Lectures 9&10
Intrabeam Scattering

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Objectives

- In a “properly built machine” the IBS typically represents the main source of emittance growth, both $\perp$ & $||$
    - Coulomb scattering cross-section diverges
    - In a beam this divergence is limited by other particle screening or size
- Conventionally, multiple and single particle scattering in a storage ring are considered to be independent. Such an approach is simple and often yields sufficiently accurate results.
    - Multiple scattering is described by Fokker-Planck equation
        - Landau collision integral
    - Single scattering – Touschek effect (important for very different T’s)
- However, there is a class of problems where such approach is not adequate; and single & multiple scatterings should to be considered together.
    - It is described by integrodifferential equation for particle distribution function, which correctly treats particle Coulomb scattering
- In this lecture we consider an evolution of particle distribution due to multiple intrabeam scattering: first in plasma then in a beam

Diffusion and Friction Force in Plasma

Multiple Scattering in Plasma

- Landau collision integral

$$\frac{df}{dt} = -2\pi n r_0^2 c^4 L_c \frac{\partial}{\partial \mathrm{v}_i} \int \left(f \frac{\partial f'}{\partial \mathrm{v}'_j} - f' \frac{\partial f}{\partial \mathrm{v}_j} \right) \frac{\left(\mathbf{v}-\mathbf{v}'\right)^2 \delta_{ij} - \left(\mathrm{v}_i - \mathrm{v}'_i\right)\left(\mathrm{v}_j - \mathrm{v}'_j\right)}{\left|\mathbf{v}-\mathbf{v}'\right|^3} d^3\mathrm{v}'$$

$$\Rightarrow \quad \frac{df}{dt} = -\frac{\partial}{\partial p_i}\left(F_i f\right) + \frac{1}{2}\frac{\partial}{\partial p_i}\left(D_{ij}\frac{\partial f}{\partial p_j}\right), \quad \begin{cases} F_i(\mathrm{v}) = -\dfrac{4\pi n e^4 L_c}{m}\displaystyle\int f(\mathrm{v}')\frac{u_i}{|\mathbf{u}|^3} d^3\mathrm{v}' , \\ D_{ij}(\mathrm{v}) = 4\pi n e^4 L_c \displaystyle\int f(\mathrm{v}')\frac{u^2\delta_{ij} - u_i u_j}{|\mathbf{u}|^3} d^3\mathrm{v}' , \end{cases} \quad \mathbf{u} = \mathbf{v} - \mathbf{v}'$$

where $L_c = \ln(\rho_{\max}/\rho_{\min})$, $\rho_{\min} = r_0 c^2 / \overline{\mathrm{v}^2}$, $\rho_{\max} = \sqrt{\overline{\mathrm{v}^2} / 4\pi n r_0 c^2}$, $r_0 = \dfrac{e^2}{mc}$, $\overline{\mathrm{v}^2} = \sigma_{\mathrm{v}x}^2 + \sigma_{\mathrm{vy}}^2 + \sigma_{\mathrm{vz}}^2$, $\int f(\mathbf{v}) d^3\mathrm{v} = 1$

and we accounted: $\dfrac{\partial u}{\partial u_i} = \dfrac{u_i}{u}$, $\dfrac{\partial}{\partial u_i}\left(\dfrac{u^2\delta_{ij} - u_i u_j}{u^3}\right) = -2\dfrac{u_i}{u^3}$, $\dfrac{\partial}{\partial \mathrm{v}'_i}\left(\dfrac{u^2\delta_{ij} - u_i u_j}{u^3}\right) = 2\dfrac{u_i}{u^3}$

- Conditions of applicability: $L_c = \ln(\rho_{\max}/\rho_{\min}) \gg 1$, or $T \gg e^2 n^{1/3}$
  - Plasma theory – a perturbation theory where we can neglect interaction of more than 2 particles
  - Lenard-Balescu equations bind low and higher order distributions

Friction and Diffusion

$$\frac{df}{dt} = -\frac{\partial}{\partial p_i}\left(F_i f\right) + \frac{1}{2}\frac{\partial}{\partial p_i}\left(D_{ij}\frac{\partial f}{\partial p_j}\right)$$

- Let's consider a single particle deceleration

  ⇨ $f = \delta\left(\mathbf{p}-\mathbf{p}_0\right)$, but D and F fixed

$$\frac{d}{dt}\overline{\delta p_i} \equiv \frac{d}{dt}\left(\overline{(p_i - p_{0i})}\right) = \int (p_i - p_{0i})\frac{\partial}{\partial p_l}\left(-F_l f + D_{kl}\frac{\partial f}{\partial p_k}\right)\mathrm{d}p^3$$

$$= -\int \delta_{il}\left(-F_l f + D_{kl}\frac{\partial f}{\partial p_k}\right)\mathrm{d}p^3 = F_i(\mathbf{p}_0) - \int D_{ki}\frac{\partial f}{\partial p_k}\mathrm{d}p^3 = F_i(\mathbf{p}_0) + \int f\frac{\partial D_{ki}}{\partial p_k}\mathrm{d}p^3 = F_i(\mathbf{p}_0) + \left.\frac{\partial D_{ki}}{\partial p_k}\right|_{\mathbf{p}=\mathbf{p}_0}$$

  i.e. the gradient in diffusion adds to deceleration:

  For Gaussian distribution it doubles the "force"

$$\boxed{\frac{d}{dt}\overline{\delta p_i} = F_i(\mathbf{p}_0) + \left.\frac{\partial D_{ki}}{\partial \mathrm{v}_k}\right|_{\mathbf{p}=\mathbf{p}_0}}$$

