Lagrangian formulation of the massless higher- and infinite- spin fields

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Aim: brief review of the canonical BRST construction for Lagrangian formulation of free massless higher spin field.

Plan

1. Introduction to basic notions
   - Irreducible representations of the Poincáre group
   - General gauge theories
   - Phase space structure of gauge theories
   - Canonical BRST charge
2. Lagrangian formulation of free massless higher-spin field theory
3. Lagrangian formulation for free infinite spin field theory
4. Generalization for infinite spin field in AdS space

Irreducible representations of the Poincáre group

Poincare group is a mathematical expression of the special relativity symmetry.

This group defined as a group of linear inhomogeneous transformations of the Minkowski space

$$x^{'\,m} = \Lambda^m{}_n x^n + a^m$$

leaving invariant interval, where a^m are the constants and the constant matrix Λ satisfies the condition $\Lambda^T \eta \Lambda = \eta$ with matrix $\eta = diag\{+,-,-,-\}$ is the Minkowski metric. Here $m, n = 0, 1, 2, 3$. At $a^m = 0$ we get the Lorentz group.

The Poincáre group is ten-parametric Lie group with generators P_m and $J_{mn} = J_{[mn]}$ satisfying the commutation relations

$$[P_m, P_n] = 0, \quad [P, J] \sim P, \quad [J, J] \sim J.$$

The group possesses two Casimir operators

$$P^2 = P^m P_m, \quad W^2 = W^m W_m,$$

where $W_m = \frac{1}{2}\epsilon_{mnkl} P^n J^{kl}$.

Irreducible representations of the Poincáre group

The irreducible representations are described by eigenvalues of the Casimir operators. They are divided into massive and massless. Usually it is said that these representations describe the elementary systems or elementary particles.

- Massive representations

$$P^2 = m^2, \quad W^2 = -m^2 s(s+1).$$

  Real positive parameter m is called mass, parameter s, taking the values $0, \frac{1}{2}, 1, \frac{3}{2}, 2 \ldots$, is called spin.

- Massless representations

$$P^2 = 0, \quad W^2 = -\mu^2,$$

  Here there are two cases.

  1. $\mu = 0$. Then $W^m = \lambda P^m$. The parameter λ, taking the values $0, \pm\frac{1}{2}, \pm 1, \pm\frac{3}{2}, \ldots$, is called helicity. Sometimes the $|\lambda|$ is called spin of massless particle
  2. $\mu \neq 0$. At each μ these representations contains infinite tower of massless representations with all (integer or half-integer) helicities. In this case it is called infinite of continuous spin particle.

Irreducible representations of the Poincáre group

The irreducible representations of the Poincáre group can be realized in terms of fields on Minkowski space.

- Massive representation with given mass m and integer spin s is realized in linear space of totally symmetric tensor fields $\varphi_{m_1 m_2 m_3 \ldots m_s}(x)$ under the conditions

$$(\Box - m^2)\varphi_{m_1 m_2 m_3 \ldots m_s} = 0, \ \partial^{m_1}\varphi_{m_1 m_2 m_3 \ldots m_s} = 0, \ \eta^{m_1 m_2}\varphi_{m_1 m_2 m_3 \ldots m_s} = 0$$

- Massless representation with $\mu = 0$ and with given integer helicity $\lambda = s$ is realized in linear space of totally symmetric tensor fields $\varphi_{m_1 m_2 m_3 \ldots m_s}(x)$ under the conditions

$$\Box\varphi_{m_1 m_2 m_3 \ldots m_s} = 0, \ \partial^{m_1}\varphi_{m_1 m_2 m_3 \ldots m_s} = 0, \ \eta^{m_1 m_2}\varphi_{m_1 m_2 m_3 \ldots m_s} = 0$$

Higher spin problem: problem of Lagrange formulation.

Irreducible representations of the Poincáre group

The infinite spin representation requires a special consideration. In this case to write the conditions defying the irreducible representation we should introduce some additional variables. As such additional variables one can take the commuting spinor coordinates ξ^{α} and $\bar{\xi}^{\dot{\alpha}}$. The spinor ξ^{α} belongs to fundamental representation of Lorentz group and the spinor $\bar{\xi}^{\dot{\alpha}} = (\xi^{\alpha})^{*}$ belongs to conjugate representation, $\alpha = 1, 2$. The conditions of irreducible representation are written in terms of fields $\Phi(x, \xi, \bar{\xi})$ in the form

$$
\begin{aligned}
\partial^{m}\partial_{m}\,\Phi(x;\xi,\bar{\xi}) &= 0\,,\\
\Big[i\left(\xi\sigma^{m}\bar{\xi}\right)\partial_{m} + \mu\Big]\,\Phi(x;\xi,\bar{\xi}) &= 0\,,\\
\left[i\left(\frac{\partial}{\partial\xi}\sigma^{m}\frac{\partial}{\partial\bar{\xi}}\right)\partial_{m} - \mu\right]\Phi(x;\xi,\bar{\xi}) &= 0\,,\\
\left[\xi\frac{\partial}{\partial\xi} - \bar{\xi}\frac{\partial}{\partial\bar{\xi}}\right]\Phi(x;\xi,\bar{\xi}) &= 0\,.
\end{aligned}
$$

Here $(\sigma^{m})^{\alpha\dot{\alpha}}$ are the invariant matrices associated with Lorentz group.

