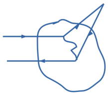
Формула Фейнмана-Каца. Равновесная классическая статфизика. Теория возмущений.

Рассеяние



$< \phi(x)\phi(y) >$, $\phi = n - \bar{n}$
$]\rho(x) \implies \Sigma = \int dx \rho(x)$, $\rho(x) \sim e^{-\frac{H}{kT}}$

$$\frac{H}{kT} \equiv S = \int d\mathbf{x}\left(\tau\phi^2(\mathbf{x}) + \partial\phi(\mathbf{x})\partial\phi(\mathbf{x}) + \frac{g}{4!}\phi^4(\mathbf{x})\right)$$

$$\int D\phi = ?$$

Формула Фейнмана-Каца (Функциональный интеграл,

Континуальный интеграл, фейнмановсаий интеграл по траекториям)

$$\hat{q}|q\rangle = |q\rangle, \hat{p}|p\rangle = e^{ipx}|p\rangle, \langle q|p\rangle = e^{ipq}, |q\rangle = \delta(x-q) \qquad (1)$$

$$\langle q|e^{-\frac{i\hat{H}\Delta t}{\hbar}}|q_0\rangle \approx \langle q|(1 - \frac{i\hat{H}\Delta t}{\hbar})|q_0\rangle \qquad (2)$$

$$1 = \int \frac{dp}{(2\pi\hbar)}|p\rangle\langle p| \qquad (3)$$

$$\langle q|(1 - \frac{i\hat{H}\Delta t}{\hbar})|q_0\rangle = \int \frac{dp}{2\pi\hbar}\langle q|p\rangle\langle p|1 - \frac{i\hat{H}\Delta t}{\hbar}|q_0\rangle \qquad (4)$$

$$\int \frac{dp}{2\pi\hbar}\langle q|p\rangle\langle p|1-\frac{i\hat{H}\Delta t}{\hbar}|q_0\rangle = \int \frac{dp}{2\pi\hbar}\langle q|p\rangle\langle p|q_0\rangle(1-\frac{iH_f\Delta t}{\hbar}) =$$
$$= \int \frac{dp}{2\pi\hbar} e^{-iH_f(p,q)\Delta t\hbar^{-1}} e^{ip(q-q_0)\hbar^{-1}} =$$
$$= \int \frac{dp}{2\pi\hbar} e^{i\hbar^{-1}p(q-q_0)-i\hbar^{-1}H_f(p,q)\Delta t} \tag{5}$$

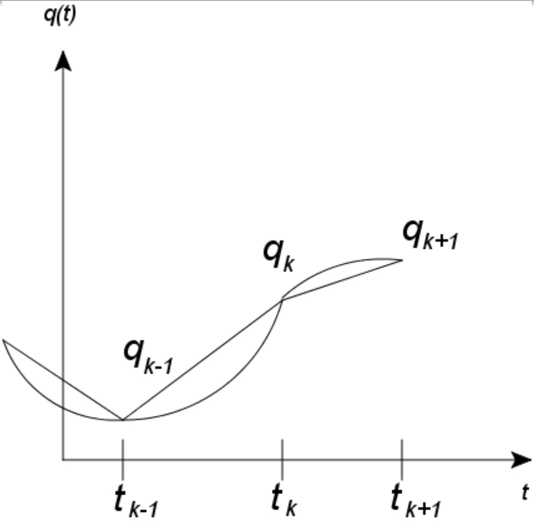
$$U(t,t') = e^{-i\hat{H}(t-t')} \tag{6}$$

$$\delta t = \frac{(t-t')}{N} \tag{7}$$

$$\langle q|e^{-\frac{i\hat{H}(t-t')}{\hbar}}|q_0\rangle = \langle q|e^{-\frac{i\hat{H}\delta t}{\hbar}} e^{-\frac{i\hat{H}\delta t}{\hbar}} \cdots e^{-\frac{i\hat{H}\delta t}{\hbar}}|q_0\rangle \tag{8}$$