- Let's consider a single particle diffusion

$$\frac{d}{dt}\overline{\delta p_i \delta p_j} = \int\left((p_i - p_{0i})(p_j - p_{0j})\right)\frac{\partial}{\partial p_l}\left(-F_l f + D_{kl}\frac{\partial f}{\partial p_k}\right)dp^3 = \int\frac{\partial}{\partial p_l}\left((p_i - p_{0i})(p_j - p_{0j})\right)\left(F_l f - D_{kl}\frac{\partial f}{\partial p_k}\right)dp^3$$

$$= \int\left(\delta_{il}(p_j - p_{0j}) + \delta_{jl}(p_i - p_{0i})\right)\left(F_l f - D_{kl}\frac{\partial f}{\partial p_k}\right)dp^3$$

$$= \int\left((p_j - p_{0j})F_i + (p_i - p_{0i})F_j\right)f dp^3 - \int\left((p_j - p_{0j})\delta_{il} + (p_i - p_{0i})\right)\delta_{jl}D_{kl}\frac{\partial f}{\partial p_k}dp^3$$

$$\xrightarrow{1^{st}\ \text{term}=0}\int f\delta_{jl}\frac{\partial}{\partial p_k}\left[\left((p_j - p_{0j})\delta_{il} + (p_i - p_{0i})\right)D_{kl}\right]dp^3 = 2\int fD_{ij}dp^3$$

$$\boxed{\frac{d}{dt}\overline{\delta p_i \delta p_j} = 2\left.D_{ij}\right|_{\mathbf{p}=\mathbf{p}_0}}$$

Temperature Exchange in Plasma

- Consider 3 temperature Gaussian distribution

$$f = \frac{1}{(2\pi)^{3/2}\sigma_{vx}\sigma_{vy}\sigma_{vz}} \exp\left(-\frac{1}{2}\left(\frac{v_x^2}{\sigma_x^2}+\frac{v_y^2}{\sigma_y^2}+\frac{v_z^2}{\sigma_z^2}\right)\right)$$

- Then the rate of rms velocity is

$$\frac{d}{dt}\overline{v_i v_j} = -\frac{2\pi e^4 n L_c}{m^2}\int v_i v_j \frac{\partial}{\partial v_k}\left(\left(f\frac{\partial f'}{\partial v'_l} - f'\frac{\partial f}{\partial v_l}\right)\frac{u^2\delta_{kl} - u_k u_l}{u^3}\right) dv'^3\, dv^3$$

$$\xrightarrow{\partial f/\partial v_j = -\left(v_j/\sigma_j^2\right) f} = -\frac{2\pi e^4 n L_c}{m^2}\int v_i v_j \frac{\partial}{\partial v_k}\left(-ff'\left(\frac{v'_l}{\sigma_l^2} - \frac{v_l}{\sigma_l^2}\right)\frac{u^2\delta_{kl} - u_k u_l}{u^3}\right) dv'^3\, dv^3$$

$$= -\frac{2\pi e^4 n L_c}{m^2}\int v_i v_j \frac{\partial}{\partial v_k}\left(ff'\frac{u_l}{\sigma_l^2}\frac{u^2\delta_{kl} - u_k u_l}{u^3}\right) dv'^3\, dv^3$$

$$= \frac{2\pi e^4 n L_c}{m^2}\int ff'\left(\delta_{ki} v_j + \delta_{kj} v_i\right)\frac{u_l}{\sigma_l^2}\frac{u^2\delta_{kl} - u_k u_l}{u^3} dv'^3\, dv^3$$

- The Tensor is diagonal. Let's consider *x*-plane.

$$\Rightarrow \quad \frac{d}{dt}\overline{v_x^2} = \frac{4\pi e^4 n L_c}{m^2}\int \frac{v_x u_x}{u^3} ff'\left(\frac{u^2}{\sigma_x^2} - \left(\frac{u_x^2}{\sigma_x^2}+\frac{u_y^2}{\sigma_y^2}+\frac{u_z^2}{\sigma_z^2}\right)\right) dv'^3\, dv^3$$

Temperature Exchange in Plasma (2)

- Substituting the distribution

$$\frac{d}{dt}\overline{\mathrm{v}_x^{\,2}} = \frac{4\pi e^4 n L_c}{(2\pi)^3\sigma_x^{\,2}\sigma_y^{\,2}\sigma_z^{\,2}m^2}\int \frac{\mathrm{v}_x u_x}{u^3}\left(\frac{u^2}{\sigma_x^{\,2}} - \left(\frac{u_x^{\,2}}{\sigma_x^{\,2}} + \frac{u_y^{\,2}}{\sigma_y^{\,2}} + \frac{u_z^{\,2}}{\sigma_z^{\,2}}\right)\right)\exp\left(-\frac{\mathrm{v}_x^{\,2} + \mathrm{v}_x'^{\,2}}{2\sigma_x^{\,2}} - \frac{\mathrm{v}_y^{\,2} + \mathrm{v}_y'^{\,2}}{2\sigma_y^{\,2}} - \frac{\mathrm{v}_z^{\,2} + \mathrm{v}_z'^{\,2}}{2\sigma_z^{\,2}}\right)d\mathrm{v}'^3\, d\mathrm{v}^3$$

- Make transition to

$\mathbf{u} = \mathbf{v} - \mathbf{v}'$, $\mathbf{w} = \mathbf{v} + \mathbf{v}'$, => $\mathbf{v} = \dfrac{\mathbf{u}+\mathbf{w}}{2}, \mathbf{v}' = \dfrac{\mathbf{w}-\mathbf{u}}{2}$,

$$\frac{\partial(\mathrm{v}_x,\mathrm{v}_y,\mathrm{v}_z,\mathrm{v}_x',\mathrm{v}_y',\mathrm{v}_z')}{\partial(u_x,u_y,u_z,w_x,w_y,w_z)} = \begin{vmatrix} 1/2 & 0 & 0 & 1/2 & 0 & 0 \\ 0 & 1/2 & 0 & 0 & 1/2 & 0 \\ 0 & 0 & 1/2 & 0 & 0 & 1/2 \\ -1/2 & 0 & 0 & 1/2 & 0 & 0 \\ 0 & -1/2 & 0 & 0 & 1/2 & 0 \\ 0 & 0 & -1/2 & 0 & 0 & 1/2 \end{vmatrix} = 1/8$$

$$\frac{d}{dt}\overline{\mathrm{v}_x^{\,2}} = \frac{4\pi e^4 n L_c}{8(2\pi)^3\sigma_x^{\,2}\sigma_y^{\,2}\sigma_z^{\,2}m^2}\int \frac{(u_x + w_x)u_x}{2u^3}\left(\frac{u^2}{\sigma_x^{\,2}} - \left(\frac{u_x^{\,2}}{\sigma_x^{\,2}} + \frac{u_y^{\,2}}{\sigma_y^{\,2}} + \frac{u_z^{\,2}}{\sigma_z^{\,2}}\right)\right)\exp\left(-\frac{u_x^{\,2} + w_x^{\,2}}{4\sigma_x^{\,2}} - \frac{u_y^{\,2} + w_y^{\,2}}{4\sigma_y^{\,2}} - \frac{u_z^{\,2} + w_z^{\,2}}{4\sigma_z^{\,2}}\right)dw^3 du^3$$

