Lectures 7&8
Emittance Growth due to Noise in RF and Magnets

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Objectives

- In a collider the beam should stay for a long time
  ⇨ The growth rates for beam emittances and bunch length should be sufficiently small
- In a properly built machine the IBS typically dominates
- However, the RF noise, if not properly addressed, may result in unacceptably large longitudinal emittance growth
  - Proton bunches are long
    - Therefore, both the phase and amplitude noises are important
  - Additional complication originates from non-linearity of potential well. It is important for hadron beams which, typically, take large fraction of RF well.
- Similar to the longitudinal degree of freedom, noise in bending magnetic field and ⊥ dampers leads to transverse emittance growth
  - Proton colliders have large circumference => small revolution frequency => more susceptible to noise due to its fast growth with frequency decrease

Fourier Transform, Spectrum, Spectral Density and Shot Noise

Correlation Function and Spectral Density

- For a complex function approaching zero at ± infinity we have

$$f_\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt \Leftrightarrow f(t) = \int_{-\infty}^{\infty} f_\omega e^{i\omega t}d\omega$$

- Introduce correlation function $K(\tau) \equiv \overline{f(t)f^*(t+\tau)}$ and express it through Fourier harmonics

$$\overline{f(t)f^*(t+\tau)} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \overline{f_\omega f_{\omega'}^*} e^{i\omega t} e^{-i\omega'(t+\tau)} d\omega d\omega'$$

- Since different harmonics in a random process do not correlate and diverge with integration time, T, we put: $\overline{f_\omega f_{\omega'}^*} = P(\omega)\delta(\omega-\omega')$

$$\Rightarrow \quad K(\tau) = \overline{f(t)f^*(t+\tau)} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \overline{f_\omega f_{\omega'}^*} e^{i\omega t} e^{-i\omega(t+\tau)} d\omega d\omega' = \int_{-\infty}^{\infty} P(\omega)e^{i\omega\tau}d\omega$$

- Thus, we obtained the Wiener-Khinchin theorem

$$K(\tau) = \int_{-\infty}^{\infty} P(\omega)e^{i\omega\tau}d\omega\,, \quad P(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} K(\tau)e^{-i\omega\tau}dt$$

Another Way to Introduce the Spectral Density

- To prevent a divergence of random function harmonic f_ω, we redefine it in the following way

$$F_\omega = \frac{1}{2\pi}\frac{1}{\sqrt{T}}\int_{-T/2}^{T/2} f(t)e^{-i\omega t}dt \Leftrightarrow f(t) = \sqrt{T}\int_{-\infty}^{\infty} F_\omega e^{i\omega t}d\omega$$

  - F_ω introduced this way stays finite with T->∞

- For the rms value of the function that yields:

$$\overline{\left|f(t)\right|^2} = \frac{1}{T}\int_{-T/2}^{T/2}\left|f(t)\right|^2 dt = \frac{1}{T}\int_{-T/2}^{T/2} dt\sqrt{T}\int_{-\infty}^{\infty} F_\omega e^{i\omega t}d\omega\sqrt{T}\int_{-\infty}^{\infty} F_{\omega'}{}^* e^{-i\omega' t}d\omega' \xrightarrow[\text{averaging}]{\text{Accounting}}$$

$$= \int_{-\infty}^{\infty}\overline{F_\omega F_{\omega'}{}^*}d\omega d\omega'\int_{-\infty}^{\infty} e^{i(\omega-\omega')t}dt = 2\pi\int_{-\infty}^{\infty}\overline{F_\omega F_{\omega'}{}^*}d\omega d\omega'\delta(\omega-\omega') = 2\pi\int_{-\infty}^{\infty}\left|F_\omega{}^2\right|d\omega$$

$$\Rightarrow \quad P(\omega) = 2\pi\left|F_\omega{}^2\right|$$

where we used that $\int_{-\infty}^{\infty} e^{i\omega t}d\omega = 2\pi\delta(t)$

  - To prove we put $f(t) = \delta(t)$ then:

$$\delta(t) = \int_{-\infty}^{\infty}\left(\delta(t)\right)_\omega e^{i\omega t}d\omega \quad \Leftrightarrow \quad \left(\delta(t)\right)_\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty}\delta(t)e^{-i\omega t}dt = \frac{1}{2\pi} \quad \Rightarrow \quad \delta(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega t}d\omega$$

Relationship between Field Energy and Spectral Energy

- Relationship between $\int_{-\infty}^{\infty} |f(t)|^2 dt$ and $\int_{-\infty}^{\infty} |f_\omega|^2 d\omega$
  - Fourier transform is applicable if $f(t) \xrightarrow{t \to \pm\infty} 0$.
    Then we can bind $f(t)$ and f_ω

$$\int_{-\infty}^{\infty} |f(t)|^2 dt = \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} d\omega d\omega' f_\omega f_{\omega'}{}^* e^{i(\omega-\omega')t} = 2\pi \int_{-\infty}^{\infty} |f_\omega|^2 d\omega$$

Shot (Schottky) Noise

- We look for a spectral density of sequence of equal pulses which are located randomly in time: $U(t) = \sum_n u(t - t_n)$

$$\overline{U^2} = \frac{1}{T}\int_{-T/2}^{T/2} dt \sum_{n,m} u(t-t_n) u^*(t-t_m) = \frac{1}{T}\int_{-T/2}^{T/2} dt \left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d\omega d\omega' \sum_{n,m} u_\omega u^*_{\omega'} \overline{e^{i(\omega(t-t_n)-\omega'(t-t_n))}} \right)$$

$$\xrightarrow{\overline{e^{i(\omega(t-t_n)-\omega'(t-t_m))}} = \delta_{nm} e^{i(\omega-\omega')(t-t_n)}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u_\omega u^*_{\omega'} d\omega d\omega' \frac{1}{T}\int_{-T/2}^{T/2} dt \sum_n e^{i(\omega-\omega')(t-t_n)}$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u_\omega u^*_{\omega'} d\omega d\omega' \left(\frac{1}{T}\sum_n e^{-i(\omega-\omega')t_n} \right) \int_{-T/2}^{T/2} dt e^{i(\omega-\omega')t} \xrightarrow{\int_{-\infty}^{\infty} dt e^{i(\omega-\omega')t} = 2\pi\delta(\omega-\omega')}$$

$$= 2\pi \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u_\omega u^*_{\omega'} d\omega d\omega' \left(\frac{1}{T}\sum_n e^{-i(\omega-\omega')t_n} \right) \delta(\omega-\omega') \xrightarrow{\frac{1}{T}\sum_n e^{-i(\omega-\omega')t_n} \to \dot{n}}$$

$$\dot{n} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u_\omega u^*_{\omega'} d\omega d\omega' \int_{-\infty}^{\infty} dt e^{i(\omega-\omega')t} \delta(\omega-\omega') = 2\pi\dot{n} \int_{-\infty}^{\infty} |u_\omega|^2 d\omega$$

=> $\boxed{P(\omega) = 2\pi\dot{n}|u_\omega|^2}$.