Problem of Lagrange formulation.

General gauge theories

General gauge theory is defined by the conditions

1. Set of fields Φ^i and action $S[\Phi]$. Index i includes space-time argument and all discrete indices if they are. The fields can have zero or non-zero Grassmann parity.
2. Filed transformations with parameters ξ^α. Index α includes space-time argument and all discrete indices if they are.

$$\delta\Phi^i = R^i{}_\alpha[\Phi]\xi^\alpha.$$

   Here, it is assumed integration over the space-time argument and summation over discrete indices corresponding to index α.
3. Invariance of action $S[\Phi]$ under above transformation,

$$S_{,i}[\Phi]R^i{}_\alpha[\Phi] = 0.$$

   Here, $S_{,i}[\phi]$ means derivative with respect of Φ^i and it is assumed integration over the space-time argument and summation over discrete indices corresponding to index i.

   The field transformations are called gauge and the $R^i{}_\alpha[\Phi]$ are called the generators of gauge transformations.

General gauge theories

The above three conditions lead to relation for the generators of gauge transformations

$$R^i{}_{\beta\,,j}[\Phi], R^j{}_\alpha[\Phi] - R^i{}_{\alpha\,,j}[\Phi], R^j{}_\beta[\Phi] = R^i{}_\gamma[\Phi] f^\gamma{}_{\alpha\beta}[\Phi] + X^{ij}{}_{\alpha\beta}[\Phi] S_{,j}[\Phi].$$

Left hand side is called the commutator of generators, the $f^\gamma{}_{\alpha\beta}[\Phi]$ are called the structure functions. This relation is said to define a general gauge algebra.

If $X^{ij}{}_{\alpha\beta}[\Phi] = 0$, the algebra is called closed, in opposite case it is called open. The generators are called linearly dependent if there exist non-zero $n^\alpha[\Phi]$ such that $R^i{}_\alpha[\Phi] n^\alpha[\Phi] = 0$. In opposite case they are called independent.

If the algebra is closed, the generators are independent and the structure functions are constants, the gauge algebra leads allows to prove that these constants satisfy the Jacobi identities and, hence, we can identify the gauge generators with generators of some Lie algebra.

Phase space structure of gauge theories

A distinctive feature of gauge theories is the presence of first-class constraints in phase space. Such constraints appear automatically when we move from the Lagrangian formulation of gauge theories to their Hamiltonian formulation.

Let (q^a, p_a) are the phase space coordinates, $H(q,p)$ is a Hamiltonian and $T_a(q,p)$ are the constraints. The constraints are called first-class if they satisfy the following relation in terms of Poisson brackets

$$\{T_a, T_b\} = C^c{}_{ab}(q,p)T_c$$

The functions $C^c{}_{ab}$ are determined by gauge algebra in Lagrangian formalism. Therefore one can say that these relations define the gauge algebra. In this case, the action in the Hamiltonian formalism is written in the form

$$S_H[q,p] = p_a\dot{q}^a - H(q,p) - \lambda^a T_a,$$

where λ^a are the Lagrange multipliers. This relation is derived when we move from Lagrangian formulation to Hamiltonian formulation. However, in principle, if we somehow define the constraints, their first class algebra and the action $S_H[q,p]$, then one can construct a Lagrangian gauge formulation corresponding to given Hamiltonian formulation with constraints. In this case, the first-class algebra of constraints is also called the gauge algebra.

Canonical BRST charge

Let T_a the first-class constraints in the phase space and let the functions $C^c{}_{ab}$ are the constants. We extend the phase space by anticommuting (fermionic) coordinates η^a and momenta $\mathcal{P}_a$ which are called the ghost variables. Define in the extended phase space the fermionic function $Q(q, p, \eta, \mathcal{P})$ by the rule

$$Q = \eta^a T_a + \frac{1}{2} C^a{}_{bc} \eta^b \eta^c \mathcal{P}_a.$$

One can show that Q is nilpotent in terms of graded Poisson brackets. This function is called BFV (Batalin-Fradkin-Vilkovisky) charge or canonical BRST (Becci-Ruet-Stora-Tyutin) charge.