- Computing integral over w_i yields $2\sqrt{\pi}\sigma_i$ for each integration. =>

$$\frac{d}{dt}\overline{\mathrm{v}_x^{\,2}} = \frac{e^4 n L_c}{4\sqrt{\pi}\sigma_x\sigma_y\sigma_z m^2}\int \frac{u_x^{\,2}}{u^3}\left(\frac{u^2}{\sigma_x^{\,2}} - \left(\frac{u_x^{\,2}}{\sigma_x^{\,2}} + \frac{u_y^{\,2}}{\sigma_y^{\,2}} + \frac{u_z^{\,2}}{\sigma_z^{\,2}}\right)\right)\exp\left(-\frac{u_x^{\,2}}{4\sigma_x^{\,2}} - \frac{u_y^{\,2}}{4\sigma_y^{\,2}} - \frac{u_z^{\,2}}{4\sigma_z^{\,2}}\right)du^3$$

Temperature Exchange in Plasma (3)

- To compute integrals, we use the identity: $\frac{1}{\theta^3}=\frac{1}{4\sqrt{\pi}}\int_0^\infty \sqrt{\lambda}e^{-\lambda\theta^2/4}d\lambda$

$$\frac{d}{dt}\overline{\mathrm{v}_x^{\,2}}=\frac{e^4 nL_c}{4\sqrt{\pi}\sigma_x\sigma_y\sigma_z m^2}\int_0^\infty \frac{\sqrt{\lambda}}{4\sqrt{\pi}}d\lambda\int\left(\frac{u^2}{\sigma_x^{\,2}}-\left(\frac{u_x^{\,2}}{\sigma_x^{\,2}}+\frac{u_y^{\,2}}{\sigma_y^{\,2}}+\frac{u_z^{\,2}}{\sigma_z^{\,2}}\right)\right)u_x^{\,2}\,\mathrm{e}^{-\frac{u_x^{\,2}+u_y^{\,2}+u_z^{\,2}}{4}\lambda}\,\mathrm{e}^{-\frac{u_x^{\,2}}{4\sigma_x^{\,2}}-\frac{u_y^{\,2}}{4\sigma_y^{\,2}}-\frac{u_z^{\,2}}{4\sigma_z^{\,2}}}d^3u$$

⇨

$$=\frac{e^4 nL_c}{4\sqrt{\pi}\sigma_x\sigma_y\sigma_z m^2}\int_0^\infty \frac{\sqrt{\lambda}}{4\sqrt{\pi}}d\lambda\int\left(\frac{u_y^{\,2}+u_z^{\,2}}{\sigma_x^{\,2}}-\frac{u_y^{\,2}}{\sigma_y^{\,2}}-\frac{u_z^{\,2}}{\sigma_z^{\,2}}\right)u_x^{\,2}\,\mathrm{e}^{-\frac{u_x^{\,2}+u_y^{\,2}+u_z^{\,2}}{4}\lambda}\,\mathrm{e}^{-\frac{u_x^{\,2}}{4\sigma_x^{\,2}}-\frac{u_y^{\,2}}{4\sigma_y^{\,2}}-\frac{u_z^{\,2}}{4\sigma_z^{\,2}}}d^3u$$

- Straightforward integration yields:

$$\frac{d}{dt}\overline{\mathrm{v}_x^{\,2}}=\frac{2\sqrt{\pi}e^4 nL_c}{\sigma_x\sigma_y\sigma_z m^2}\int_0^\infty\left(\frac{\left(\frac{1}{\sigma_x^{\,2}}-\frac{1}{\sigma_y^{\,2}}\right)}{\lambda+\frac{1}{\sigma_y^{\,2}}}+\frac{\left(\frac{1}{\sigma_x^{\,2}}-\frac{1}{\sigma_z^{\,2}}\right)}{\lambda+\frac{1}{\sigma_z^{\,2}}}\right)\frac{\sqrt{\lambda}d\lambda}{\left(\lambda+\frac{1}{\sigma_x^{\,2}}\right)^{3/2}\sqrt{\lambda+\frac{1}{\sigma_y^{\,2}}}\sqrt{\lambda+\frac{1}{\sigma_z^{\,2}}}}$$

- Finally, we rewrite:

$$\frac{d}{dt}\overline{\mathrm{v}_x^{\,2}}=\frac{(2\pi)^{3/2}e^4 nL_c}{\sqrt{\sigma_x^{\,2}+\sigma_y^{\,2}+\sigma_z^{\,2}}\,m^2}\psi(\sigma_x,\sigma_y,\sigma_z)$$

$$\psi(\sigma_x,\sigma_y,\sigma_z)=\frac{\sqrt{\sigma_x^{\,2}+\sigma_y^{\,2}+\sigma_z^{\,2}}}{\sqrt{2\pi}\sigma_x\sigma_y\sigma_z}\int_0^\infty\left(\frac{\left(\frac{1}{\sigma_x^{\,2}}-\frac{1}{\sigma_y^{\,2}}\right)}{\lambda+\frac{1}{\sigma_y^{\,2}}}+\frac{\left(\frac{1}{\sigma_x^{\,2}}-\frac{1}{\sigma_z^{\,2}}\right)}{\lambda+\frac{1}{\sigma_z^{\,2}}}\right)\frac{\sqrt{\lambda}d\lambda}{\left(\lambda+\frac{1}{\sigma_x^{\,2}}\right)^{3/2}\sqrt{\lambda+\frac{1}{\sigma_y^{\,2}}}\sqrt{\lambda+\frac{1}{\sigma_z^{\,2}}}}$$

