_k|^2 e^{-i(\omega(k)-\Omega)t} dk$ converge and define continuous function $G(t)$ with finite moments up to $(m+1)$-th one for some $m \in \mathbb{N}$. Then their exist $\Delta\Omega^{(m)} \in \mathbb{R}$ and $\Gamma^{(m)} \geqslant 0$ such that for some solution $\rho^{(m)}(t;\lambda)$ of the equation

$$\frac{d}{dt}\rho^{(m)}(t;\lambda) = \mathcal{L}^{(m)}(\rho^{(m)}(t;\lambda)),$$

$$\mathcal{L}^{(m)}(\rho) = -i[\Delta\Omega^{(m)}\sigma_+\sigma_-, \rho] + \Gamma^{(m)} \left(\sigma_-\rho\sigma_+ - \frac{1}{2}\sigma_+\sigma_-\rho - \frac{1}{2}\rho\sigma_+\sigma_-\right),$$

one has $\rho(t;\lambda) = \rho^{(m)}(t;\lambda) + O(\lambda^{2m+2})$ as $\lambda \to 0$ and fixed $t > 0$.

Initial condition

$$\rho(0;\lambda) = \begin{pmatrix} \sigma_{11}(0) & \sigma_{10}(0) \\ \sigma_{01}(0) & \sigma_{00}(0) \end{pmatrix}$$

$$\rho^{(m)}(0;\lambda) = \begin{pmatrix} |r^{(m)}(\lambda)|^2 \sigma_{11}(0) & r^{(m)}(\lambda)\sigma_{10}(0) \\ (r^{(m)}(\lambda))^* \sigma_{01}(0) & \sigma_{00}(0) + (1 - |r^{(m)}(\lambda)|^2)\sigma_{11}(0) \end{pmatrix}$$

Initial condition

$$\rho(0;\lambda) = \begin{pmatrix} \sigma_{11}(0) & \sigma_{10}(0) \\ \sigma_{01}(0) & \sigma_{00}(0) \end{pmatrix}$$

$$\rho^{(m)}(0;\lambda) = \begin{pmatrix} |r^{(m)}(\lambda)|^2\sigma_{11}(0) & r^{(m)}(\lambda)\sigma_{10}(0) \\ (r^{(m)}(\lambda))^*\sigma_{01}(0) & \sigma_{00}(0) + (1 - |r^{(m)}(\lambda)|^2)\sigma_{11}(0) \end{pmatrix}$$

If $\int_0^\infty tG(t)dt \neq 0$ and $m \geqslant 1$, then

$$\rho(0;\lambda) - \rho^{(m)}(0;\lambda) = O(\lambda^2)$$

Initial condition

$$\rho(0;\lambda) = \begin{pmatrix} \sigma_{11}(0) & \sigma_{10}(0) \\ \sigma_{01}(0) & \sigma_{00}(0) \end{pmatrix}$$

$$\rho^{(m)}(0;\lambda) = \begin{pmatrix} |r^{(m)}(\lambda)|^2\sigma_{11}(0) & r^{(m)}(\lambda)\sigma_{10}(0) \\ (r^{(m)}(\lambda))^*\sigma_{01}(0) & \sigma_{00}(0) + (1 - |r^{(m)}(\lambda)|^2)\sigma_{11}(0) \end{pmatrix}$$

If $\int_0^\infty tG(t)dt \neq 0$ and $m \geqslant 1$, then

$$\rho(0;\lambda) - \rho^{(m)}(0;\lambda) = O(\lambda^2)$$

Moreover, if $\mathrm{Re}\int_0^\infty tG(t)dt > 0$, then $\rho^{(m)}(0;\lambda)$ **is even not a density matrix** for some density matrices σ (there are σ such that $\rho^{(m)}(0;\lambda)$ differs from any density matrix by terms of order $O(\lambda^2)$).