- For electric current $u(t) = e\delta(t)$ => $P_I(\omega) = \frac{e^2\dot{n}}{2\pi} = \frac{eI}{2\pi}$ => $\boxed{\overline{\Delta I^2} = 2eI\Delta f}$

Longitudinal Emittance Growth due to RF Noise

Equations of Longitudinal Motion

- In the absence of perturbations

$$\frac{d^2\varphi}{dt^2}+\Omega_s{}^2\sin\varphi=0\,,\quad \Omega_s{}^2=\frac{eZV_0\eta q}{2\pi Amc^2\gamma\beta^2}$$

- Fluctuations of RF phase and amplitude result in

$$\frac{d^2\varphi}{dt^2}+\Omega_s{}^2\left(1+\frac{\delta V(t)}{V_0}\right)\sin\left(\varphi-\psi(t)\right)=0\quad\xrightarrow[u(t)\equiv\delta V(t)/V_0]{\text{exp}ending}\quad\frac{d^2\varphi}{dt^2}+\Omega_s{}^2\sin\varphi=-\Omega_s{}^2\left(\sin(\varphi)u(t)+\cos(\varphi)\psi(t)\right)$$

- First, we consider a small amplitude (i.e.) linear motion and RF phase fluctuations. Then

$$\frac{d^2\varphi}{dt^2}+\Omega_s{}^2\varphi=-\Omega_s{}^2\psi(t)$$

The solution is well-known

$$\varphi(t)=-\Omega_s\int_0^t\psi(t')\sin\left(\Omega_s(t-t')\right)dt'$$

The rms particle deviation is

$$\overline{\varphi^2(t)}=\Omega_s{}^2\int_0^t\int_0^t\overline{\psi(t_1)\psi(t_2)}\sin\left(\Omega_s(t-t_1)\right)\sin\left(\Omega_s(t-t_2)\right)dt_1dt_2$$

Spectral Density and Correlation Function

- Correlation function

$$K(\tau) = \overline{\psi(t)\psi(t+\tau)} \quad \Rightarrow \quad K(t_1 - t_2) = \overline{\psi(t_1)\psi(t_2)}$$

- Wiener-Khinchin theorem

$$K(\tau) = \int_{-\infty}^{\infty} P(\omega) e^{i\omega\tau} d\omega \,, \quad P(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} K(\tau) e^{-i\omega\tau} dt$$

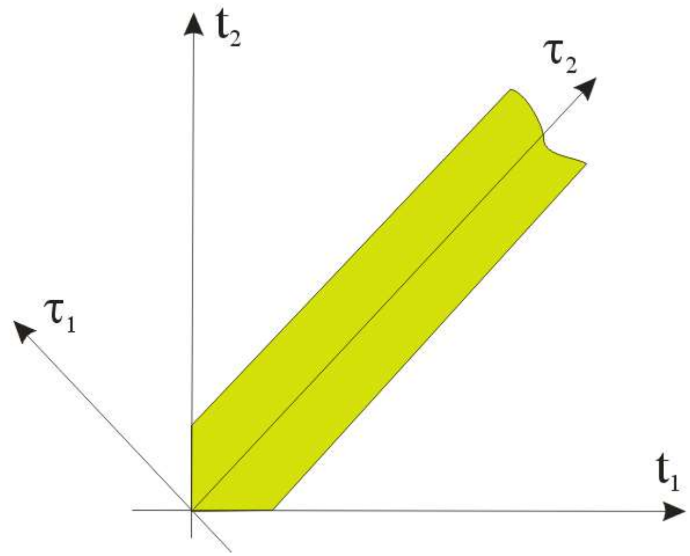
$$\overline{\psi(t_1)\psi(t_2)} = K(t_1 - t_2) \quad \Rightarrow \quad \overline{\psi^2} = K(0) = \int_{-\infty}^{\infty} P(\omega) d\omega$$

- Particle motion under random phase fluctuations

$$\overline{\varphi^2(t)} = \Omega_s^{\;2} \int_0^t \int_0^t \overline{\psi(t_1)\psi(t_2)} \sin\left(\Omega_s (t - t_1)\right) \sin\left(\Omega_s (t - t_2)\right) dt_1 dt_2$$

$$= \Omega_s^{\;2} \int_0^t \int_0^t K_\psi(t_1 - t_2) \sin\left(\Omega_s (t - t_1)\right) \sin\left(\Omega_s (t - t_2)\right) dt_1 dt_2$$

Computation of integral



Make substitution $\begin{cases}\tau_1 = t_1 - t_2 \\ \tau_2 = t_1 + t_2\end{cases}$ $\begin{cases} t_1 = (\tau_1 + \tau_2)/2 \\ t_2 = (\tau_2 - \tau_1)/2 \end{cases}$

Corresponding Jacobian is:

$$\frac{\partial(t_1,t_2)}{\partial(\tau_1,\tau_2)} = \begin{bmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{bmatrix} = \frac{1}{2}$$