Properties of Function $\psi(x,y,z)$

- Function $\psi(x,y,z)$ can be reduced to the sum off symmetric elliptic integrals

$$\psi(x,y,z)=\frac{\sqrt{2}r}{3\pi}\left(y^2 \mathrm{R_D}\left(z^2,x^2,y^2\right)+z^2 \mathrm{R_D}\left(x^2,y^2,z^2\right)-2x^2 \mathrm{R_D}\left(y^2,z^2,x^2\right)\right)$$

where: $$\mathrm{R_D}(x,y,z)=\frac{3}{2}\int_0^\infty \frac{dt}{\sqrt{(t+x)(t+y)(t+z)^3}}\ ,\quad r=\sqrt{x^2+y^2+z^2}$$

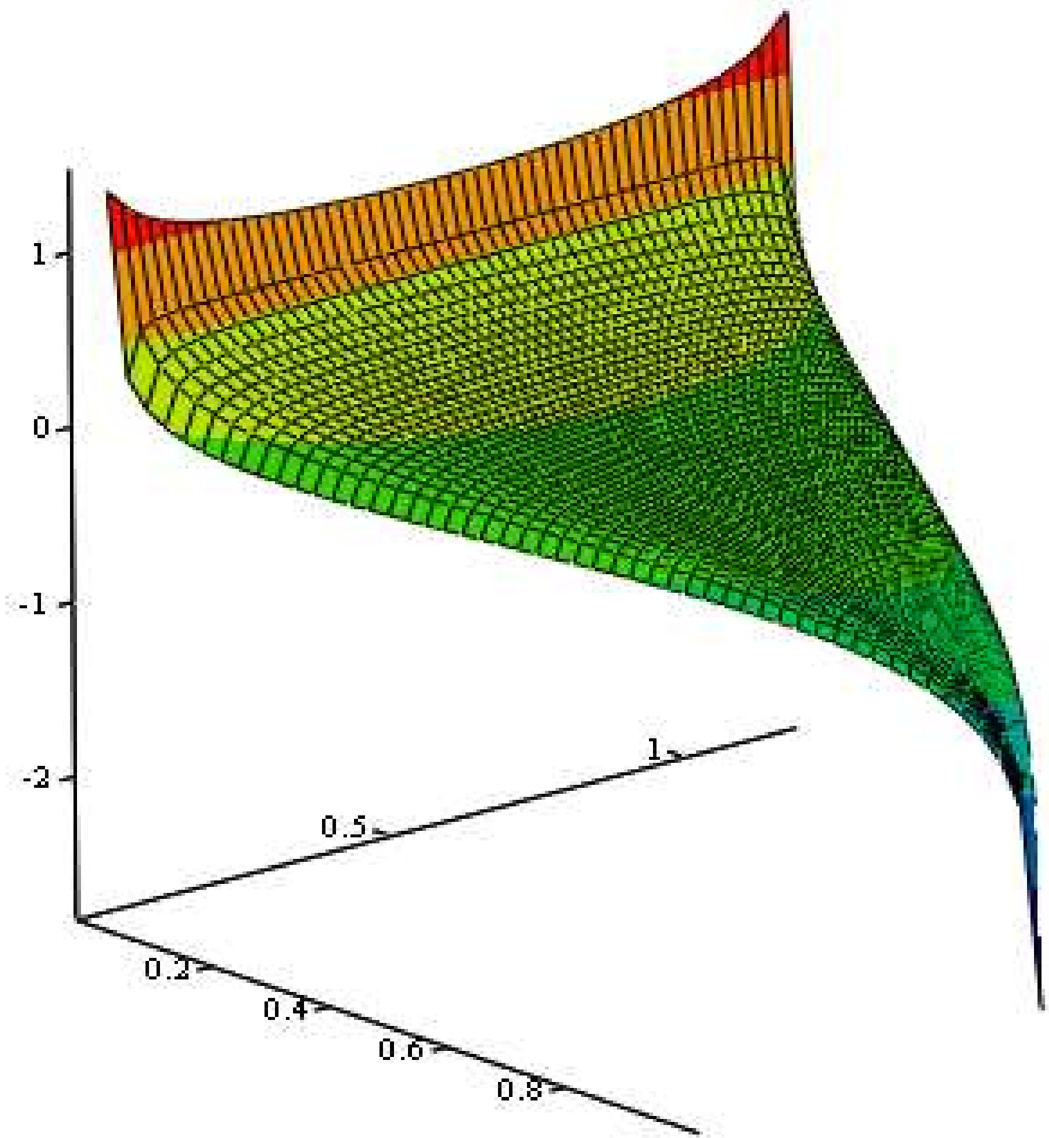
see algorithm for fast computation of $\psi(x,y,z)$ in Appendix to the lecture

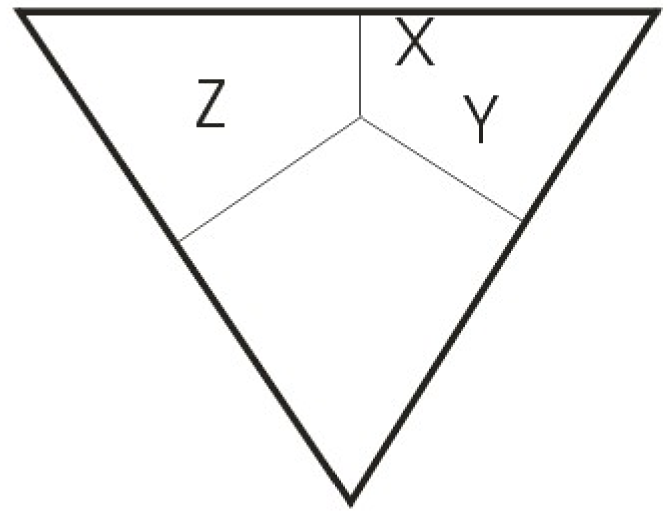
- $\psi(x,y,z)$ depends on the ratios of its variables but not on r.
- $\psi(x,y,z)$ is symmetric relative to the variables y and z, and is normalized so that $\psi(0,1,1) = 1$.
- The energy conservation requires: $\psi(x,y,z)+\psi(y,z,x)+\psi(z,x,y)=0$