Initial condition

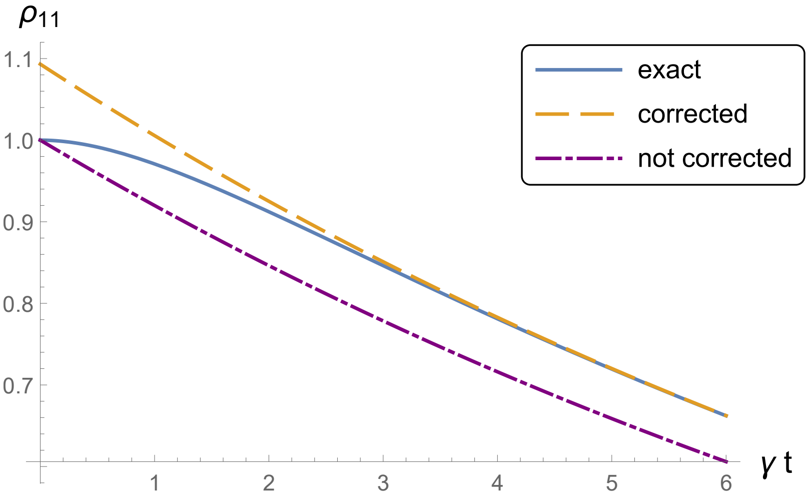
$$\rho(0;\lambda) = \begin{pmatrix} \sigma_{11}(0) & \sigma_{10}(0) \\ \sigma_{01}(0) & \sigma_{00}(0) \end{pmatrix}$$

$$\rho^{(m)}(0;\lambda) = \begin{pmatrix} |r^{(m)}(\lambda)|^2\sigma_{11}(0) & r^{(m)}(\lambda)\sigma_{10}(0) \\ (r^{(m)}(\lambda))^*\sigma_{01}(0) & \sigma_{00}(0) + (1 - |r^{(m)}(\lambda)|^2)\sigma_{11}(0) \end{pmatrix}$$

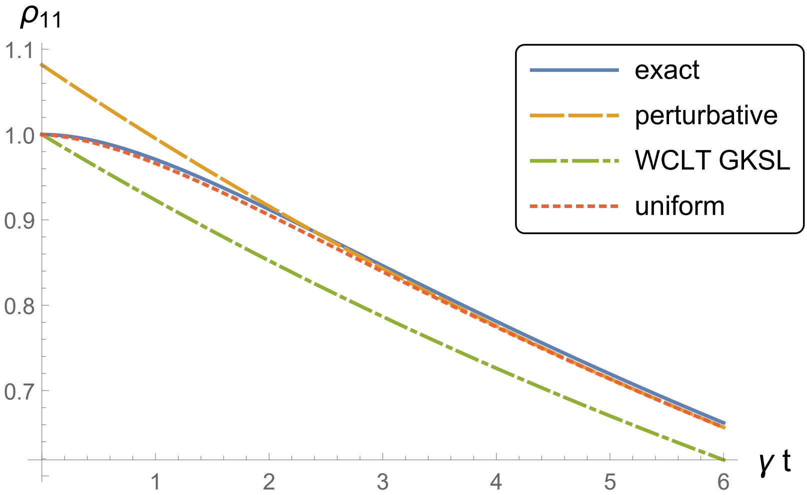
If $\int_0^\infty tG(t)dt \neq 0$ and $m \geqslant 1$, then