$$\langle q|e^{-\frac{i\hat{H}(t-t')}{\hbar}}|q_0\rangle = \prod_{k=1}^{N-1} \int dq_k \langle q|e^{-\frac{i\hat{H}\delta t}{\hbar}}|q_{k-1}\rangle\langle q_{k-1}|e^{-\frac{i\hat{H}\delta t}{\hbar}}|q_{k-2}\rangle \cdots \langle q_2|e^{-\frac{i\hat{H}\delta t}{\hbar}}|q_0\rangle \tag{9}$$

$$\langle q|e^{-\frac{i\hat{H}(t-t')}{\hbar}}|q_0\rangle = \prod_{k=1}^{N-1}\prod_{j=1}^{N} \int dq_k \frac{dp_k}{2\pi\hbar} e^{\sum_{i=1}^{N-1}(i\hbar^{-1}p_j\frac{(q_{k+1}-q_k)}{\delta t} - i\hbar^{-1}H_f(p,q))\delta t} \tag{10}$$


q(t)
q_{k+1}
q_k
q_{k-1}
t_{k-1}
t_k
t_{k+1}
t

$N \to \infty$

$$(i\hbar^{-1} p_j \frac{(q_{k+1} - q_k)}{\delta t} - i\hbar^{-1} H_f(p,q))\delta t \to \int dt i\hbar^{-1}(p\dot{q} - H_f(p,q)) = \int dt i\hbar^{-1} S_{classica} \tag{11}$$

$$\langle q|U(t,t')|q_0\rangle = \int_{q(t')=q_0}^{q(t)=q} \mathcal{D}q(t) \int \mathcal{D}\frac{p(t)}{2\pi} e^{iS_{classical}(t,t')\hbar^{-1}} \tag{12}$$

Полные функции Грина

$$<<\phi(x_1)...\phi(x_n)>>\equiv\int D\phi\phi(x_1)...\phi(x_n)e^{-S(\phi)}$$

$$S=\int dx(\frac{1}{2}\partial_i\phi\partial_i\phi+\frac{\tau}{2}\phi^2+\frac{g}{4!}\phi^4)$$

$$\int dxe^{-\frac{1}{2}x^2-\frac{g}{4!}x^4}$$

$$Z(A,b) = \int d^n x e^{-\frac{1}{2}(x,Kx)+bx}, K = K^T \tag{13}$$

$$x = K^{-1}b + y \tag{14}$$

$$\frac{-1}{2}(x,Ax) + bx = -\frac{1}{2}\left((K^{-1}b,b) + (K^{-1}b,Ky) + (y,b) + (y,Ky)\right) + \\ + (b,K^{-1}b) + by = -\frac{1}{2}(b,K^{-1}b) - \frac{1}{2}(y,Ky) \tag{15}$$

$$Z(K,b) = e^{\frac{1}{2}(b,K^{-1}b)} \int dy e^{-\frac{1}{2}(y,Ky)} \tag{16}$$

$$K' = OKO^T, K'_{ij} = a_i\delta_{ij}, y = Oz, |Det(O)| = 1, O^T O = I \tag{17}$$

$$
\begin{aligned}
&-\frac{1}{2}(y, Ky) = -\frac{1}{2}(y, OK'O^T y) = -\frac{1}{2}(O^T y, K'O^T y) = \\
&= -\frac{1}{2}(O^{-1}y, K'O^{-1}y) = -\frac{1}{2}(z, K'z) = -\frac{1}{2}\sum_i a_i z_i^2
\end{aligned} \tag{18}
$$

$$
Z(K, b) = e^{\frac{1}{2}(b, K^{-1}b)} \prod_{i=1}^{N} \int dz_i e^{-\frac{1}{2}a_i z_i^2} \tag{19}
$$

$$
Z(K, b) = det^{-\frac{1}{2}}(\frac{K}{2\pi}) e^{\frac{1}{2}(b, K^{-1}b)} \tag{20}
$$