  ⇨ $\psi(1,0,1) = \psi(1,1,0) = -1/2$
- The thermal equilibrium corresponds to $\psi(1,1,1) = 0$.
- The function $\psi(0,y,z)$ can be approximated with ~0.5% accuracy by:

$$\psi(0,y,z)\simeq 1+\frac{\sqrt{2}}{\pi}\ln\left(\frac{y^2+z^2}{2yz}\right)-0.055\left(\frac{y^2-z^2}{y^2+z^2}\right)^2$$

Properties of Function $\psi(x,y,z)$





Function $\psi(0,y,z)$

Boersch Effect

- In the course of the beam electrostatic acceleration its longitudinal temperature decreases as 1/E
  - Energy conservation yields || temperature in the beam frame

$$\begin{cases} E \\ E+T \end{cases} \xrightarrow[\text{velosities}]{\text{Corresponding}} \begin{cases} \mathrm{v}_0 = \sqrt{2E/m} \\ \mathrm{v}_0 + \Delta\mathrm{v} = \sqrt{2(E+T)/m} \approx \sqrt{\dfrac{2E}{m}}\left(1+\dfrac{T}{2E}\right) \end{cases} \Rightarrow \begin{cases} \Delta\mathrm{v} = \dfrac{T}{\sqrt{2Em}} \\ T' = \dfrac{m\Delta\mathrm{v}^2}{2} = \dfrac{T^2}{4E} \end{cases}$$

  - Transverse temperature does not change much
    - For a long transport the beam size and transverse temperature can be stabilized by accompanying magnetic field

⇨ $T_{||} \ll T_{\perp}$

- IBS results in the energy transfer from ⊥ to || degree of freedom:

$$\frac{d}{dt}\overline{\mathrm{v}_z^{\,2}} = \frac{(2\pi)^{3/2} e^4 n L_c}{\sqrt{\sigma_x^{\,2}+\sigma_y^{\,2}+\sigma_z^{\,2}}\, m^2}\psi(\sigma_z,\sigma_y,\sigma_y) \xrightarrow[\sigma_z^{\,2} \ll \sigma_\perp^{\,2} \Rightarrow \sigma_z^{\,2}=0]{\sigma_x^{\,2}=\sigma_y^{\,2}=\overline{\mathrm{v}_\perp^{\,2}}} \frac{d}{dt}\overline{\mathrm{v}_{||}^{\,2}} = \frac{2\pi\sqrt{\pi}e^4 n L_c}{m^2\sqrt{\overline{\mathrm{v}_\perp^{\,2}}}}$$

Suppression of IBS by Strong Magnetic field

- Longitudinal-longitudinal relaxation set longitudinal temperature to

$$T_{\|} \approx \frac{T_c^{\,2}}{2W} + 1.9 e^2 n_e^{\,-1/3}$$

  after quarter of plasma period
- When $r_L \le n_e^{\,-1/3}$ magnetic field strongly suppresses IBS

INFLUENCE ON THE SIGN OF AN ION CHARGE ON FRICTION FORCE AT ELECTRON COOLING

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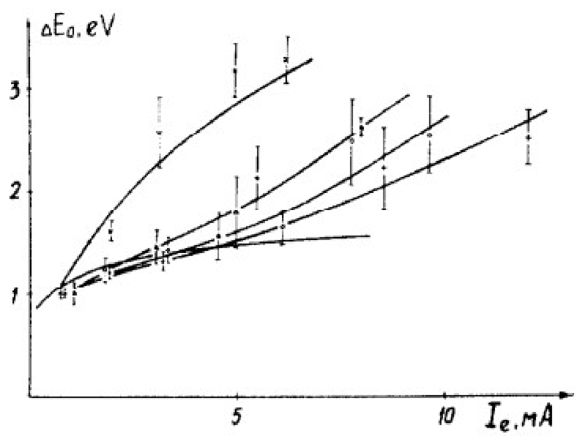


Fig. 6. The energy width ΔE_0 vs the electron current for different magnetic fields: 4 (+), 3 (o), 2 (Δ) and 1 kGs (×), for positive and negative ions the values of ΔE_0 coincide accurate within the measurements. The dotted curve corresponds to expression $\Delta E_0 = \sqrt{32 W e^2 n^{1/3}}$.

Intrabeam Scattering in a Storage Ring

RMS Velocities in Smooth Lattice Approximation

$$\beta_x = \frac{R_0}{\nu_x}, \quad \beta_y = \frac{R_0}{\nu_y}, \quad D = \frac{R_0}{\nu_x^{\ 3}}, \quad \alpha = \frac{1}{\nu_x^{\ 2}}$$

- RMS velocities and angles
  - Trivial in vert. plane $\theta_y = \sqrt{\varepsilon_y / \beta_y}\ , \quad \mathrm{v}_y = \theta_y \beta\gamma c$
  - Radial and horizontal planes are coupled

$$f \propto \exp\left(-\frac{1}{2}\left(\frac{\left(x - D\theta_s\right)^2}{\varepsilon_x \beta_x} + \frac{\beta_x}{\varepsilon_x}\theta_x^{\ 2} + \frac{\theta_s^{\ 2}}{\sigma_p^{\ 2}}\right)\right)$$