Then we have

$$\overline{\varphi^2(t)} = \Omega_s{}^2 \int_0^t \int_0^t K_\psi(t_1 - t_2)\sin\left(\Omega_s(t-t_1)\right)\sin\left(\Omega_s(t-t_2)\right) dt_1 dt_2$$

$$= \Omega_s{}^2 \int_0^t \int_0^t K_\psi(t_1 - t_2)\left[\frac{\cos\left(\Omega_s(t_1-t_2)\right) + \cos\left(\Omega_s(2t - t_1 - t_2)\right)}{2}\right]\left(\frac{1}{2}\right) dt_1 dt_2$$

$$\approx \frac{\Omega_s{}^2}{4}\int_{-\infty}^{\infty} d\tau_1 \int_0^{2t} K_\psi(\tau_1)\left[\cos\left(\Omega_s\tau_1\right) + \cos\left(\Omega_s(2t-\tau_2)\right)\right] d\tau_2 \xrightarrow[\text{oscillating term in } t_2 \text{ integration}]{\text{drop fast}}$$

$$\approx \frac{\Omega_s{}^2}{2} t \int_{-\infty}^{\infty} K_\psi(\tau_1)\cos\left(\Omega_s\tau_1\right) d\tau_1$$

Recollecting connection between the correlation function and the spectral density we finally obtain: $\boxed{\frac{d}{dt}\overline{\varphi^2(t)} = \pi\Omega_s{}^2 P_\psi(\Omega_s)}$

Bunch Lengthening due to Amplitude Noise

- Equation of motion for the small amplitude RF voltage fluctuations:

$$\frac{d^2\varphi}{dt^2} + \Omega_s{}^2\varphi = -\Omega_s{}^2\varphi u(t)$$

In perturbation theory we replace φ in RH side by $\varphi_0 \sin(\Omega_s t)$

$$\Rightarrow \quad \overline{\varphi^2(t)} \approx \Omega_s{}^2\varphi_0{}^2 \int_0^t\int_0^t \overline{u(t_1)u(t_2)} \sin(\Omega_s t_1)\sin(\Omega_s t_2)\sin(\Omega_s(t-t_1))\sin(\Omega_s(t-t_2))\,dt_1 dt_2$$

Acting similar to the case of phase noise, accounting that $K_u(t_1 - t_2) = \overline{u(t_1)u(t_2)}$ and dropping fast oscillating terms we obtain

$$\overline{\varphi^2(t)} \approx \frac{\Omega_s{}^2}{4} t\varphi_0{}^2 \int_{-\infty}^{\infty} K_u(\tau)\cos(2\Omega_s\tau)\,d\tau$$

Accounting also $\overline{\varphi^2(t)} = \varphi_0{}^2/2$ we finally obtain

$$\boxed{\frac{d}{dt}\overline{\varphi^2(t)} = \pi\Omega_s{}^2\,\overline{\varphi^2(t)}P_u(2\Omega_s)}$$

Practical Estimate

- Let's consider Tevatron: $f_s = \Omega_s / 2\pi$ = 35 Hz, initial bunch length 30 cm and RF bucket length of 5.65 m (53.1 MHz)
- Require the bunch lengthening 10% after in 10 hours

$$\varphi_{fin} = \sqrt{\varphi_0^{\,2} + T \frac{d}{dt}\overline{\varphi^2}} \approx \varphi_0 + \frac{T}{2\varphi_0}\frac{d}{dt}\overline{\varphi^2}$$

$$\Rightarrow \frac{d}{dt}\overline{\varphi^2} = \frac{\varphi_{fin} - \varphi_0}{\varphi_0 T} 2\varphi_0^{\,2} = \frac{0.05}{10 \cdot 3600} 2 \left(2\pi \frac{35}{565} \right)^2 = 6.5 \cdot 10^{-7} \frac{\text{rad}^2}{\text{s}}$$

⇨ Corresponding spectral densities

$$P_\psi = 4.3 \cdot 10^{-12} \text{s}^{-1}, \quad P_u = 3.8 \cdot 10^{-11} \text{s}^{-1}$$

⇨ Corresponding rms fluctuations for the white noise in 100 Hz band

$$\sqrt{\overline{\psi^2}} = \sqrt{4\pi P_\psi \Delta f} = 7.3 \cdot 10^{-5} \text{rad}, \quad \sqrt{\overline{u^2}} = \sqrt{4\pi P_{u\psi} \Delta f} = 2.2 \cdot 10^{-4} \text{rad}$$

Here 4π accounts transition from "physical" to "technical" definition of the spectral density

- Next, we see how it works in non-linear RF motion

Action-Phase Variables

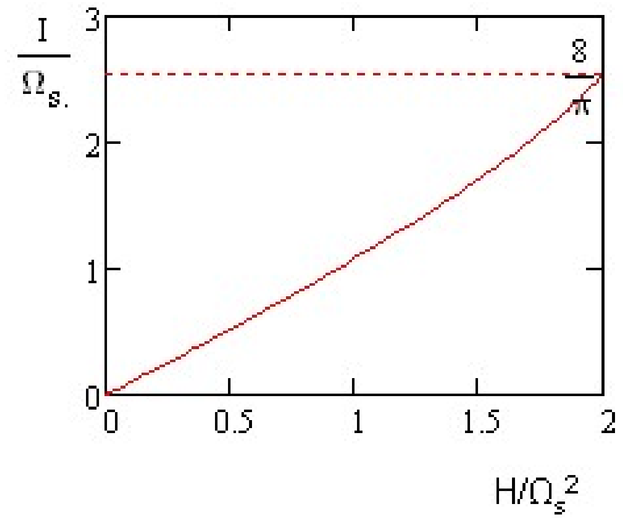
$$H=\frac{\hat{p}^2}{2}+U(\varphi)\xrightarrow[\text{harmonic RF}]{\text{Single}}\frac{\hat{p}^2}{2}+2\Omega_s^{\;2}\left(\sin\frac{\varphi}{2}\right)^2$$

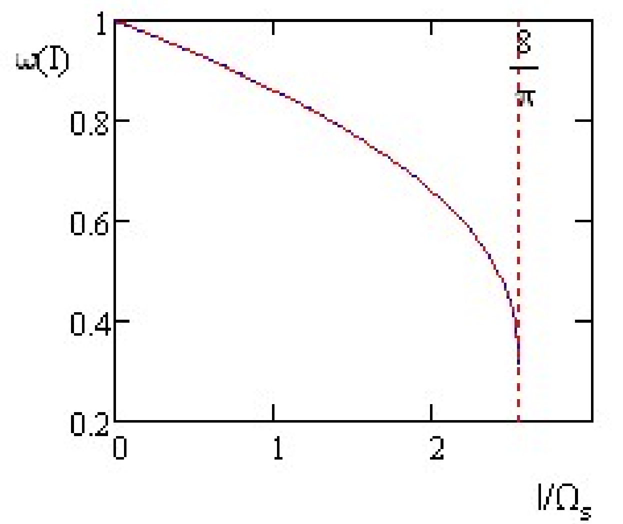
$$I=\frac{1}{2\pi}\oint\hat{p}\,d\varphi\,,\qquad \frac{dH}{dI}=\omega\,,\qquad \phi=\omega t$$