$$
K^{-1} = \Delta = G \tag{21}
$$

$$
\langle\langle x_{k_1} \cdots x_{k_l} \rangle\rangle = det^{-\frac{1}{2}}(\frac{K}{2\pi}) \left[\frac{\partial}{\partial b_{k_1}} \cdots \frac{\partial}{\partial b_{k_l}} Z(K, b)\right]\Bigg|_{b=0} \tag{22}
$$

$$
\langle\langle x_{k_1} \cdots x_{k_l} \rangle\rangle = \sum_P \langle\langle x_{k_{P_1}} x_{k_{P_2}} \rangle\rangle \cdots \langle\langle x_{k_{P_{l-1}}} x_{k_{P_l}} \rangle\rangle \tag{23}
$$

Доказательство:

$$det \equiv det^{-\frac{1}{2}}(\frac{K}{2\pi})$$

$$<< x_1x_2 >>= \partial_{b_1}\partial_{b_2} det\, e^{\frac{1}{2}bGb}|_{b=0} = det\partial_{b_1}\left(G_{1k}b_k e^{\frac{1}{2}bGb}\right)|_{b=0} =$$

$$det\left(G_{12} + G_{1k}b_k G_{1k'}b'_k\right)e^{\frac{1}{2}bGb}|_{b=0}$$

$$\langle\langle x_{i_1}x_{i_2}x_{i_3}x_{i_4}\rangle\rangle = G_{i_1 i_2}G_{i_3 i_4} + G_{i_1 i_3}G_{i_2 i_4} + G_{i_1 i_4}G_{i_3 i_2} \qquad (24)$$

$$S = \frac{1}{2}(\varphi, K\varphi) + \frac{g}{4!}\varphi^4 \tag{25}$$

$$\varphi K\varphi \equiv \int dxdy\varphi(x)K(x,y)\varphi(y) \quad \varphi A \equiv \int dx\varphi(x)A(x)$$

Производящий функционал полных функций Грина

$$\bar{G}(A) \equiv \int D\varphi e^{-S+A\varphi}$$

$$\partial_{b_i} \to \frac{\delta}{\delta A(x)}, \quad \frac{\delta A(y)}{\delta A(x)} = \delta(x-y) = 1$$

$$S_0 = \int dx(\partial_i\varphi\partial_i\varphi + \frac{\tau}{2}\varphi^2) \implies$$

$$\varphi K\varphi = \int dxdy\delta(x-y)(\partial_{x_i}\varphi(x)\partial_{y_i}\varphi(y) + \frac{\tau}{2}\varphi(x)\varphi(y))$$

$$K = (-\Delta + \tau)\delta(x-y)$$

$$KK^{-1} = 1 \implies \int dz K(x-z) K^{-1}(z-y) = \delta(x-y)$$

$$(-\Delta + \tau) K^{-1}(x-y) = \delta(x-y) \implies K^{-1} = G$$

$$]f_n(x): \quad Kf_n = \lambda_n f_n \quad \varphi = \sum c_n f_n(x) \implies \varphi K \varphi = \sum \lambda_n c_n^2$$

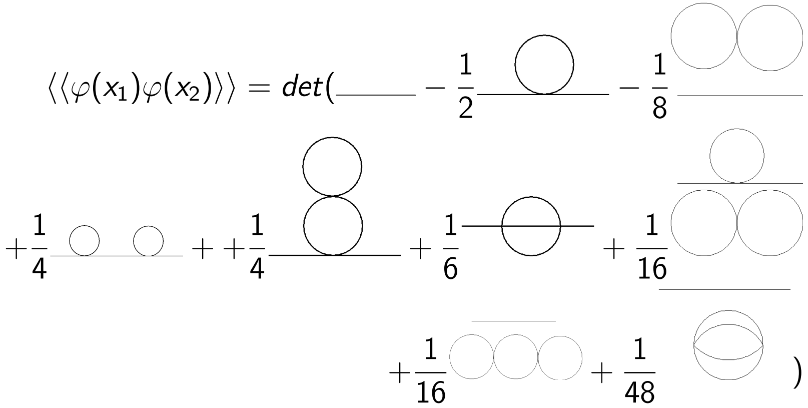
$K^{-1}\varphi \equiv \int dy K^{-1}(x-y)\varphi(y)$