For Gaussian distribution temperatures across the beam do not depend on location. Therefore, it is sufficient to see in the beam center

$$f \propto \exp\left(-\frac{1}{2}\left(\left(\frac{D^2}{\varepsilon_x \beta_x} + \frac{1}{\sigma_p^{\ 2}}\right)\theta_s^{\ 2} + \frac{\beta_x}{\varepsilon_x}\theta_x^{\ 2}\right)\right) = \exp\left(-\frac{\theta_x^{\ 2}}{2\sigma_{\theta x}^{\ 2}} - \frac{\theta_s^{\ 2}}{2\sigma_{\theta s}^{\ 2}}\right)$$

where $\sigma_{\theta x} = \sqrt{\frac{\varepsilon_x}{\beta_x}}\ , \quad \sigma_{\theta s} = \sqrt{\frac{\varepsilon_x \beta_x}{\varepsilon_x \beta_x + D^2 \sigma_p^{\ 2}}}\sigma_p$

  - In the beam frame: $\sigma_{\mathrm{v}x} = \gamma\beta c\sqrt{\frac{\varepsilon_x}{\beta_x}}\ , \quad \sigma_{\mathrm{v}s} = \beta c\sqrt{\frac{\varepsilon_x \beta_x}{\varepsilon_x \beta_x + D^2 \sigma_p^{\ 2}}}\sigma_p$

Thermal Equilibrium in Smooth Lattice Approximation

- Thermal equilibrium implies

$$\sigma_{vx} = \sigma_{vs} \Rightarrow \gamma\sqrt{\frac{\varepsilon_x}{\beta_x}} = \sqrt{\frac{\varepsilon_x\beta_x}{\varepsilon_x\beta_x + D^2\sigma_p^{\ 2}}}\sigma_p \Rightarrow \gamma^2\left(\varepsilon_x\beta_x + D^2\sigma_p^{\ 2}\right) = \beta_x^{\ 2}\sigma_p^{\ 2}$$

⇨ Momentum spread in equilibrium:

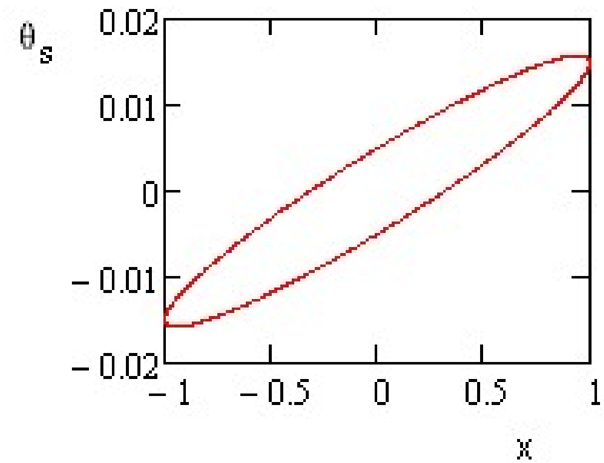
$$\boxed{\sigma_p = \gamma\sqrt{\frac{\varepsilon_x\beta_x}{\beta_x^{\ 2} - \gamma^2 D^2}}}$$

- ♦ Denominator equal to zero at

$$\beta_x = \gamma_{tr} D \Rightarrow \frac{R_0}{\nu_x^{\ 2}} = \gamma_{tr}\frac{R_0}{\nu_x^{\ 3}} \Rightarrow \gamma_{tr} = \nu_x \xrightarrow{\alpha = 1/\nu_x^{\ 2}} \gamma_{tr} = \frac{1}{\sqrt{\alpha}}$$

  i.e. at the transition energy
  => Equilibrium is impossible above transition

- ♦ In other words, the longitudinal particle mass changes its sign at the transition what makes the thermal equilibrium impossible above transition



IBS in Smooth Lattice Approximation

- In plasma: $\frac{d}{dt}\overline{\mathrm{v}_x{}^2} = \frac{(2\pi)^{3/2} e^4 n L_c}{m^2 \sqrt{\overline{\mathrm{v}_x{}^2} + \overline{\mathrm{v}_y{}^2} + \overline{\mathrm{v}_z{}^2}}} \psi\left(\sqrt{\overline{\mathrm{v}_x{}^2}}, \sqrt{\overline{\mathrm{v}_y{}^2}}, \sqrt{\overline{\mathrm{v}_z{}^2}}\right)$
- In the beam we need
  - to substitute velocity spreads in the beam frame

$$\sigma_{\mathrm{v}x} = \gamma\beta c\sqrt{\frac{\varepsilon_x}{\beta_x}}\ , \quad \sigma_{\mathrm{v}y} = \gamma\beta c\sqrt{\frac{\varepsilon_y}{\beta_y}}\ , \quad \sigma_{\mathrm{v}s} = \beta c\sqrt{\frac{\varepsilon_x\beta_x}{\varepsilon_x\beta_x + D^2\sigma_p{}^2}}\sigma_p$$

  - average across the beam volume. For continues beam

$$n = \left(\frac{1}{\sqrt{2}}\right)^2 \frac{N}{2\pi\sigma_x\sigma_y C\gamma}$$

  - make transition to the lab frame and divide by 2 for oscillatory degrees of freedom

$$\frac{dp_\perp}{dt} = \frac{dp_\perp}{\gamma dt_{bf}} = \frac{1}{\gamma}\frac{d\mathrm{v}_\perp}{dt_{bf}}\ , \quad \frac{dp_\parallel}{dt} = \frac{\gamma dp_\parallel}{\gamma dt_{bf}} = m\frac{d\mathrm{v}_\parallel}{dt_{bf}}$$

  - We also need to account that the longitudinal kicks contribute to the transverse emittance growth

$$\frac{d}{dt}\overline{\delta a_x{}^2} = \beta_x\frac{d\varepsilon_x}{dt} = D^2\frac{d}{dt}\frac{\overline{\delta p^2}}{p^2} \Rightarrow \frac{d\varepsilon_x}{dt} = \frac{D^2}{\beta_x}\frac{d}{dt}\frac{\overline{\delta p^2}}{p^2}$$

IBS in Smooth Lattice Approximation (2)