- For single harmonic RF

$$I=\frac{2\sqrt{2}\Omega_s}{\pi}\int_0^{\varphi_{max}}\sqrt{\cos\varphi-\cos\varphi_{max}}\,d\varphi\,,\qquad H=2\Omega_s^{\;2}\left(\sin\frac{\varphi_{max}}{2}\right)^2$$

$$T=\frac{4}{\sqrt{2}\Omega_s}\int_0^{\varphi_{max}}\frac{d\varphi}{\sqrt{\cos\varphi-\cos\varphi_{max}}}\,,\qquad \omega=\frac{2\pi}{T}$$





Fokker-Planck Equation

- Introduce the diffusion equation in the following form:

$$\frac{\partial f}{\partial t} = \frac{1}{2}\frac{\partial}{\partial I}\left(\frac{D(I)}{\omega(I)} I \frac{\partial f}{\partial I}\right)$$

- Changes in average action/energy

$$\frac{d\bar{I}}{dt} = \int_0^\infty I \frac{\partial f}{\partial t} dI = \frac{1}{2}\int_0^\infty I \frac{\partial}{\partial I}\left(\frac{D(I)}{\omega(I)} I \frac{\partial f}{\partial I}\right) dI = -\frac{1}{2}\int_0^\infty \frac{D(I)}{\omega(I)} I \frac{\partial f}{\partial I} dI = \frac{1}{2}\int_0^\infty f \frac{d}{dI}\left(I\frac{D(I)}{\omega(I)}\right) dI$$

$$\xrightarrow{f(I)=\delta(I-I_0)} \frac{1}{2}\left(\frac{D(I_0)}{\omega(I_0)} + I_0 \frac{d}{dI}\left(\frac{D(I_0)}{\omega(I_0)}\right)\right) \Rightarrow \frac{d\bar{H}}{dt} = \omega \frac{d\bar{I}}{dt} = \frac{1}{2}\left(D(I) + I\omega(I)\frac{d}{dI}\left(\frac{D(I)}{\omega(I)}\right)\right)$$

For linear RF: $\frac{d\bar{H}}{dt} = \frac{1}{2} D \quad \Rightarrow \quad \frac{d}{dt}\overline{p^2} = \frac{1}{2} D$

1/2 accounts reduction of momentum growth in linear oscillator

Thus, for linear RF: $D(I) = \pi\Omega_s^{\ 3}\left(2\Omega_s P_\varphi(\Omega_s) + P_u(2\Omega_s) I\right)$

- Widening of the distribution in the action space

$$\frac{d}{dt}\overline{\delta I^2} \equiv \frac{d}{dt}\overline{(I-I_0)^2} = \frac{1}{2}\int_0^\infty (I-I_0)^2 \frac{\partial}{\partial I}\left(\frac{D(I)}{\omega(I)} I \frac{\partial f}{\partial I}\right) dI = -\int_0^\infty (I-I_0)\frac{D(I)}{\omega(I)} I \frac{\partial f}{\partial I} dI$$

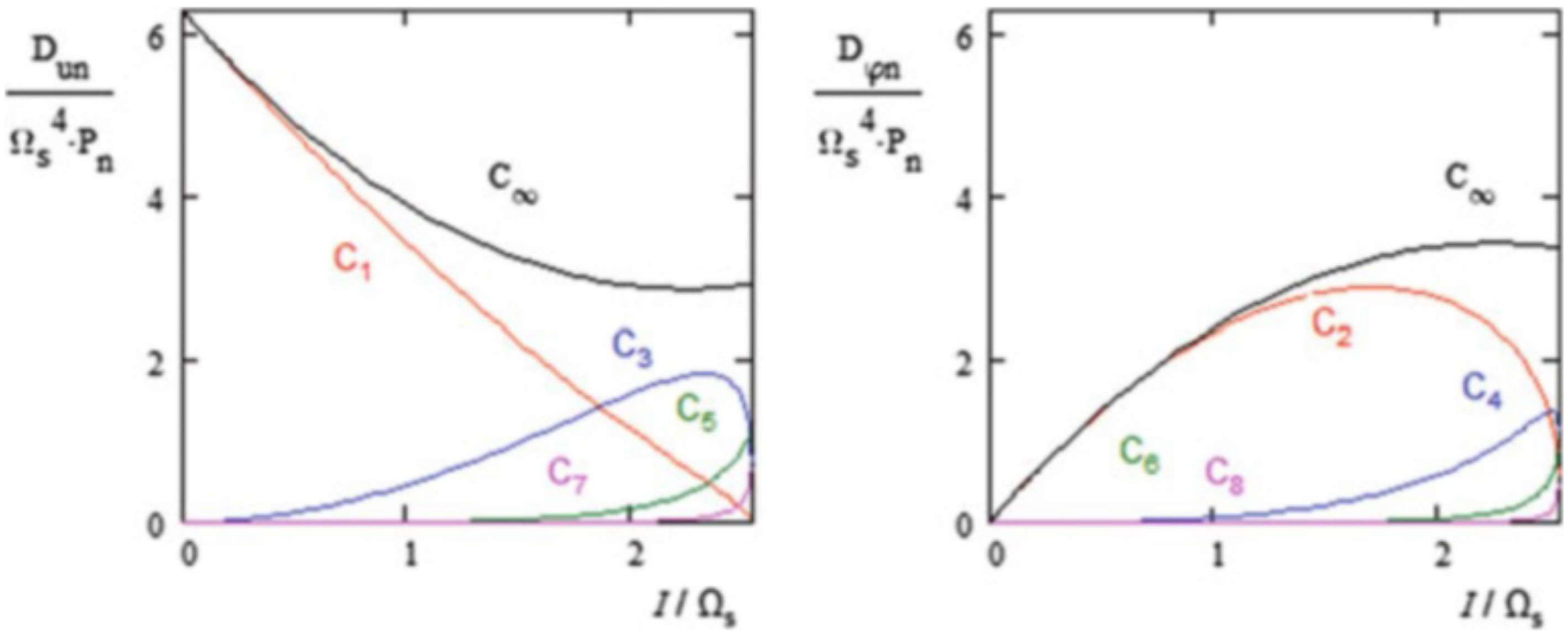
$$= \int_0^\infty f \frac{d}{dI}\left((I-I_0) I \frac{D(I)}{\omega(I)}\right) f dI \xrightarrow{f(I)=\delta(I-I_0)} \int_0^\infty \delta(I-I_0) I \frac{D(I)}{\omega(I)} dI = I_0 \frac{D(I_0)}{\omega(I_0)}$$