$$\int D\varphi \varphi_(x_1) \ldots \varphi_(x_n) e^{-\frac{1}{2}\varphi K \varphi - \frac{g}{4!}\int dx \varphi^4(x)} =$$

$$\int D\varphi \varphi_(x_1) \ldots \varphi(x_n) e^{-\frac{1}{2}\varphi K \varphi} \sum_k \frac{1}{k!} (-\frac{g}{4!} \int dx \varphi^4(x))^k$$

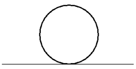
$\phi = \varphi + K^{-1}A$

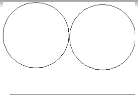
$$\langle\langle \varphi(x_1)\varphi(x_2) \rangle\rangle = \int D\varphi \varphi_1 \varphi_2 e^{-\frac{1}{2}\varphi K \varphi} (1 + \frac{1}{2!}(-\frac{g}{4!}\int dx \varphi^4(x)) + \ldots)$$

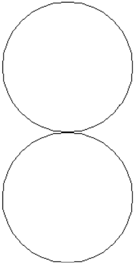
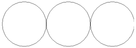
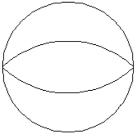
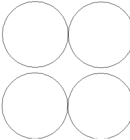


(26)

Симметрийные коэффициенты, матрица смежности, классификация Никеля

 $= G(x-y)$  $= \int dz G(x-z)G(z-z)G(z-y)$ (27)

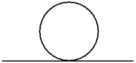
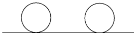
 $= G(x-y)\int dzG(z-z)G(z-z)$ (28)

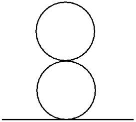
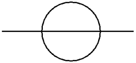
$\langle\langle 1\rangle\rangle = det(1-\frac{1}{8}$  $+\frac{1}{16}$  $+\frac{1}{48}$  $+\frac{1}{64}$  (29)

Корреляционные функции

$$\langle\varphi(x_1)\dots\varphi(x_n)\rangle = \frac{\langle\langle\varphi(x_1)\dots\varphi(x_n)\rangle\rangle}{\Sigma} = \frac{\langle\langle\varphi(x_1)\dots\varphi(x_n)\rangle\rangle}{\langle\langle 1\rangle\rangle}$$

$$\langle\varphi(x_1)\varphi(x_2)\rangle = \qquad (30)$$

$=$ $-\frac{1}{2}$  $+\frac{1}{4}$  $+$

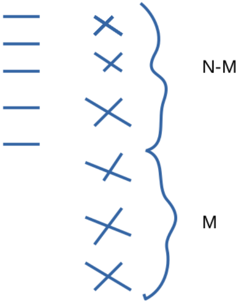
$+\frac{1}{4}$  $+\frac{1}{6}$  (31)

Док-во:

$$\frac{\underline{\qquad\qquad} - \frac{1}{8}\,\overline{\bigcirc\bigcirc}}{\dfrac{\bigcirc}{1 - \frac{1}{8}\bigcirc}} \approx \underline{\qquad\qquad} - \frac{1}{8}\,\bigcirc\bigcirc + \frac{1}{8}\,\bigcirc\bigcirc \tag{32}$$

$$\langle\varphi(x_1)\dots\varphi(x_n)\rangle\langle\langle 1\rangle\rangle = \langle\langle\varphi(x_1)\dots\varphi(x_n)\rangle\rangle$$

n – хвостов, N – вершин, M – в вакуумной петле



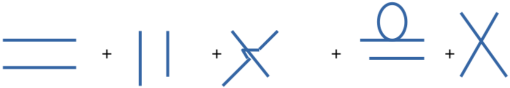
(33)