- Performing the previous slide actions, one obtains the emittance growth rates for continuous beam:

$$\frac{d}{dt}\begin{bmatrix}\varepsilon_x\\ \varepsilon_y\\ {\sigma_p}^2\end{bmatrix}=\frac{\sqrt{\pi}}{2\sqrt{2}}\frac{e^4 N L_c}{M^2c^3\sigma_x\sigma_y C\beta^3\gamma^5\sqrt{{\theta_x}^2+{\theta_y}^2+{\theta_p}^2}}\begin{bmatrix}\beta_x\psi\left(\theta_x,\theta_y,\theta_p\right)+\gamma^2\frac{D^2}{\beta_x}\psi\left(\theta_p,\theta_x,\theta_y\right)\\ \beta_y\psi\left(\theta_y,\theta_x,\theta_p\right)\\ 2\gamma^2\psi\left(\theta_p,\theta_x,\theta_y\right)\end{bmatrix}$$

where

$$\sigma_x=\sqrt{\varepsilon_x\beta_x+D^2{\sigma_p}^2}\ ,\quad \sigma_y=\sqrt{\varepsilon_y\beta_y}\ ,\quad \theta_x=\sqrt{\frac{\varepsilon_x}{\beta_x}}\ ,\quad \theta_y=\sqrt{\frac{\varepsilon_y}{\beta_y}}\ ,\quad \theta_p=\sqrt{\frac{\varepsilon_x\beta_x}{\varepsilon_x\beta_x+D^2{\sigma_p}^2}}\frac{\sigma_p}{\gamma}\equiv\frac{\sqrt{\varepsilon_x\beta_x}}{\sigma_x}\frac{\sigma_p}{\gamma}$$

- For the bunched beam with linear RF one needs to replace

$$\frac{1}{C}\to\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2\pi}\sigma_s}$$

and $2\gamma^2\to\gamma^2$ in the bottom row of the matrix (because the energy is equally divided between potential and kinetic energies)

IBS in Relativistic Hadron Colliders

- At present energies the proton beam is non-relativistic in the beam frame. It enables to consider considered above non-relativistic collisions and greatly simplifies formulas
- For ultra-relativistic beam one can neglect the longitudinal velocity (in the beam frame) and set it to zero.
- In the absence of coupling the vertical emittance growth is suppressed as $(\nu_x/\gamma)^2$ and is negligible in comparison to the horizontal emittance growth
- Then in the smooth lattice approximation we have

$$\frac{d}{dt}\begin{bmatrix}\varepsilon_x \\ \varepsilon_y \\ {\sigma_p}^2\end{bmatrix} = \frac{1}{4\sqrt{2}}\frac{e^4 N L_c \psi\left(0,\theta_x,\theta_y\right)}{M^2 c^3 \sigma_x \sigma_y \sigma_s \beta^3 \gamma^3 \sqrt{{\theta_x}^2 + {\theta_y}^2}}\begin{bmatrix}\dfrac{D^2}{\beta_x} \\ 0 \\ 1\end{bmatrix}$$

  - ♦ *As one can see both x and s planes are heated due to scattering from transverse planes to the longitudinal plane*

IBS in Relativistic Hadron Colliders (2)

- It is straightforward to account for the actual beta-functions
- In this case we need to accurately account the hor. emittance heating only from || kicks

$$\begin{cases} x = D\theta_s \\ \theta_x = D'\theta_s \end{cases} \Rightarrow \quad \Delta\varepsilon = \frac{x^2}{\beta_x}\left(1+\alpha_x^{\ 2}\right) + 2\alpha_x x\theta + \beta_x \theta_x^{\ 2} = \left(\frac{D^2}{\beta_x}\left(1+\alpha_x^{\ 2}\right) + 2\alpha_x DD' + \beta_x D'^2\right)\theta_s^{\ 2}$$

$$\Rightarrow \quad \frac{d\varepsilon}{dt} = A_x \frac{d}{dt}\sigma_p^{\ 2}\ , \quad A_x = \frac{D^2}{\beta_x}\left(1+\alpha_x^{\ 2}\right) + 2\alpha_x DD' + \beta_x D'^2 = \frac{D^2 + \left(D\alpha_x + D'\beta_x\right)^2}{\beta_x}$$

- Finally averaging over machine circumference, we obtain

$$\frac{d}{dt}\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \sigma_p^{\ 2} \end{bmatrix} = \frac{1}{4\sqrt{2}} \frac{e^4 N}{M^2 c^3 \sigma_s \beta^3 \gamma^3} \left\langle \frac{\psi\left(0,\theta_x,\theta_y\right)}{\sigma_x \sigma_y \sqrt{\theta_x^{\ 2} + \theta_y^{\ 2}}} \begin{bmatrix} A \\ 0 \\ 1 \end{bmatrix} \right\rangle_s$$

where $\sigma_x = \sqrt{\varepsilon_x \beta_x + D^2 \sigma_p^{\ 2}}\ , \quad \sigma_y = \sqrt{\varepsilon_y \beta_y}\ , \quad \theta_x = \sqrt{\frac{\varepsilon_x}{\beta_x}\left(1 + \frac{\sigma_p^{\ 2}\left(\beta_x D'_x + \alpha_x D_x\right)^2}{\sigma_x^{\ 2}}\right)}\ , \quad \theta_y = \sqrt{\frac{\varepsilon_y}{\beta_y}}$

and we can approximately write that

$$\psi\left(0,y,z\right) \simeq 1 + \frac{\sqrt{2}}{\pi}\ln\left(\frac{y^2+z^2}{2yz}\right) - 0.055\left(\frac{y^2-z^2}{y^2+z^2}\right)^2$$

IBS and Transverse Noise

- Noise in the magnetic field made significant contribution to the emittance growth