Diffusion in Harmonic RF

- Motion non-linearity couples the diffusion to higher harmonics of synchrotron frequency [*]

$$D_\varphi = \sum_{n=1}^{\infty} P_\varphi(n\omega) C_{\varphi n}(I) \ , \quad C_{\varphi n}(I) = \frac{\omega \Omega_s{}^4}{\pi I} \left| \oint d\phi \cos\phi \exp\left(in\omega t(\phi)\right) \right|^2 ,$$

$$D_u = \sum_{n=1}^{\infty} P_u(n\omega) C_{un}(I) , \quad C_{un}(I) = \frac{\omega \Omega_s{}^4}{\pi I} \left| \oint \frac{d\phi}{2\pi} \sin\phi \exp\left(in\omega t(\phi)\right) \right|^2 .$$



[*]"Accelerator Physics at the Tevatron Collider", edited by V. Lebedev and V. Shiltsev, Springer, 2014.

Final Remarks to the RF noise

- To prevent the longitudinal emittance growth a hadron collider requires high quality RF, both in the RF phase and the RF amplitude
- Modern high quality RF generators are well within these requirements for the master oscillator
  - Microphonics in RF cavities as well as noise in power amplifiers may excite RF noise to unacceptable level
    - To address this problem in the Tevatron Run II the phase feedback was used. It stabilized the RF phase relative to the master (reference) oscillator
  - As will be seen in the second half of the lecture the longitudinal damper may be helpful to reduce effect of phase noise
    - However, it will require very small noise floor in detecting of synchrotron motion
- Noise in the bending magnetic field at synchrotron frequency harmonics works the same way as RF!!!

Transverse Emittance Growth due to Noise in Magnets and its Suppression by Transverse Damper

Equations of Motion and their Solution*

- In difference to synchrotron tune the betatron tunes are large (close or above f_0). That completely changes beam response to a perturbation
- First, we consider one point-like dipole perturbation
- To simplify equations, we transit to new (normalized) variables

$$x=\frac{X}{\sqrt{\beta}},\quad p=\beta\frac{d}{ds}\frac{X}{\sqrt{\beta}}=\beta\left(\frac{1}{\sqrt{\beta}}\frac{dX}{ds}-\frac{X}{2\beta^{3/2}}\frac{d\beta}{ds}\right)=\sqrt{\beta}\theta+\alpha\frac{X}{\sqrt{\beta}}$$

- In new variables a particle position after N turns is:

$$x_N=x_0\cos\left(\mu N+\psi_0\right)+\sum_{n=0}^{N-1}\Delta p_n\sin\left(\mu\left(N-n\right)\right),\quad \Delta p_n=\sqrt{\beta}\theta_n$$

- Further we imply that: $\Delta p_n\equiv\Delta p(nT)$, $\overline{p(t_1)p(t_2)}=K_p\left(t_1-t_2\right)$, $K_p\left(\tau\right)=\int_{-\infty}^{\infty}P_p(\omega)e^{i\omega\tau}d\omega$
- Then

$$\overline{x_N{}^2}=x_0{}^2\cos^2\left(\mu N+\psi_0\right)+\sum_{n,m=0}^{N-1}\overline{\Delta p_n\Delta p_m}\sin\left(\mu\left(N-n\right)\right)\sin\left(\mu\left(N-m\right)\right)$$

$$=x_0{}^2\cos^2\left(\mu N+\psi_0\right)+\sum_{n,m=0}^{N-1}K\left(T(n-m)\right)\sin\left(\mu\left(N-n\right)\right)\sin\left(\mu\left(N-m\right)\right)$$

* V. Lebedev, et.al. "EMITTANCE GROWTH DUE TO NOISE AND ITS SUPPRESSION WITH THE FEEDBACK SYSTEM IN LARGE HADRON COLLIDERS", Particle Accelerators, 1994, Vol. 44, pp. 147-164; http://cds.cern.ch/record/248620/files/p147.pdf

Equations of Motion and their Solution (2)

- Express the correlation function through the spectral density

$$\overline{x_N{}^2} = x_0{}^2\cos^2\left(\mu N+\psi_0\right)+\int_{-\infty}^{\infty} d\omega \sum_{n,m=0}^{N-1} P(\omega)e^{i\omega T(n-m)}\sin\left(\mu\left(N-n\right)\right)\sin\left(\mu\left(N-m\right)\right)$$

- Perform summation

$$\Sigma = \sum_{n,m=0}^{N-1} e^{i\omega T(n-m)}\sin\left(\mu\left(N-n\right)\right)\sin\left(\mu\left(N-m\right)\right) = \frac{1}{4}\sum_{n,m=0}^{N-1} e^{i\omega T(n-m)}\left(e^{i\mu(N-n)}-e^{-i\mu(N-n)}\right)\left(e^{-i\mu(N-m)}-e^{i\mu(N-m)}\right)$$

$$=\frac{1}{4}\sum_{n,m=0}^{N-1} e^{i\omega T(n-m)}\left(e^{i\mu(m-n)}+e^{-i\mu(m-n)}-e^{i\mu(2N-n-m)}-e^{-i\mu(2N-n-m)}\right)\xrightarrow[\text{drop last 2 terms}]{N\to\infty}$$

$$=\frac{1}{4}\left(\left|\sum_{n=0}^{N-1} e^{i(\omega T-\mu)n}\right|^2+\left|\sum_{n=0}^{N-1} e^{i(\omega T+\mu)n}\right|^2\right)=\frac{1}{4}\left(\left|\frac{1-e^{i(\omega T-\mu)N}}{1-e^{i(\omega T-\mu)}}\right|^2+\left|\frac{1-e^{i(\omega T+\mu)N}}{1-e^{i(\omega T+\mu)}}\right|^2\right)$$

$$=\frac{1}{4}\left(\frac{\sin^2\left((\omega T-\mu)N/2\right)}{\sin^2\left((\omega T-\mu)/2\right)}+\frac{\sin^2\left((\omega T+\mu)N/2\right)}{\sin^2\left((\omega T+\mu)/2\right)}\right)$$