After quantization, the BRST charge becomes an nilpotent operator acting in the state space of vectors $|\Psi\rangle, \quad |\Psi\rangle' = Q|\Psi\rangle$. This relation is invariant under transformation $|\Psi\rangle \to |\Psi\rangle + Q|\Lambda\rangle$. Last relation is realization of gauge invariance in state space.

Lagrangian formulation of free massless higher-spin field theory

Derivation of the Lagrangian formulation for free massless integer spin filed. Begin with conditions defying the irreducible massless representation of the Poincáre group with definite helicity s. Representation is realized in space of fields $\varphi_{m_1 \dots m_s}$ under conditions

$$\Box\varphi = 0, \quad \mathrm{tr}\varphi = 0, \quad \mathrm{div}\varphi = 0.$$

Using the Lorentz group invariant matrices $(\sigma^m)^{\alpha\dot{\alpha}}$ we can convert each vector index m into two spinor indices $\alpha, \dot{\alpha}$ as follows $v^m = (\sigma^m)^{\alpha\dot{\alpha}} v_{\alpha\dot{\alpha}}$. As a result, one gets the field $\varphi_{\alpha_1 \dots \alpha_s \dot{\alpha}_1 \dots \dot{\alpha}_s}(x)$. Then make total symmetrization over all dotted indices and all undotted indices. It gives the field $\varphi_{\alpha(s)\dot{\alpha}(s)}(x)$. If to convert the spinor indices of this field into vector ones, we obtain the tensor field automatically satisfying relation $\mathrm{tr}\varphi = 0$. Further we work with field $\varphi_{\alpha(s)\dot{\alpha}(s)}(x)$ satisfying relations

$$\Box\varphi_{\alpha(s)\dot{\alpha}(s)}(x) = 0,$$

$$\partial^{\alpha_1\dot{\alpha}_1}\varphi_{\alpha(s)\dot{\alpha}(s)}(x) = 0,$$

where $\partial^{\alpha_1\dot{\alpha}_1} \sim (\sigma^m)_{\alpha\dot{\alpha}}\partial_m$

Lagrangian formulation of free massless higher-spin field theory

Let us now begin to interpret the relations that define the irreducible representations of the Poincáre group for the field $\varphi_{\alpha(s)\dot{\alpha}(s)}(x)$ as first-class constraints in the phase space of some as yet unknown gauge theory and construct the corresponding Lagrangian formulation.

Realization

1. Let us introduce two sets of bosonic annihilation $a_\alpha, \bar{a}^{\dot{\alpha}}$ and creation $c^\alpha, \bar{c}_{\dot{\alpha}}$ operators with standard commutation relations and define the Fock space of the vectors $|\varphi_s\rangle = \frac{1}{s!}\varphi_{\alpha(s)}{}^{\dot{\alpha}(s)}(x)c^{\alpha(s)}\bar{c}_{\dot{\alpha}(s)}|0\rangle$, where $|0\rangle$ is the corresponding vacuum vector.
2. Let us introduce the operators, acting on Fock space vectors $L_0 = \Box, \quad L = i(a\sigma^m\bar{a})\partial_m$ and the conjugate operator L^+, acting on conjugate vectors
3. The relations defying the irreducible representation can be rewritten in the form $L_0|\varphi_s\rangle = 0, \quad L|\varphi_s\rangle = 0.$

Lagrangian formulation of free massless higher-spin field theory

Commutations relations for the operators L_0, L, L^+ have the form
$[L, L_0] = 0, \quad [L^+, L_0] = 0, \quad [L^+, L] = KL_0,$
where $K = c^\alpha a_\alpha + \bar{c}_{\dot{\alpha}} \dot{a}^{\dot{\alpha}} + 2$.

- Using the operators L_0, L, L^+ as the constrains we construct the nilpotent BRST operator by the general rule in the form

$$Q = \eta_0 L_0 + \eta^+ L + \eta L^+ + K\eta^+\eta\mathcal{P}_0,$$

  where $\eta_) = \eta_0^+, \eta, \eta^+$ are the anticommuting ghost coordinates and $\mathcal{P}_0 = \mathcal{P}_0^+, \mathcal{P}, \mathcal{P}^+$ are the corresponding anticommuting ghost momenta. Both ghost coordinates and ghost momenta annihilate vacuum vector.
- Define the extended Fock space of the vectors $|\Phi\rangle_s = |\varphi_s\rangle + \eta_0\mathcal{P}^+|\varphi_{1,(s-1)}\rangle + \eta^+\mathcal{P}^+|\varphi_{2,(s-2)}\rangle$
- Consider the equation of motion $Q|\Phi\rangle_s = 0$, which is invariant under the transformation $|\Phi\rangle_s^{'} = |\Phi\rangle_s + Q|\Lambda_s\rangle$. One can show that this equation reproduce the relations defying irreducible representation.