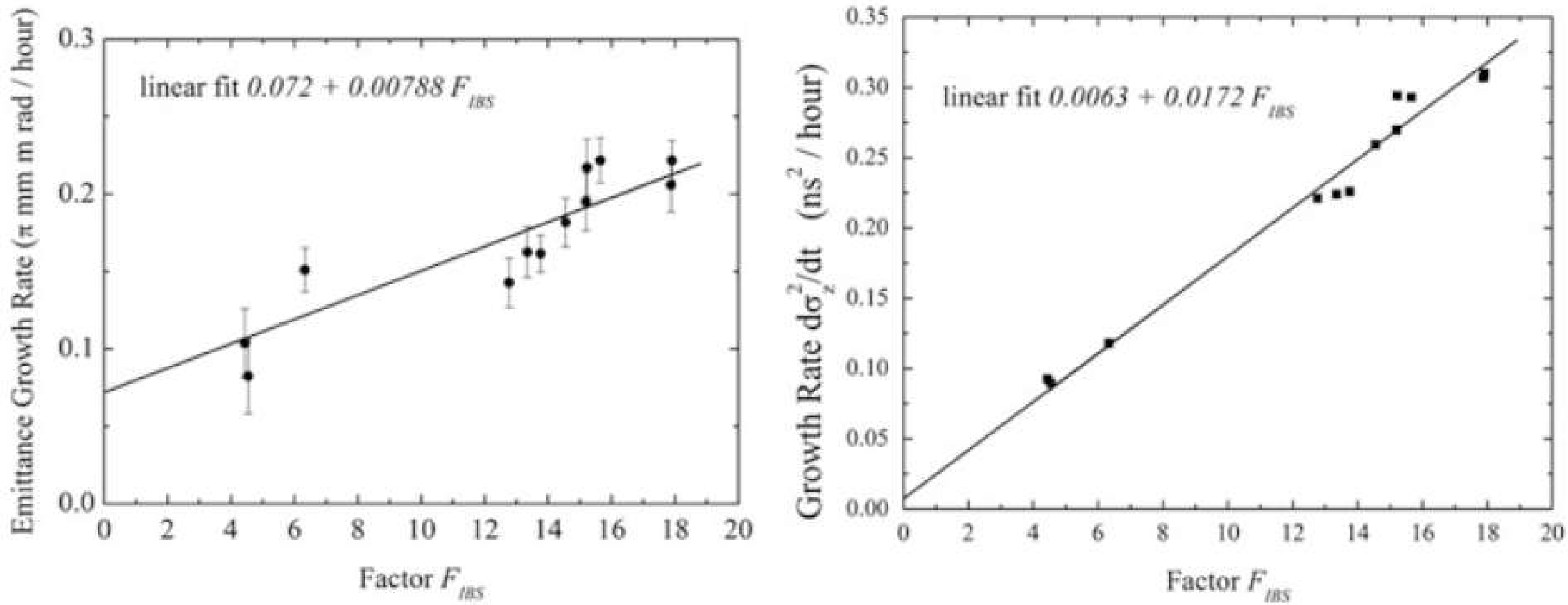


Fig. 6.12 Vertical emittance growth rates (rms, norm.) of proton bunches vs the IBS factor F_{IBS} (*left*); the rms bunch length growth rates vs the IBS factor F_{IBS} (*right*) [20]

- Measurements at injection energy showed that magnetic noise is smaller than then scattering at the residual gas. It is not right at the top energy (150->1000 GeV). $d\varepsilon / dt_{gas} \propto 1/\gamma^2, \quad d\varepsilon / dt_{noise} \propto 1/\gamma^0$

Reading

- Lenard-Balescu equations bind low and higher order distributions. Details can be found in any good plasma textbook.

References

1. "Accelerator Physics at the Tevatron Collider", edited by V. Lebedev and V. Shiltsev, Springer, 2014.
2. S. Nagaitsev, Phys. Rev. ST Accel. Beams 8, 064403 (2005).
3. H. Boersch, Z. Phys. 139, 115 (1954)

Problems

1. Prove that for gaussian distribution with equal temperatures for all degrees of freedom the Landau collision integral in plasma yields $df/dt=0$.
2. Electron beam with 2 mm diameter, energy 500 V and the beam current of 5 mA passes distance of 2 mm. Find rms energy spread at the exit. Weak magnetic field keeps the transverse beam size constant, but does not affect on the intrabeam scattering.
3. Prove that for ultra-relativistic beam the vertical emittance growth is suppressed as $(\nu_x/\gamma)^2$ relative to the horizontal emittance growth. Use the smooth lattice approximation.
4. Prove that the rms local horizontal angular spread in the beam is

$$\theta_x = \sqrt{\frac{\varepsilon_x}{\beta_x}\left(1+\frac{{\sigma_p}^2\left(\beta_x D_x' + \alpha_x D_x\right)^2}{{\sigma_x}^2}\right)}$$

Appendix: Algorithm for fast computation of symmetric elliptic integral

$C1 := \frac{3}{14}$ $C2 := \frac{1}{6}$ $C3 := \frac{9}{22}$ $C4 := \frac{3}{26}$ $C5 := \frac{9}{88}$ $C6 := \frac{9}{52}$

$ERRTOL := 0.05$

$R_D(x,y,z) :=$

```
xt ← x
yt ← y
zt ← z
delx ← 1
dely ← 1
delz ← 1
sum ← 0.0
fac ← 1.0
while max(|delx|, |dely|, |delz|) > ERRTOL
    sqrtx ← √xt
    sqrty ← √yt
    sqrtz ← √zt
    alamb ← sqrtx·(sqrty + sqrtz) + sqrty·sqrtz
    sum ← sum + fac / (sqrtz·(zt + alamb))
    fac ← 0.25·fac
    xt ← 0.25·(xt + alamb)
    yt ← 0.25·(yt + alamb)
    zt ← 0.25·(zt + alamb)
    ave ← 0.2·(xt + yt + 3.0·zt)
    delx ← (ave − xt) / ave
    dely ← (ave − yt) / ave
    delz ← (ave − zt) / ave
ea ← delx·dely
eb ← delz·delz
ec ← ea − eb
ed ← ea − 6.0·eb
ee ← ed + ec + ec
return 3.0·sum + fac·(1.0 + ed·(−C1 + C5·ed − C6·delz·ee) + delz·[C2·ee + delz·(−C3·ec + delz·C4·ea)]) / (ave·√ave)
```