- Account that: $\dfrac{\sin^2(\xi/2N)}{\sin^2\left(\xi/2\right)}\xrightarrow{N\to\infty} 2\pi N\sum_{n=-\infty}^{\infty}\delta(\xi-2\pi n)$

⇨ $$\Sigma = \frac{2\pi N}{4}\left(\sum_{n=-\infty}^{\infty}\delta((\omega T-\mu)-2\pi n)+\sum_{n=-\infty}^{\infty}\delta((\omega T+\mu)-2\pi n)\right)$$

Transverse Emittance Growth due to Noise

- Combining we obtain

$$\overline{x_N{}^2} = x_0{}^2\cos^2(\mu N+\psi_0) + \int_{-\infty}^{\infty} d\omega \sum_{n,m=0}^{N-1} P_p(\omega) e^{i\omega T(n-m)} \sin(\mu(N-n))\sin(\mu(N-m))$$

$$= x_0{}^2\cos^2(\mu N+\psi_0) + \frac{\pi N}{2}\int_{-\infty}^{\infty} P_p(\omega)d\omega\left(\sum_{n=-\infty}^{\infty}\delta((\omega T-\mu)-2\pi n) + \sum_{n=-\infty}^{\infty}\delta((\omega T+\mu)-2\pi n)\right)$$

$$= x_0{}^2\cos^2(\mu N+\psi_0) + \frac{\pi N}{2T}\sum_{n=-\infty}^{\infty}\left(P_p\left(\frac{2\pi n+\mu}{T}\right) + P_p\left(\frac{2\pi n-\mu}{T}\right)\right)$$

Returning to initial variables and accounting that $\mu = 2\pi\nu \;\&\; \omega_0 = 2\pi/T$

$$\overline{X_N{}^2} = X_0{}^2\cos^2(\mu N+\psi_0) + \frac{N\omega_0\beta^2}{4}\sum_{n=-\infty}^{\infty}\left(P_\theta\left(\omega_0(n+\nu)\right) + P_\theta\left(\omega_0(\nu-n)\right)\right)$$

Averaging over all particles, accounting that both terms in the sum make equal contribution and returning to "dimensional variables" we finally obtain

$$\varepsilon_N \equiv \frac{1}{\beta}\left[\overline{X_N{}^2}\right]_{\substack{all\\part}} = \varepsilon_0 + \frac{N\omega_0\beta}{2}\sum_{n=-\infty}^{\infty} P_\theta\left(\frac{2\pi n-\mu}{T}\right)$$

- If all sources of perturbation are statistically independent then for the entire ring we obtain

$$\boxed{\frac{d\varepsilon}{dn} = \frac{\omega_0}{2}\sum_{k}^{\substack{all\\sorces}} \beta_k \sum_{n=-\infty}^{\infty} P_{\theta k}\left(\frac{2\pi n-\mu}{T}\right)}$$

Transverse Emittance Growth due to White Noise

- For the white noise in the band $\Delta f >> n_{max}\omega_0/2\pi$, n_{max} >> 1

$$\overline{\theta^2} = \int_{-n_{\max}\omega_0}^{n_{\max}\omega_0} P_\theta(\omega)\, d\omega = \omega_0 \sum_{n=-n_{\max}}^{n_{\max}} P_\theta(\omega_0 n) \approx \omega_0 \sum_{n=-n_{\max}}^{n_{\max}} P_\theta(\omega_0(n+\nu))$$

Accounting this we obtain

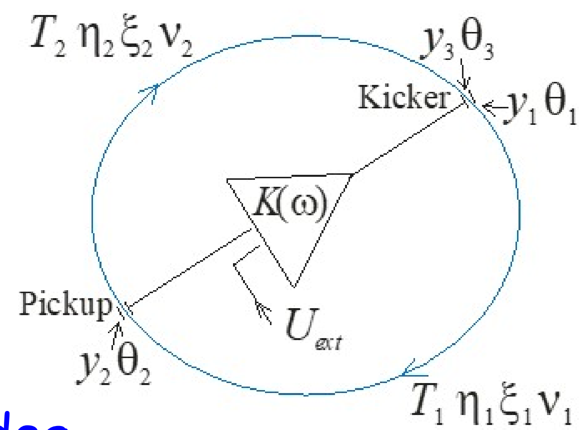
$$\varepsilon_N = \varepsilon_0 + \frac{N\omega_0\beta}{2} \sum_{n=-\infty}^{\infty} P_\theta\left(\frac{2\pi n - \mu}{T}\right) = \varepsilon_0 + \frac{1}{2}\beta\overline{\theta^2}N$$

That we could obtain much easier

Suppression of Emittance Growth by Damper

- If in the above consideration all particles have the same betatron tune, the actual emittance of the beam does not increase. Only the beam centroid oscillation grows
- In real world, and in a collider in particular, different particles have different betatron tunes and therefore beam decoheres with typical decoherence time ~1000 turns.
  - a transverse damper suppresses the coherent beam oscillations and could suppress the emittance growth
  - However, to suppress the emittance growth the damper should damp the beam faster than it decoheres.
- Steps in our calculations
  - Consider damping of the entire beam.
  - Make a transition from matrix formalism to ODE
  - Find a solution for a single kick of the entire beam
  - Find solution for a single particle
  - Obtain equation for the emittance growth