$$C_N^M = \frac{N!}{M!(N-M)!}\quad\frac{1}{N!} \implies \frac{1}{M!(N-M)!}$$

$$\frac{1}{(4!)^N} = \frac{1}{(4!)^M(4!)^{N-M}}$$

1-я теорема Майера

$$< \phi_1 ... \phi_4 >= \quad (34)$$



Производящий функционал корреляционных функций
$G(A) = \frac{\int D\phi e^{-S+\phi A}}{\int D\phi e^{-S}}$

$$G(A) = \sum \frac{1}{n!} A^n G_n$$

Производящий функционал связных функций $W(A)$

$$W(A) = \sum \frac{1}{n!} A^n W_n$$

$$G(a) = \quad\quad + \ldots \tag{35}$$

$$W(a) = \quad\quad + \ldots \tag{36}$$

$$G(A) = e^{W(A)} \implies W(A) = \ln G(A)$$

$$G_4 = \quad\quad + \ldots \tag{37}$$

Пропагаторы теории

$$(-\Delta + \tau)G = \delta(x - x')$$

$$d = 3 \implies G(x) = \frac{1}{4\pi r} e^{-r\sqrt{\tau}}$$

$$\forall d \implies G(k) = \frac{1}{k^2 + \tau}$$

Перейдем в импульсное представление:

$$\int dx_1 dx_2 G(x-x_1)G^3(x_1-x_2)G(x_2-y)$$

$\int dxdy e^{-ip_1x-ip_2y},\ G(a-b)=\int \frac{dk_i}{(2\pi)^d}G(k_i)e^{ik(a-b)}$

$$=\int dxdydx_1dx_2\frac{dk_1dk_2dk_3dk_4dk_5}{(2\pi)^{5d}}G(k_1)G(k_2)G(k_3)G(k_4)G(k_5)$$

$\exp(-ip_1x-ip_2y+ik_1(x-x_1)+i(k_2+k_3+k_4)(x_1-x_2)+ik_5(x_2-y))$

$$\int dx\to(2\pi)^d\delta(p_1-k_1),\quad \int\frac{dk_1}{(2\pi)^d}\to k_1=p_1$$

$$\int dy\to(2\pi)^d\delta(p_2-k_5),\quad \int\frac{dk_5}{(2\pi)^d}\to k_5=-p_2$$

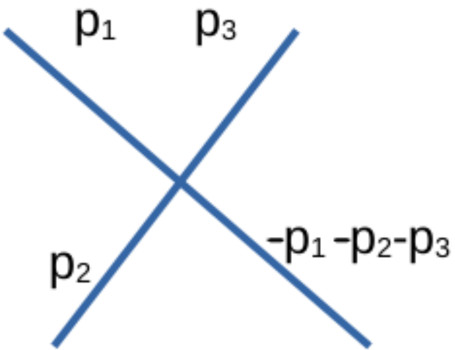
$$\int dx_1\to(2\pi)^d\delta(p_1-k_2-k_3-k_4),\quad \int\frac{dk_4}{(2\pi)^d}\to k_4=p_1-k_2-k_3$$

$$\int dx_2 \to (2\pi)^d \delta(p_2 + k_2 + k_3 + k_4), \quad = \delta(p_1 + p_2)$$

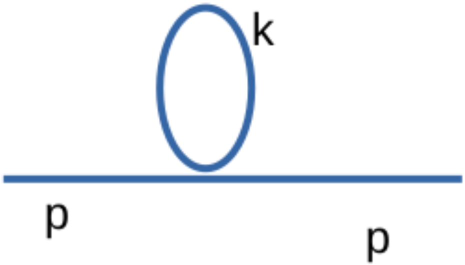
$$Arbuz = \int \frac{dkdq}{(2\pi)^{2d}} G(p)G(k)G(q)G(p-k-q)G(p)\underline{\delta(p_1+p_2)}$$

пояснение – трансляционная инвариантность:

$$\int dxdye^{-ip_1x-ip_2y}f(x-y) = \int d(x-y)dye^{-ip_1(x-y)-i(p_1+p_2)y} \sim \delta(p_1+p_2)$$



$$= G(p_1)G(p_2)G(p_3)G(-p_1-p \tag{38}$$



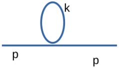
$$= G(p)G(p)\int \frac{dk}{(2\pi)^d} G(k) \quad (39)$$