$$\rho(0;\lambda) - \rho^{(m)}(0;\lambda) = O(\lambda^2)$$

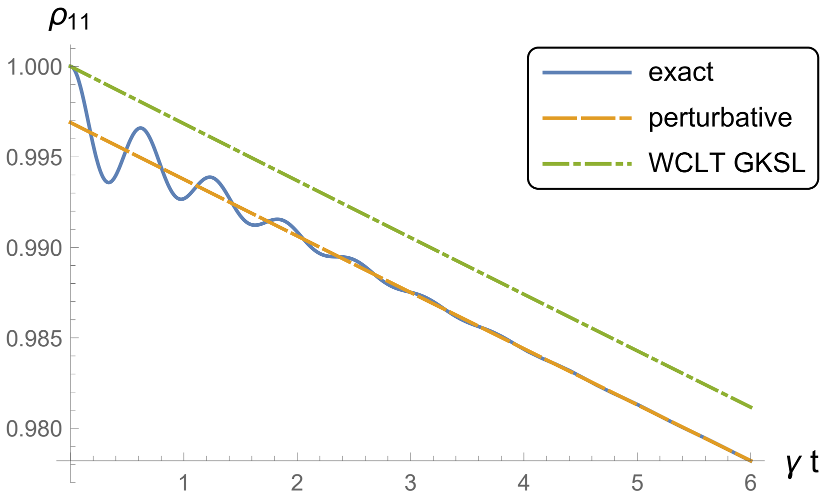
Moreover, if $\mathrm{Re} \int_0^\infty tG(t)dt > 0$, then $\rho^{(m)}(0;\lambda)$ **is even not a density matrix** for some density matrices σ (there are σ such that $\rho^{(m)}(0;\lambda)$ differs from any density matrix by terms of order $O(\lambda^2)$). But $\exists t^* = O(\lambda^2)$ such that $\forall t > t^*$ $\rho^{(m)}(t;\lambda)$ is a density matrix for arbitrary σ.

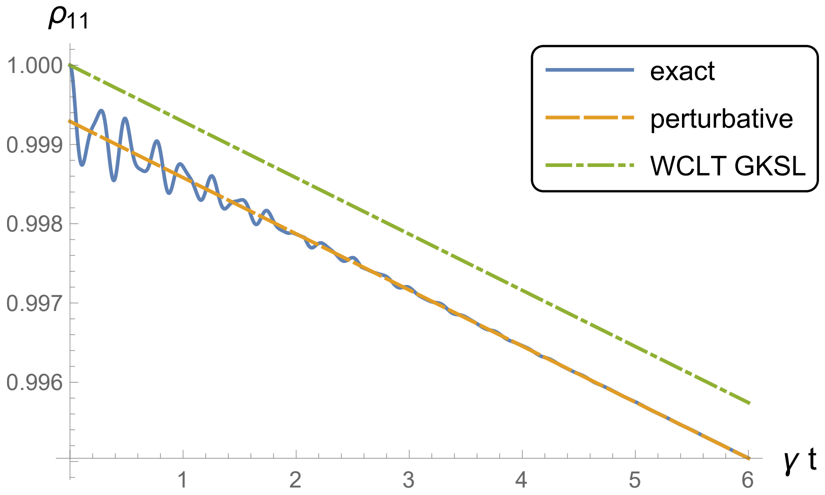


ρ_{11}
1.1
1.0
0.9
0.8
0.7
1
2
3
4
5
6
γ t
exact
corrected
not corrected



ρ_{11}
1.1
1.0
0.9
0.8
0.7
1
2
3
4
5
6
γ t
exact
perturbative
WCLT GKSL
uniform


ρ_{11}
1.000
0.995
0.990
0.985
0.980
exact
perturbative
WCLT GKSL
1
2
3
4
5
6
γ t


ρ_{11}
1.000
0.999
0.998
0.997
0.996
1
2
3
4
5
6
γ t
exact
perturbative
WCLT GKSL

Correlation function

The Markovian (based on the regression formula) correlation function has the form

$$\rho(t_2) = \Phi_{t_1}^{t_2} \rho(t_1)$$

$$\langle \sigma_-(t_2)\sigma_+(t_1)\rangle_{\mathrm{M}} \equiv \mathrm{Tr}(\sigma_- \Phi_{t_1}^{t_2}(\sigma_+ \Phi_0^{t_1}(|0\rangle\langle 0|)))$$

The exact correlation function has the form

$$\langle \sigma_-(t_2)\sigma_+(t_1)\rangle \equiv \mathrm{Tr}\, U_{t_2}^\dagger |0\rangle\langle 1| U_{t_2} U_{t_1}^\dagger |1\rangle\langle 0| U_{t_1} |0\rangle\langle 0| \otimes |\mathrm{vac}\rangle\langle \mathrm{vac}|,$$

where U_t is unitary evolution of the system and reservoir in the interaction picture.