Lagrangian formulation of free massless higher-spin field theory

The Lagrangian leading to the equation of motion $Q|\Phi\rangle_s = 0$ is written in the form

$$\mathcal{L}_s = \int d\eta_0 \langle\Phi_s|Q|\Phi_s\rangle$$

Substituting the explicit form of the operator Q and vectors $|\Phi_s$ into above expression one gets the Lagrangian

$$\mathcal{L}_s = \langle\varphi_s|(L_0|\varphi_s\rangle - L^+|\varphi_{1(s-1)}\rangle) - \langle\varphi_{1(s-1)}|(L|\varphi_s - L^+|\varphi_{2(s-2)}\rangle + K|\varphi_{1(s-1)}\rangle) -$$
$$-\langle\varphi_{2(s-2)}|(L_0|\varphi_{2(s-2)}\rangle - L|\varphi_{1(s-1)}\rangle)$$

and gauge transformations

$$\delta|\varphi_s\rangle = L^+|\lambda_{s-1}\rangle, \quad \delta|\varphi_{1(s-1)}\rangle = L_0|\lambda_{s-1}\rangle, \quad \delta|\varphi_{2(s-2)}\rangle = L|\lambda_{s-1}\rangle.$$

BRST construction automatically solves a problem of gauge invariant Lagrangian formulation for free higher spin field theory on the based of relations defying the irreducible representation. The formulation possesses extremely high gauge symmetry. Using the various gauge fixing conditions we can derive the various equivalent Lagrangian formulations.

Lagrangian formulation for free infinite spin field theory

In this case the Lagrangian formulation is constructed analogously to previous case

1. Introduce the Fock space of the vectors $|\varphi\rangle = \sum_{s=0}^{\infty} |\varphi_s\rangle$.
2. Introduce the operators $L_0, \quad L - \mu, \quad L^+ - \mu$.
3. Construct the nilpotent BRST charge
$$Q = \eta_0 L_0 + \eta^+(L - \mu) + \eta(L^+ - \mu) + K\eta^+\eta\mathcal{P}_0.$$
4. Introduce the extended Fock space of the vectors
$$|\Phi\rangle = |\varphi\rangle + \eta_0\mathcal{P}^+|\varphi_1\rangle + \eta^+\mathcal{P}^+|\varphi_2\rangle.$$
5. Define the Lagrangian
$$\mathcal{L} = \int d\eta_0 \langle\Phi|Q|\Phi\rangle$$
Lagrangian is invariant under the gauge transformation $|\Phi'\rangle = |\Phi\rangle + Q|\Lambda\rangle$.

One can show that this Lagrangian is a sum of Lagrangians for free massless fields plus μ-dependent cross terms responsible for infinite spin field description. The formulation possesses extremely high gauge symmetry. Using the various gauge fixing conditions we can derive the various equivalent Lagrangian formulations.

Generalization for infinite spin fields in AdS_4 space

Taking into account the Lagrangian free infinite spin field theory, it is naturally to try construct a theory in curved space-time. Further I consider only a generic scheme.

- As it was demonstrated, a central element the BRST construction is some algebra of constraints which became to be a gauge algebra of final Lagrangian theory. In the previous cases, the constraints and their algebra was stipulated the conditions defying the irreducible representation of the Poincáre group. Since the Poincáre group is not a symmetry group of arbitrary curved space, we should find another way to study.
- If we want to preserve the concept of spin in a curved spacetime, we should restrict ourselves only by the space AdS, where the concept of spin exists.
- However, the infinite spin representation of the AdS group (group $SO(3,2)$) does not exist.
- However, one can put an aim to construct a formal generalization of free infinite spin filed Lagrangian formulation to one in AdS space. It can be do if to construct algebra of constraints in AdS space which will have a known flat limit in form of algebra of constrains for infinite spin filed theory in Minkowski space.

Generalization for infinite spin fields in AdS_4 space

Surprisingly it can be done!

1. Minimal inclusion of interaction with gravity. We change in the constraints of flat theory the partial derivatives by covariant ones and then we check whether the algebra of new constraints is closed. The algebra is not closed!
2. We add non-minimal terms containing curvature and look for them from the algebra closure condition. We add non-minimal terms containing curvature and look for them from the algebra closure condition. This is not a trivial job at all. A priori there are no guarantees that this can be done. It turned out that it can be done (I.L.B, S.A. Fedoruk, A.P. Isaev, V.A. Krykhtin arXiv:2403.14446 [hep-th])!
3. . Taking into account this constraint algebra we apply the BRST formalism and derive the Lagrangian formulation. As in all previous cases, this formulation has a large gauge freedom and by applying different gauges we can obtain formally different but equivalent Lagrangians.

Thank you very much for attention.