Ампутированные диаграммы —

$\alpha-$ представление

$$\frac{1}{a^\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^\infty dt t^{\alpha-1} e^{-at}$$

$$\frac{1}{a_1^{\alpha_1} a_2^{\alpha_2}} = \frac{\Gamma(\alpha_1+\alpha_2)}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^1 dz z^{\alpha_1-1}(1-z)^{\alpha_2-1} \frac{1}{(za_1+(1-z)a_2)^{\alpha_1+\alpha_2}}$$



$$|_{amp} = \int \frac{dk}{(2\pi)^d(k^2+\tau)} = \int_0^\infty dt \int \frac{dk}{(2\pi)^d} e^{-t(k^2+\tau)} = \tag{40}$$

$$= \int_0^\infty dt \frac{\pi^{d/2}}{(2\pi)^d t^{d/2}} e^{-t(\tau)} = \frac{\pi^{d/2}}{(2\pi)^d} \tau^{d/2-1} \Gamma(1-d/2)$$

$$\Big|_{amp} = \int \frac{dk}{(2\pi)^d (k^2+\tau)((p-k)^2+\tau)} = \tag{41}$$

$$= \int_0^1 dz \int \frac{dk}{(2\pi)^d ((k^2+\tau)z + ((p-k)^2+\tau)(1-z))^2}$$

$$= \int_0^1 dz \int \frac{dk}{(2\pi)^d} \int_0^\infty dt t e^{-t((k-p(1-z))^2 + p^2 z(1-z) + \tau)} =$$

$$= \frac{\pi^{d/2}\Gamma(2-d/2)}{(2\pi)^d} \int_0^1 dz (p^2 z(1-z)+\tau)^{d/2-2}$$

$]\tau = 0$

$$\int dx \frac{e^{-ik(x-x')}}{(x-x')^{2\alpha}} = \int dx \frac{1}{\Gamma(\alpha)} \int_0^\infty dt t^{\alpha-1} e^{-t(x-x')^2 - ik(x-x')} =$$

$$= \frac{\pi^{d/2}}{\Gamma(\alpha)} \int_0^\infty dt t^{\alpha-1-d/2} e^{-\frac{k^2}{4t}} =$$

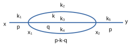
$t_1 = k^2/(4t)$

$$= \frac{\pi^{d/2} 2^{2(d/2-\alpha)}}{\Gamma(\alpha)} \int_0^\infty dt_1 t_1^{d/2-\alpha-1} e^{-t_1} \frac{1}{k^{2(d/2-\alpha)}} \equiv c(\alpha) \frac{1}{k^{2(d/2-\alpha)}}$$

$$c(\alpha) = \frac{\pi^{d/2} 2^{2(d/2-\alpha)} \Gamma(d/2-\alpha)}{\Gamma(\alpha)}$$

$$\int \frac{dk}{(2\pi)^d} \frac{e^{ik(x-x')}}{k^{2\alpha}} = \bar{c}(\alpha) \frac{1}{(x-x')^{2(d/2-\alpha)}} \quad \bar{c}(\alpha) = \frac{1}{(2\pi)^d} c(\alpha)$$

$$\bar{c}(\alpha) c(d/2-\alpha) = 1$$



$$\to \bar{c}(1)^5 \int dx_1 dx_2 \frac{1}{(x-x_1)^{2(d/2-1)}} \left(\frac{1}{(x_1-x_2)^{2(d/2-1)}} \right)^3 \frac{1}{(x_2-y)^{2(d/2-1)}}$$

$$\to \bar{c}(1)^5 c(d/2-1)^2 c(3d/2-3) \frac{1}{p^2} \frac{1}{p^{2(3-d)}} \frac{1}{p^2} = \bar{c}(1)^3 c(3d/2-3) \frac{1}{p^{2(5-d)}}$$