Correlation function

Markovianity in terms of correlations functions:

$$\langle \sigma_-(t_2)\sigma_+(t_1)\rangle_{\mathrm{M}} = \langle \sigma_-(t_2)\sigma_+(t_1)\rangle$$

G. Lindblad, Comm. Math. Phys. 65 (3) 281–294 (1979).
L. Accardi, A. Frigerio, J. T. Lewis, Publ. Res. Inst. Math. Sci. 18 (1) 97–133 (1982).
L. Li, M. J. W. Hall, and H. M. Wiseman, Phys. Rep. 759, 1–51 (2018).

Correlation function

Under the conditions of the above theorem we have

$$\langle \sigma_-(t_2)\sigma_+(t_1)\rangle_{\mathrm{M}} = r^{(m)}(\lambda)\langle \sigma_-(t_2)\sigma_+(t_1)\rangle + O(\lambda^{2m+2}), \quad t_2 > t_1.$$

Generally, only in the case $m = 0$ one has $r^{(0)} = 1$.
R. Dümcke, J. Math. Phys. 24 (2), 311–315 (1983).
So strictly speaking the dyanmics is non-Markovian:
N.L. Gullo, I. Sinayskiy, T. Busch, F. Petruccione, arXiv:1401.1126 (2014).
But all the non-Markovianity is localized at correlation time and the defintion is renormalized..

Non-existence of moments

$$G(t) = g^2 \Bigg(\chi e^{-\gamma|t|} + \\ + \frac{1-\chi}{2} \left(e^{-\gamma|t|} (1 - \mathrm{Im}\, \mathrm{erf}\, (i\sqrt{\gamma|t|})) + e^{\gamma|t|} (1 - \mathrm{erf}\, (\sqrt{\gamma|t|})) \right) \Bigg),$$

where $\chi \in (0,1)$. It has the finite zeroth moment $\tilde{G}_0 = \chi \frac{g^2}{\gamma}$, but integral for $\tilde{G}_1$ does not converge.

$$\rho_{11} = |x(t;\lambda)|^2, \qquad x(t;\lambda) = x_0(t) + \lambda x_{\frac{1}{2}}(t) + O(\lambda^2),$$

where

$$x_0(t) = e^{-\chi \frac{g^2}{\gamma} t}, \qquad x_{\frac{1}{2}}(t) = O(t^{-\frac{1}{2}}), t \to +\infty$$

Conclusions

We have shown:

1. In the considered model we asymptotically describe the dynamics of the reduced density matrix by the GKSL equation with constant coefficients.
2. The accuracy to which this can be done is determined by the number of finite moments of the reservoir correlation function.
3. The initial conditions for the asymptotic GKSL equation do not coincide with the initial conditions for exact dynamics.
4. One can explicitly calculate the generator with the corresponding accuracy.

Conclusions

- A. E. Teretenkov, Non-perturbative effects in corrections to quantum master equations arising in Bogolubov–van Hove limit, J. Phys. A, 54:26 (2021), 265302, 24 pp., arXiv: 2008.02820.
- A. E. Teretenkov, Long-time Markovianity of multi-level systems in the rotating wave approximation, Lobachevskii J. Math., 42:10 (2021), 2455–2465 , arXiv: 2105.02443.
- A. E. Teretenkov, Quantum Markovian Dynamics after the Bath Correlation Time, Computational Mathematics and Mathematical Physics, 63:1 (2023), 135–145.

Thank you for your attention!