Entire Beam Damping



- Turn-by-turn transformation referenced to the pickup location

$$\mathbf{x}_{n+1} = \mathbf{M}_{kp}\left(\mathbf{M}_{pk}\mathbf{x}_n + \mathbf{G}\sum_{k=0}^{K-1} A_k \mathbf{x}_{n-k}\right) , \qquad \mathbf{G} = \begin{bmatrix} 0 & 0 \\ g & 0 \end{bmatrix}$$

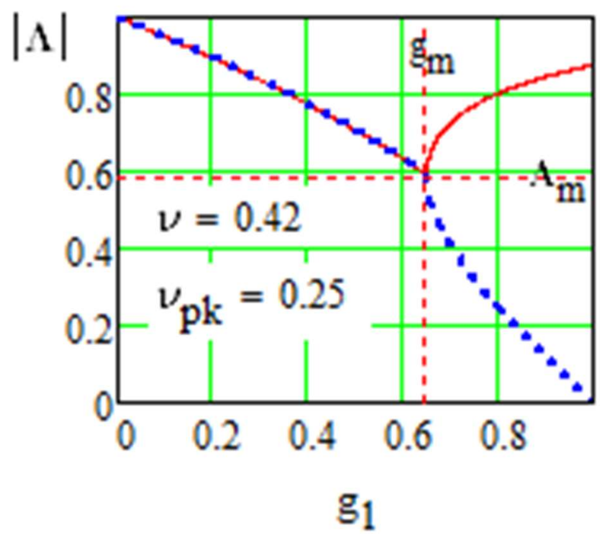
- Consider the simplest one-turn model with μ_{pk}=90 deg.

$$\mathbf{x}_{n+1} = \begin{bmatrix} c & s \\ -s & c \end{bmatrix}\left(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\mathbf{x}_n + \begin{bmatrix} 0 & 0 \\ g & 0 \end{bmatrix}\mathbf{x}_n\right) = \begin{bmatrix} c(1-g) & s \\ -s(1-g) & c \end{bmatrix}\mathbf{x}_n \ , \quad c=\cos\mu,\ s=\sin\mu$$

- Solution

$$\left\| \begin{bmatrix} c(1-g)-\Lambda & s \\ -s(1-g) & c-\Lambda \end{bmatrix} \right\| = 0$$

$$\Rightarrow \quad \Lambda_{1,2} = c\left(1-\frac{g}{2}\right) \pm \sqrt{c^2\left(1-\frac{g}{2}\right)^2 - (1-g)}$$



- For small gain: $\Lambda_{1,2} = \left(1-\dfrac{g}{2}\right)e^{\pm i\mu}$

- The optimal gain decreases with number of turns participating in correction computation , K, as $\approx 1/K$

Transition from Matrix Formalism to ODE

$$\Lambda_{1,2} = \left(1 - \frac{g}{2}\right) e^{\pm i\mu} \approx e^{\pm i\mu - g/2} \Rightarrow \mathbf{x}_n = \mathbf{x}_0 e^{(i\mu - g/2)n} \quad \Rightarrow \quad \mathbf{x}(t) = \mathbf{x}(0) e^{(i\mu - g/2)t/T}, \quad \mathbf{x} = x + ip$$

- For small g we can use ODE for description of motion

$$\mathbf{x}_n = \mathbf{x}_0 e^{(i\mu - g/2)n} \quad \Rightarrow \quad \mathbf{x}(t) = \mathbf{x}(0) e^{(i\mu - g/2)t/T}, \quad \mathbf{x} = x + ip$$

$$\Rightarrow \quad \boxed{\frac{d^2 x}{d\theta^2} + \frac{g}{\mu}\frac{dx}{d\theta} + x = 0, \quad \theta \in [0, \mu]}$$

- The solution is

$$x(\theta) \approx e^{-g\theta/2\mu}\left(x(0)\cos\theta + p(0)\sin\theta\right), \quad g/2\mu \ll 1$$

Solution for a Single kick of the Entire Beam

$$\bar{x} \approx \left(\Delta x \cos\theta + \Delta p \sin\theta\right) e^{-g\theta/2\mu}$$

$$\bar{p} \approx \left(-\Delta x \sin\theta + \Delta p \cos\theta\right) e^{-g\theta/2\mu}$$

where we assume that the decoherence time is much longer than the damping time

Damping of a Single Particle

- Single particle does not produce sufficient signal => no damping
  - Particle with $\Delta\nu=0$ and $x_0=\Delta x$ is damped together with the beam

$$\frac{d^2x}{d\theta^2}+\left(1+\frac{\Delta\nu}{\nu}\right)^2 x=-\frac{g}{\mu}\overline{p}\equiv-\frac{g}{\mu}e^{-g\theta/2\mu}\left(-\Delta x\sin\theta+\cos\theta\,\Delta p\right)$$

where we accounted that $d\overline{x}/d\theta=\overline{p}\approx\left(-\Delta x\sin\theta+\Delta p\cos\theta\right)e^{-g\theta/2\mu}$

- The general solution for initial conditions $x=x_0+\Delta x,\ p=p_0+\Delta p$:

$$x=\left(x_0+\Delta x\right)\cos\left(\nu_p\theta\right)+\left(p_0+\Delta p\right)\sin\left(\nu_p\theta\right)-\frac{1}{\nu_p}\int_0^\theta\left(\frac{g}{\mu}e^{-g\theta'/2\mu}\left(-\Delta x\sin\theta'+\cos\theta'\,\Delta p\right)\right)\sin\left(\nu_p\left(\theta-\theta'\right)\right)d\theta'$$

where we accounted $\varphi(t)=\frac{1}{\Omega_s}\int_0^t f(t')\sin\left(\Omega_s(t-t')\right)dt'$ for $\frac{d^2\varphi}{dt^2}+\Omega_s{}^2\varphi=f(t)$, and $\nu_p=1+\frac{\Delta\nu}{\nu}$

- Lengthy integration in the limit of large θ (see below) yields:

$$x=\left(x_0+\frac{2\mu\left(\nu_p-1\right)}{g}\Delta x\right)\cos\left(\nu_p\theta\right)+\left(p_0-\frac{2\mu\left(\nu_p-1\right)}{g}\Delta p\right)\cos\left(\nu_p\theta\right)$$

- Thus, detuning results in a single particle emittance increase

$$\delta\varepsilon=\frac{\delta p^2+\delta x^2}{2}=\left(\frac{4\pi\Delta\nu}{g}\right)^2\frac{\Delta x^2+\Delta p^2}{2}=\left(\frac{4\pi\Delta\nu}{g}\right)^2\Delta\varepsilon \quad \Rightarrow \quad \boxed{\frac{d\varepsilon}{dt}=\frac{16\pi^2\overline{\Delta\nu^2}}{g^2}\left(\frac{d\varepsilon}{dt}\right)_0}$$

where we accounted that $\left(\nu_p-1\right)\mu=2\pi\Delta\nu$

Computation of Integrals

- Consider only term with Δx (term with Δp done similarly, same result)

$$I=\cos\left(\nu_p\theta\right)+\frac{g}{\nu_p\mu}\int_0^\theta e^{-g\theta'/2\mu}\sin\theta'\sin\left(\nu_p\left(\theta-\theta'\right)\right)d\theta'$$

$$=\cos\left(\nu_p\theta\right)+\frac{g}{\nu_p\mu}\left(\sin\left(\nu_p\theta\right)\int_0^\theta e^{-g\theta'/2\mu}\sin\theta'\cos\left(\nu_p\theta'\right)d\theta'-\cos\left(\nu_p\theta\right)\int_0^\theta e^{-g\theta'/2\mu}\sin\theta'\sin\left(\nu_p\theta'\right)d\theta'\right)$$

- Account that: $2\sin x\cos y=\sin(x+y)+\sin(x-y),\quad 2\sin x\sin y=\cos(x-y)-\cos(x+y)$

  and drop fast oscillating terms

$$I=\cos\left(\nu_p\theta\right)-\frac{g}{\nu_p\mu}\left(\sin\left(\nu_p\theta\right)\int_0^\theta e^{-g\theta'/2\mu}\sin\left((\nu_p-1)\theta'\right)d\theta'+\cos\left(\nu_p\theta\right)\int_0^\theta e^{-g\theta'/2\mu}\cos\left((\nu_p-1)\theta'\right)d\theta'\right)$$

- In the limit of large θ

$$I=\cos\left(\nu_p\theta\right)-\frac{g}{\nu_p\mu}\left(\frac{\sin\left(\nu_p\theta\right)}{2i}\left(\frac{1}{1-e^{-g/2\mu+i\left(\nu_p-1\right)}}-CC\right)+\frac{\cos\left(\nu_p\theta\right)}{2}\left(\frac{1}{1-e^{-g/2\mu+i\left(\nu_p-1\right)}}+CC\right)\right)$$

$$\xrightarrow[\nu_p-1\ll1]{g/2\mu\ll1}\cos\left(\nu_p\theta\right)-\frac{g}{\nu_p\mu}\left(\frac{\sin\left(\nu_p\theta\right)}{2i}\left(\frac{1}{-\left(-g/2\mu+i\left(\nu_p-1\right)\right)}-CC\right)+\frac{\cos\left(\nu_p\theta\right)}{2}\left(\frac{1}{\left(-g/2\mu+i\left(\nu_p-1\right)\right)}+CC\right)\right)$$

$$=\cos\left(\nu_p\theta\right)-\frac{1}{\nu_p}\left(\frac{2\mu g(\nu_p-1)\sin\left(\nu_p\theta\right)+g^2\cos\left(\nu_p\theta\right)}{g^2+4\mu^2(\nu_p-1)^2}\right)\xrightarrow[\nu_p-1\ll1]{g\gg\mu(\nu_p-1)}\frac{2\mu(\nu_p-1)}{g}\sin\left(\nu_p\theta\right)$$

Suppression of Emittance Growth by Damper

- In the above calculations we assumed that $g \gg \Delta\nu$. Otherwise, the problem would be much more complicated because we would need to account the beam decoherence in computation of damper response.
- Therefore, the obtained answer

$$\frac{d\varepsilon}{dt}=\frac{16\pi^2\overline{\Delta\nu^2}}{g^2}\left(\frac{d\varepsilon}{dt}\right)_0$$

  is justified for $g \gg \Delta\nu$ only.
- For practical estimates, since there is no suppression for small g, we use an interpolation

$$\frac{d\varepsilon}{dt}\approx\frac{16\pi^2\overline{\Delta\nu^2}}{g^2+16\pi^2\overline{\Delta\nu^2}}\left(\frac{d\varepsilon}{dt}\right)_0$$

References

1. V. Lebedev, et.al. "Emittance growth due to noise and its suppression with feedback system in large hadron colliders", SSCL-Preprint-188, (1993); https://inis.iaea.org/collection/NCLCollectionStore/_Public/26/066/26066808.pdf
2. "Accelerator Physics at the Tevatron Collider", edited by V. Lebedev and V. Shiltsev, Springer, 2014.

Problems

1. Prove that $\varphi(t)=\frac{1}{\Omega_s}\int_0^t f(t')\sin\left(\Omega_s(t-t')\right)dt'$ is the solution of the following equation

   $\frac{d^2\varphi}{dt^2}+\Omega_s{}^2\varphi=f(t)$ with zero initial coordinates

2. Rewrite equations of longitudinal motion for low frequency noise in bending magnets which can drive the longitudinal emittance growth. Make estimates for the LHC (C=30 km, Q=40) and Tevatron (C=6 km, Q=20)
3. Prove that $\frac{\sin^2(\xi/2N)}{\sin^2\left(\xi/2\right)}\xrightarrow{N\to\infty}2\pi N\sum_{n=-\infty}^{\infty}\delta(\xi-2\pi n)$
4. Compute the rms tune spread due to head-on beam-beam effects in round beams. Estimate corresponding decoherence time. Assume round beams of the same rms sizes and $\beta_x=\beta_y$.
5. Estimate acceptable value of white noise in the LHC dipole in the absence of emittance growth suppression by transverse damper (N_{dip}=1232). Assume noise in different dipoles independent. Compute corresponding spectral density assuming 5 kHz band.
6. Extend the equation for the emittance growth suppression by damper so that in addition to external noise it would include the damper noise. Reference the damper noise to the rms resolution of damper pickup.